# The $\mathcal{S}$-Transform of Sub-fBm and an Application to a Class of Linear Subfractional BSDEs 

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#### Abstract

Let $S^{H}$ be a subfractional Brownian motion with index $0<H<1$. Based on the $\mathcal{\delta}$-transform in white noise analysis we study the stochastic integral with respect to $S^{H}$, and we also prove a Girsanov theorem and derive an Itô formula. As an application we study the solutions of backward stochastic differential equations driven by $S^{H}$ of the form $-d Y_{t}=f\left(t, Y_{t}, Z_{t}\right) d t-Z_{t} d S_{t}^{H}, t \in[0, T], Y_{T}=\xi$, where the stochastic integral used in the above equation is Pettis integral. We obtain the explicit solutions of this class of equations under suitable assumptions.


## 1. Introduction

As an extension of Brownian motion, Bojdecki et al. [1, 2] introduced and studied a rather special class of self-similar Gaussian processes which preserves many properties of the fractional Brownian motion of the Weyl type here and below. This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition. This process is called the subfractional Brownian motion (sub$\mathrm{fBm})$. The so-called sub-fBm with index $H \in(0,1)$ is a mean zero Gaussian process $S^{H}=\left\{S_{t}^{H}, t \geq 0\right\}$ with $S_{0}^{H}=0$ and the covariance

$$
\begin{equation*}
E\left[S_{t}^{H} S_{s}^{H}\right]=s^{2 H}+t^{2 H}-\frac{1}{2}\left[(s+t)^{2 H}+|t-s|^{2 H}\right] \tag{1}
\end{equation*}
$$

for all $s, t \geq 0$. For $H=1 / 2, S^{H}$ coincides with the standard Brownian motion B. $S^{H}$ is neither a semimartingale nor a Markov process unless $H=1 / 2$. So many of the powerful techniques from stochastic analysis are not available when dealing with $S^{H}$. As a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to $S^{H}$ (see, e.g., Alòs et al. [3] and Nualart [4]). The sub-fBm has
properties analogous to those of fractional Brownian motion and satisfies the following estimates:

$$
\begin{align*}
{\left[\left(2-2^{2 H-1}\right) \wedge 1\right](t-s)^{2 H} } & \leq E\left[\left(S_{t}^{H}-S_{s}^{H}\right)^{2}\right] \\
& \leq\left[\left(2-2^{2 H-1}\right) \vee 1\right](t-s)^{2 H} \tag{2}
\end{align*}
$$

Thus, Kolmogorov's continuity criterion implies that subfractional Brownian motion is Hölder continuous of order $\gamma$ for any $\gamma<H$. But its increments are not stationary. More works for sub- fBm can be found in Bojdecki et al. [5], Liu and Yan [6], Shen and Chen [7], Tudor [8-11], Yan et al. [12-14], and the references therein.

On the other hand, it is well known that general backward stochastic differential equations (BSDEs) driven by a Brownian motion were first studied by Pardoux and Peng [15], where they also gave a probabilistic interpretation for the viscosity solution of semilinear partial differential equations. Because of their important value in various areas including probability theory, finance, and control, BSDEs have been subject to the attention and interest of researchers. A survey and complete literature for BSDEs could be found in Peng [16]. Recently, motivated by stochastic control problems, Biagini et al. [17]
first studied linear BSDEs driven by a fractional Brownian motion, where existence and uniqueness were discussed in order to study a maximum principle. Bender [18] gave explicit solutions for a linear BSDEs driven by a fractional Brownian motion, and Hu and Peng [19] studied the linear and nonlinear BSDEs driven by a fractional Brownian motion using the quasi-conditional expectation. More works for the BSDEs driven by Brownian motion and fractional Brownian motion can be found in Bisumt [20], Geiss et al. [21], Karoui et al. [22], Ma et al. [23], Maticiuc and Nie [24], Peng [25], and the references therein. In this paper, we study the BSDEs driven by a sub- $\mathrm{fBm} S^{H}$ of the form

$$
\begin{gather*}
-d Y_{t}=f\left(t, Y_{t}, Z_{t}\right) d t-Z_{t} d S_{t}^{H}, \quad t \in[0, T]  \tag{3}\\
Y_{T}=\xi
\end{gather*}
$$

where the stochastic integral used in above equation is Pettis integral.

In recent years, there has been considerable interest in studying fractional Brownian motion due to its applications in various scientific areas including telecommunications, turbulence, image processing, and finance and also due to some of its compact properties such as long-range dependence, self-similarity, stationary increments, and Hölder's continuity (see, e.g., Mandelbrot and van Ness [26], Biagini et al. [27], Hu [28], Mishura [29], Li [30], Li and Zhao [31, 32], and Lim and Muniandy [33]). Moreover, many authors have proposed to use more general self-similar Gaussian processes and random fields as stochastic models. Such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. Therefore, other generalizations of Brownian motion have been introduced such as sub-fBm, bifractional Brownian motion, and weighted-fractional Brownian motion. However, in contrast to the extensive studies on fractional Brownian motion, there has been little systematic investigation on other self-similar Gaussian processes. The main reason for this is the complexity of dependence structures for self-similar Gaussian processes which do not have stationary increments. The sub- fBm has properties analogous to those of fractional Brownian motion (self-similarity, long-range dependence, Hölder paths, the variation, and the renormalized variation). However, in comparison with fractional Brownian motion, the sub-fBm has nonstationary increments and the increments over nonoverlapping intervals are more either weakly or strongly correlated and their covariance decays polynomially as a higher rate in comparison with fractional Brownian motion (for this reason in Bojdecki et al. [1] is called subfractional Brownian motion). The above mentioned properties make sub-fBm a possible candidate for models which involve long-range dependence, self-similarity, and nonstationary. Thus, it seems interesting to study the BSDEs driven by a sub-fBm.

This paper is organized as follows. Section 2 contains some basic results. In Section 3, we give a definition of subfractional Itô integral based on an $\mathcal{S}$-transform in white noise analysis. As an application we establish a Girsanov theorem for this integral. In Section 4, we give an Itô formula for functionals of a Wiener integral for a sub-fBm. We also
discuss the geometric sub- fBm in this section. Section 5 considers the BSDEs (3). Finally, we will conclude the paper in Section 6.

## 2. Preliminaries

In this section, we briefly recall some basic definitions and results of sub- fBm . Throughout this paper we assume that $0<$ $H<1$ is arbitrary but fixed and let $S^{H}=\left\{S_{t}^{H}, 0 \leq t \leq T\right\}$ be a one-dimensional sub-fBm with Hurst index $H$ defined on $\left(\Omega, \mathscr{F}^{H}, P\right)$. To simplify, we denote $\alpha=H-1 / 2$, and let $B=\left\{B_{t}\right\}_{t \in \mathbb{R}}$ be a two-sides Brownian motion and

$$
1_{(a, b)}(t)= \begin{cases}1, & \text { if } a \leq t<b  \tag{4}\\ -1, & \text { if } b \leq t<a \\ 0, & \text { others }\end{cases}
$$

We also denote
(i) $|f|_{2}$ : the usual $L^{2}(\mathbb{R})$-norm, and the corresponding inner product is denoted by $(f, g)_{2}$;
(ii) $S(\mathbb{R})$ : the Schwartz space of rapidly decreasing smooth functions of real valued;
(iii) $I(f)$ : the Wiener integral $\int_{\mathbb{R}} f(s) d B_{s}$ of the function $f \in L^{2}(\mathbb{R}) ;$
(iv) $\mathscr{G}$ : the $\sigma$-field generated by $\left\{I(f), f \in L^{2}(\mathbb{R})\right\}$;
(v) $\|\Phi\|_{2}$ : the $L^{2}(\Omega, \mathscr{G}, P)$-norm.
$S^{H}$ can be written as a Volterra process with the following moving average representation:

$$
\begin{equation*}
S_{t}^{H}=C_{H}^{\alpha} \int_{\mathbb{R}}\left[(t-s)_{+}^{\alpha}+(t+s)_{-}^{\alpha}-2(-s)_{+}^{\alpha}\right] d B_{s}, \tag{5}
\end{equation*}
$$

where $C_{H}^{\alpha}=\Gamma(H+1 / 2) / \sqrt{H \sin \pi H \Gamma(2 H)}, x_{+}=\max (x, 0)$, $x_{-}=\max (-x, 0)$. The sub- $\mathrm{fBm} S^{H}$ is also possible to construct a stochastic calculus of variations with respect to the Gaussian process $S^{H}$, which will be related to the Malliavin calculus. Some surveys and complete literatures for Malliavin calculus of Gaussian process could be found in Alòs et al. [3], Nualart [4] and Tudor [9, 10], Zähle [34], and the references therein.

Let $0<\beta<1$. Consider Weyl's type fractional integrals $I_{ \pm}^{\beta}$ of order $\beta$

$$
\begin{align*}
& \left(I_{-}^{\beta} f\right)(x):=\frac{1}{\Gamma(\beta)} \int_{x}^{\infty} f(t)(t-x)^{\beta-1} d t \\
& \left(I_{+}^{\beta} f\right)(x):=\frac{1}{\Gamma(\beta)} \int_{-\infty}^{x} f(t)(x-t)^{\beta-1} d t \tag{6}
\end{align*}
$$

if the integrals exist for almost all $x \in \mathbb{R}$, and Marchand's type fractional derivatives $D_{ \pm}^{\beta}$ of order $\beta$

$$
\begin{equation*}
\left(D_{ \pm}^{\beta} f\right):=\lim _{\varepsilon \downarrow 0^{+}}\left(D_{ \pm, \varepsilon}^{\beta} f\right) \tag{7}
\end{equation*}
$$

if the limit exists in $L^{p}(\mathbb{R})$ for some $p>1$, where

$$
\begin{equation*}
\left(D_{ \pm, \varepsilon}^{\beta} f\right)(x):=\frac{\beta}{\Gamma(1-\beta)} \int_{\varepsilon}^{\infty} \frac{f(x)-(x \mp t)}{t^{1+\beta}} d t \tag{8}
\end{equation*}
$$

for $\varepsilon>0$. Define the operator

$$
M_{ \pm}^{H} f:= \begin{cases}C_{H} D_{ \pm}^{-\alpha} f, & \text { if } 0<H<\frac{1}{2}  \tag{9}\\ f, & \text { if } H=\frac{1}{2} \\ C_{H} I_{ \pm}^{\alpha} f, & \text { if } \frac{1}{2}<H<1\end{cases}
$$

where $C_{H}=\sqrt{2 H \sin \pi H \Gamma(2 H)}$ and $\Gamma(\cdot)$ denotes the gamma function defined by

$$
\begin{equation*}
\Gamma(z)=\int_{0}^{\infty} t^{z-1} e^{-t} d t, \quad z>0 \tag{10}
\end{equation*}
$$

Recall that we now give a stochastic version of the HardyLittlewood theorem as follows.

Theorem 1 (Theorem 2.10 in [35]). Let $1 / 2<H<1$ and let the operators $M_{ \pm}^{H}$ be defined as above. Then $M_{ \pm}^{H}$ is a continuous operator from $L^{p}(\mathbb{R} ; \Omega)$ into $L^{q}(\mathbb{R} ; \Omega)$ if $1<p<$ $2 /(2 H-1)$ and $q=2 p /(2-p(2 H-1))$.

Define the function

$$
f^{0}(x):= \begin{cases}f(x), & x \geq 0  \tag{11}\\ -f(-x), & x<0\end{cases}
$$

for any Borel function $f$ on $\mathbb{R}_{+}$. Then the function $f^{0}$ is odd, which is called the odd extension of $f$. Based on the moving average representation (5), we can show the following proposition.

Proposition 2. Let the operators $M_{ \pm}^{H}$ be defined as above. Then $M_{-}^{H}\left(1_{[0, t)}^{0}\right) \in L^{2}(\mathbb{R})$ and $S^{H}$ admits the following integral representation:

$$
\begin{equation*}
S_{t}^{H}=\frac{1}{\sqrt{2}} \int_{\mathbb{R}} M_{-}^{H}\left(1_{[0, t)}^{0}\right)(s) d B_{s} \tag{12}
\end{equation*}
$$

for all $t \geq 0$.
We finally recall the $\mathcal{S}$-transform. The $\mathcal{S}$-transform is an important tool in white noise analysis. Here we give a definition and state some results that do not depend on properties of the white noise space. Denote the $\mathcal{S}$-transform of $\Phi \in L^{2}(\Omega, \mathscr{G}, P)$ (see, e.g., [35,36] for more details) by

$$
\begin{equation*}
\mathcal{S} \Phi(\eta):=E\left[\Phi \cdot: e^{I(\eta)}:\right], \quad \eta \in \mathcal{S}(\mathbb{R}), \tag{13}
\end{equation*}
$$

where the Wick exponential : $e^{I(\eta)}$ : of $I(\eta)$ is given by

$$
\begin{equation*}
: e^{I(\eta)}:=e^{I(\eta)-(1 / 2)|\eta|_{2}^{2}} \tag{14}
\end{equation*}
$$

The $\mathcal{S}$-transform has the following important properties.
$\left(A_{1}\right)$ The $\mathcal{S}$-transform is injective; that is, $\mathcal{S} \Phi(\eta)=\mathcal{S} \Psi(\eta)$ for all $\eta \in \mathcal{S}(\mathbb{R})$, implies that $\Phi=\Psi$.
$\left(A_{2}\right)$ Let $f_{n}$ be a sequence that converges to $f$ in $L^{2}(\mathbb{R})$; then $: e^{I\left(f_{n}\right)}:$ converges to $: e^{I(f)}:$ in $L^{2}(\mathbb{R})$.
$\left(A_{3}\right) E\left[: e^{I(f)}:\right]=1$ for $f \in L^{2}(\mathbb{R})$. Hence it can deduce a probability measure on $\mathscr{F}$ by

$$
\begin{equation*}
d Q_{f}=: e^{I(f)}: d P \tag{15}
\end{equation*}
$$

especially, for $\eta \in \mathcal{S}(\mathbb{R})$, we can rewrite the $\mathcal{S}$-transform as

$$
\begin{equation*}
\mathcal{S} \Phi(\eta)=E^{\mathrm{Q}_{\eta}}[\Phi] \tag{16}
\end{equation*}
$$

$\left(A_{4}\right)$ Let $X: \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ be a progressively measurable process such that

$$
\begin{equation*}
E \int_{\mathbb{R}}\left|X_{t}\right|^{2} d t<\infty \tag{17}
\end{equation*}
$$

Then $\int_{\mathbb{R}} X_{t} d B_{t}$ is the unique element in $L^{2}(\Omega, \mathscr{G}, P)$ with $\mathcal{S}$-transform given by

$$
\begin{equation*}
\int_{\mathbb{R}}\left(\delta X_{t}\right)(\eta) \eta(t) d t . \tag{18}
\end{equation*}
$$

$\left(A_{5}\right)$ The Wiener integral $I(f), f \in L^{2}(\mathbb{R})$ is the unique element in $L^{2}(\Omega, \mathscr{G}, P)$ with $\mathcal{S}$-transform given by

$$
\begin{equation*}
\int_{\mathbb{R}} f(t) \eta(t) d t . \tag{19}
\end{equation*}
$$

The following result points out that the operators $M_{ \pm}^{H}$ interchanges with the $\mathcal{S}$-transform.

Lemma 3 (Lemma 2.9 in [35]). Let $M_{ \pm}^{H} X$ exist for some $X$ : $\mathbb{R} \rightarrow L^{2}(\Omega, \mathscr{G}, P)$. Then one has

$$
\begin{equation*}
E\left[\left(M_{ \pm}^{H} X\right)_{t} \Psi\right]=M_{ \pm}^{H}\left(E\left[X_{t} \Psi\right]\right) \tag{20}
\end{equation*}
$$

for all $\Psi \in L^{2}(\Omega, \mathscr{G}, P)$. In the case $H<1 / 2$ the convergence of the fractional derivative on the right-hand side is in the $L^{p}(\mathbb{R})$ sense, if $M_{ \pm}^{-(H-1 / 2)} X \in L^{p}\left(\mathbb{R} ; L^{2}(\Omega, \mathscr{G}, P)\right)$. In particular, the operators $M_{ \pm}^{H}$ interchange with the $\mathcal{S}$-transform.

## 3. A Subfractional Itô Integral

In this section, based on the $\mathcal{S}$-transform we aim to define the subfractional Itô integral, denoted by $\Phi=\int_{a}^{b} X_{t} d S_{t}^{H}$ with $0 \leq a<b$, and introduce the Girsanov theorem. To this end, inspired by the Hitsuda-Skorohod integral, we define the subfractional Itô integral as the unique random variable $\Phi$ such that

$$
\begin{equation*}
\mathcal{S} \Phi(\eta)=\int_{a}^{b} \mathcal{S}\left(X_{t}\right)(\eta) \frac{d}{d t} \mathcal{S}\left(S_{t}^{H}\right)(\eta) d t \tag{21}
\end{equation*}
$$

for all $\eta \in \mathcal{S}(\mathbb{R})$, provided the integral exists under suitable conditions. According to (12) and Property ( $A_{5}$ ), we have

$$
\begin{align*}
\frac{d}{d t} & \mathcal{S}\left(S_{t}^{H}\right)(\eta) \\
& =\frac{1}{\sqrt{2}} \frac{d}{d t} \int_{\mathbb{R}} M_{-}^{H}\left(1_{[0, t)}^{0}\right) \eta(s) d s \\
& =\frac{1}{\sqrt{2}} \frac{d}{d t} \int_{\mathbb{R}}\left(1_{[0, t)}^{0}\right) M_{+}^{H} \eta(s) d s  \tag{22}\\
& =\frac{1}{\sqrt{2}} \frac{d}{d t} \int_{0}^{t} M_{+}^{H} \eta(s) d s-\frac{1}{\sqrt{2}} \frac{d}{d t} \int_{-t}^{0} M_{+}^{H} \eta(s) d s \\
& =\frac{1}{\sqrt{2}}\left[M_{+}^{H} \eta(t)-M_{+}^{H} \eta(-t)\right] .
\end{align*}
$$

Combining this with the fact $\left(A_{1}\right)$ in Section 2, we give the following definition.

Definition 4. Let $M \subset \mathbb{R}^{+}$be a Borel set. A mapping $X$ : $M \rightarrow L^{2}(\Omega, \mathscr{G}, P)$ is said to be subfractional Itô integrable on $M$ if

$$
\begin{equation*}
(\mathcal{S} X .)(\eta)\left[\left(M_{+}^{H} \eta\right)(\cdot)-\left(M_{+}^{H} \eta\right)(-\cdot)\right] \in L^{1}(M) \tag{23}
\end{equation*}
$$

for any $\eta \in \mathcal{S}(\mathbb{R})$, and there is a $\Phi \in L^{2}(\Omega, \mathscr{G}, P)$ such that

$$
\begin{equation*}
\mathcal{S} \Phi(\eta)=\frac{1}{\sqrt{2}} \int_{M} \mathcal{S}\left(X_{t}\right)(\eta)\left[\left(M_{+}^{H} \eta\right)(t)-\left(M_{+}^{H} \eta\right)(-t)\right] d t \tag{24}
\end{equation*}
$$

for all $\eta \in S(\mathbb{R})$.
It is important to note that $\Phi$ in the above definition is unique because the $\mathcal{\delta}$-transform is injective, which is called the subfractional Itô integral of $X$ on $M$ and we denote it by

$$
\begin{equation*}
\Phi=\int_{M} X_{t} d S_{t}^{H} \tag{25}
\end{equation*}
$$

In this paper, sub-fractional Itô integralalways refers to the $\mathcal{S}$-transform approach proposed in Definition 4.

Proposition 5. The following statements hold.
(1) For any $a<b$ one has

$$
\begin{equation*}
S_{b}^{H}-S_{a}^{H}=\int_{a}^{b} d S_{t}^{H} \tag{26}
\end{equation*}
$$

(2) Let $X:[a, b] \rightarrow L^{2}(\Omega, \mathscr{G}, P)$ be subfractional Itô integrable for $0 \leq a<b$. Then

$$
\begin{gather*}
\int_{a}^{b} X_{t} d S_{t}^{H}=\int_{\mathbb{R}} 1_{[a, b]}(t) X_{t} d S_{t}^{H} \\
E\left[\int_{a}^{b} X_{t} d S_{t}^{H}\right]=0 \tag{27}
\end{gather*}
$$

Proof. These results are some simple examples.

Recall that the Wick product $F \diamond G$ of $F, G \in L^{2}(\Omega, \mathscr{G}, P)$ is an element $F \diamond G \in L^{2}(\Omega, \mathscr{G}, P)$ such that

$$
\begin{equation*}
\mathcal{S}(F \diamond G)(\eta)=\mathcal{S}(F)(\eta) \mathcal{S}(G)(\eta) \tag{28}
\end{equation*}
$$

for all $\eta \in \mathcal{S}(\mathbb{R})$. The following theorem expresses the relationship between the subfractional Itô integral defined as above and the integral based on Wick product $\diamond$.

Theorem 6. Let $X: \mathbb{R}^{+} \rightarrow L^{2}(\Omega, \mathscr{G}, P)$ and $Y \in L^{2}(\Omega, \mathscr{G}, P)$; then

$$
\begin{equation*}
Y \diamond \int_{\mathbb{R}^{+}} X_{s} d S_{s}^{H}=\int_{\mathbb{R}^{+}} Y \diamond X_{s} d S_{s}^{H} \tag{29}
\end{equation*}
$$

in the sense that if one side is well defined then so is the other, and both coincide.

We can obtain it by calculating the $\mathcal{\delta}$-transform of both sides. In particular, for $Y \in L^{2}(\Omega, \mathscr{G}, P)$, this theorem implies that

$$
\begin{equation*}
Y \diamond\left(S_{b}^{H}-S_{a}^{H}\right)=\int_{\mathbb{R}^{+}} 1_{(a, b)}(s) Y d S_{s}^{H} . \tag{30}
\end{equation*}
$$

It means that the subfractional Itô integral is the $L^{2}(\Omega, \mathscr{G}, P)$ limit of Wick-Riemann sums for some suitable processes. That is,

$$
\begin{equation*}
\int_{0}^{T} X_{s} d S_{s}^{H}=\lim _{\left|\pi_{n}\right| \rightarrow 0} \sum_{i=0}^{n} X_{s_{i}} \diamond\left(S_{s_{i+1}}^{H}-S_{s_{i}}^{H}\right) \tag{31}
\end{equation*}
$$

for some suitable processes $X$, where $\pi_{n}=\left\{0=s_{0}<s_{1}<\right.$ $\left.\cdots<s_{n+1}=T\right\}$ is a partition of $[0, T]$ with $\left|\pi_{n}\right|:=\max \left\{s_{i+1}-\right.$ $\left.s_{i}\right\}$ and the convergence is in $L^{2}(\Omega, \mathscr{G}, P)$.

Now we calculate the expectation of a subfractional Itô integral under a measure $Q_{f}, f \in L^{2}(\mathbb{R})$.

Theorem 7. Let $0<H<1$ and $Q_{f}, f \in L^{2}(\mathbb{R})$ be given by (15). If the following assumptions hold:
(1) $X: \mathbb{R}^{+} \rightarrow L^{2}(\Omega, \mathscr{G}, P)$ is subfractional Itô integrable, and $X \in L^{1 / H}\left(\mathbb{R}^{+}, L^{2}(\Omega, \mathscr{G}, P)\right)$;
(2) $M_{+}^{H} f \in L^{1 /(1-H)}$ and $M_{-}^{H} X \in L^{2}\left(\mathbb{R}^{+}\right)$for $H<1 / 2$,

One then has

$$
\begin{align*}
E^{Q_{f}} & {\left[\int_{\mathbb{R}^{+}} X_{t} d S_{t}^{H}\right] } \\
& =\frac{1}{\sqrt{2}} \int_{\mathbb{R}^{+}} E^{Q_{f}}\left[X_{t}\right]\left[\left(M_{+}^{H} f\right)(t)-\left(M_{+}^{H} f\right)(-t)\right] d t \tag{32}
\end{align*}
$$

Proof. Let $\left(\eta_{n}\right)_{n \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R})$ be given such that $\eta_{n}$ converges to $f$ in $L^{2}(\mathbb{R})$, we have the identity

$$
\begin{align*}
E^{Q_{\eta_{n}}} & {\left[\int_{\mathbb{R}^{+}} X_{t} d S_{t}^{H}\right] } \\
& =\frac{1}{\sqrt{2}} \int_{\mathbb{R}^{+}} E^{Q_{\eta_{n}}}\left[X_{t}\right]\left[\left(M_{+}^{H} \eta_{n}\right)(t)-\left(M_{+}^{H} \eta_{n}\right)(-t)\right] d t . \tag{33}
\end{align*}
$$

It can be easily obtained that the left-hand side of (33) converges to the same side of (32) by Theorem 1 and $\left(A_{2}\right)$ in Section 2.

Then we just need to prove the right-hand side of (33) converges to (32) correspondingly. By Lemma 3, applying the fractional integration by parts rule, we have

$$
\begin{align*}
& \mid \int_{\mathbb{R}^{+}} E^{\mathrm{Q}_{\left(\eta_{n}\right)}}\left[X_{t}\right]\left[\left(M_{+}^{H} \eta_{n}\right)(t)-\left(M_{+}^{H} \eta_{n}\right)(-t)\right] \\
& \quad-E^{Q_{f}}\left[X_{t}\right]\left[\left(M_{+}^{H} f\right)(t)-\left(M_{+}^{H} f\right)(-t)\right] d t \mid \\
& \quad=\mid \int_{\mathbb{R}^{+}} E\left[X_{t}: e^{I\left(\eta_{n}\right)}:\right]\left[\left(M_{+}^{H} \eta_{n}\right)(t)-\left(M_{+}^{H} \eta_{n}\right)(-t)\right] \\
& \quad-E\left[X_{t}: e^{I(f)}:\right]\left[\left(M_{+}^{H} f\right)(t)-\left(M_{+}^{H} f\right)(-t)\right] d t \mid \tag{34}
\end{align*}
$$

which is bounded by

$$
\begin{align*}
& \left|\int_{\mathbb{R}^{+}} E\left[\left(M_{-}^{H} X\right)_{t}: e^{I\left(\eta_{n}\right)}:\right] \eta_{n}(t)-E\left[\left(M_{-}^{H} X\right)_{t}: e^{I(f)}:\right] f(t) d t\right| \\
& +\left|\int_{\mathbb{R}^{+}} E\left[\left(M_{-}^{H} X\right)_{t}: e^{I(f)}:\right] f(-t)-E\left[\left(M_{-}^{H} X\right)_{t}: e^{I\left(\eta_{n}\right)}:\right] \eta_{n}(-t) d t\right| \\
& \quad \leq\left|\int_{\mathbb{R}^{+}} E\left[\left|\left(M_{-}^{H} X\right)_{t}\left(: e^{I\left(\eta_{n}\right)}:-: e^{I(f)}:\right)\right|\right]\right| \eta_{n}(t)|d t| \\
& \quad+\left|\int_{\mathbb{R}^{+}} E\left[\left|\left(M_{-}^{H} X\right)_{t}: e^{I(f)}:\right|\right]\right| \eta_{n}(t)-f(t)|d t| \\
& \quad+\left|\int_{\mathbb{R}^{+}} E\left[\left|\left(M_{-}^{H} X\right)_{t}\left(: e^{I\left(\eta_{n}\right)}:-: e^{I(f)}:\right)\right|\right]\right| \eta_{n}(-t)|d t| \\
& \quad+\left|\int_{\mathbb{R}^{+}} E\left[\left|\left(M_{-}^{H} X\right)_{t}: e^{I(f)}:\right|\right]\right| \eta_{n}(-t)-f(-t)|d t| \\
& \quad \equiv I_{1}+I_{2}+I_{3}+I_{4} . \tag{35}
\end{align*}
$$

We can easily show that $I_{1}, I_{2}, I_{3}, I_{4}$ converge to zero, as $n \rightarrow$ $\infty$, respectively, by Hölder's inequality. This completes the proof.

Remark 8. Under the assumptions of Theorem 7, $\int_{\mathbb{R}^{+}} X_{t}$ $\left[\left(M_{+}^{H} f\right)(t)-\left(M_{+}^{H} f\right)(-t)\right] d t$ exists as a Pettis integral (see Definition 2.3 in [35]). In fact, for all $\Phi \in L^{2}(\Omega, \mathscr{G}, P)$,

$$
\begin{aligned}
\int_{\mathbb{R}^{+}} \mid E & {\left[X_{t}\left[\left(M_{+}^{H} f\right)(t)-\left(M_{+}^{H} f\right)(-t)\right] \Phi\right] \mid d t } \\
& \leq\left(\int_{\mathbb{R}^{+}}\left|\left(M_{+}^{H} f\right)(t)-\left(M_{+}^{H} f\right)(-t)\right|^{1 /(1-H)} d t\right)^{1-H} \\
& \times\left(\int_{\mathbb{R}^{+}} E\left[\left|X_{t}\right|^{2}\right]^{1 / 2 H} d t\right)^{H} E\left[|\Phi|^{2}\right]^{1 / 2}
\end{aligned}
$$

$$
\begin{align*}
\leq & {\left[\left(\int_{\mathbb{R}^{+}}\left|\left(M_{+}^{H} f\right)(t)\right|^{1 /(1-H)} d t\right)^{1-H}\right.} \\
& \left.+\left(\int_{\mathbb{R}^{+}}\left|\left(M_{+}^{H} f\right)(-t)\right|^{1 /(1-H)} d t\right)^{1-H}\right] \\
& \times\left(\int_{\mathbb{R}^{+}} E\left[\left|X_{t}\right|^{2}\right]^{1 / 2 H} d t\right)^{H} E\left[|\Phi|^{2}\right]^{1 / 2} \\
< & \infty \tag{36}
\end{align*}
$$

Thus, the property of the Pettis integral deduces

$$
\begin{align*}
E^{\mathrm{Q}_{f}} & {\left[\int_{\mathbb{R}^{+}} X_{t} d S_{t}^{H}\right] } \\
& \quad=\frac{1}{\sqrt{2}} E^{\mathrm{Q}_{f}} \int_{\mathbb{R}^{+}} X_{t}\left[\left(M_{+}^{H} f\right)(t)-\left(M_{+}^{H} f\right)(-t)\right] d t \tag{37}
\end{align*}
$$

Now, we establish a Girsanov theorem for subfractional Itô integral. Consider the measure $Q_{f}, f \in L^{2}(\mathbb{R})$, the probability space $\left(\Omega, \mathscr{F}, Q_{f}\right)$ carries a two-side Brownian motion given by

$$
\begin{equation*}
\widetilde{B}_{t}=B_{t}-\int_{0}^{t} f(s) d s \tag{38}
\end{equation*}
$$

according to the classical Girsanov theorem. On this probability space, we denote $\mathcal{S}_{\mathrm{Q}_{f}}$ the $\mathcal{S}$-transform with respect to the measure $Q_{f}, f \in L^{2}(\mathbb{R})$; that is,

$$
\begin{equation*}
\left(\mathcal{S}_{\mathrm{Q}_{f}} X\right)(\eta):=E^{\mathrm{Q}_{f}}\left[: e^{I^{\tilde{B}}(\eta)}: X\right] \tag{39}
\end{equation*}
$$

and the following identity holds:

$$
\begin{equation*}
: e^{I^{\tilde{B}}(g)}: \cdot: e^{I(f)}:=: e^{I(f+g)}: \tag{40}
\end{equation*}
$$

for all $g \in L^{2}(\mathbb{R})$.
Theorem 9. Let the assumptions of Theorem 7 be satisfied, and
$E^{Q_{f}}\left[\left|\int_{\mathbb{R}^{+}} X_{t} d S_{t}^{H}-\frac{1}{\sqrt{2}} \int_{\mathbb{R}^{+}} X_{t}\left[\left(M_{+}^{H} f\right)(t)-\left(M_{+}^{H} f\right)(-t)\right] d t\right|^{2}\right]$
$<\infty$.

Then, the identity

$$
\begin{align*}
& \int_{\mathbb{R}^{+}} X_{t} d \widetilde{S}_{t}^{H} \\
& \quad=\int_{\mathbb{R}^{+}} X_{t} d S_{t}^{H}-\frac{1}{\sqrt{2}} \int_{\mathbb{R}^{+}} X_{t}\left[\left(M_{+}^{H} f\right)(t)-\left(M_{+}^{H} f\right)(-t)\right] d t \tag{42}
\end{align*}
$$

holds in $L^{2}\left(\Omega, \mathscr{F}, Q_{f}\right)$-almost surely.

Proof. We apply Theorem 7 to $f+\eta, \eta \in \mathcal{S}(\mathbb{R})$. It is easy to check that $M_{+}^{H}(f+\eta) \in L^{1 /(1-H)}(\mathbb{R})$ according to Lemma 2.5 in [36]. By Theorem 7 and (40), it follows

$$
\begin{align*}
& \mathcal{S}_{\mathrm{Q}_{f}}\left(\int_{\mathbb{R}^{+}} X_{t} d S_{t}^{H}-\int_{\mathbb{R}^{+}} X_{t}\left[\left(M_{+}^{H} f\right)(t)-\left(M_{+}^{H} f\right)(-t)\right] d t\right)(\eta) \\
&= E^{Q_{f+\eta}}\left[\int_{\mathbb{R}^{+}} X_{t} d S_{t}^{H}-\int_{\mathbb{R}^{+}} X_{t}\left[\left(M_{+}^{H} f\right)(t)-\left(M_{+}^{H} f\right)(-t)\right] d t\right] \\
&= \frac{1}{\sqrt{2}} \int_{\mathbb{R}^{+}} E^{Q_{f+\eta}}\left[X_{t}\right]\left[M_{+}^{H}(f+\eta)(t)-M_{+}^{H}(f+\eta)(-t)\right] d t \\
&-\frac{1}{\sqrt{2}} \int_{\mathbb{R}^{+}} E^{Q_{f+\eta}}\left[X_{t}\right]\left[\left(M_{+}^{H} f\right)(t)-\left(M_{+}^{H} f\right)(-t)\right] d t \\
&= \frac{1}{\sqrt{2}} \int_{\mathbb{R}^{+}} E^{Q_{f+\eta}}\left[X_{t}\right]\left[\left(M_{+}^{H} \eta\right)(t)-\left(M_{+}^{H} \eta\right)(-t)\right] d t \\
&= \frac{1}{\sqrt{2}} \int_{\mathbb{R}^{+}} \mathcal{S}_{\mathrm{Q}_{f}} X_{t}(\eta)\left[\left(M_{+}^{H} \eta\right)(t)-\left(M_{+}^{H} \eta\right)(-t)\right] d t . \tag{43}
\end{align*}
$$

The second identity based on the fact that $\int_{\mathbb{R}^{+}} X_{t}\left[\left(M_{+}^{H} f\right)(t)-\right.$ $\left.\left(M_{+}^{H} f\right)(-t)\right] d t$ exists as a Pettis integral which is proved in Remark 8. The proof is complete.

## 4. An Itô Formula

In this section, we prove an Itô formula for a subfractional Wiener integral using the $\mathcal{S}$-transform approach. An indefinite subfractional Wiener integral is understood as a process

$$
\begin{equation*}
X_{t}=\int_{0}^{t} \varphi(s) d S_{s}^{H} \equiv \int_{\mathbb{R}} 1_{[0, t]}(s) \varphi(s) d S_{s}^{H} \tag{44}
\end{equation*}
$$

for all $0 \leq t \leq T$ provided $\varphi$ is a deterministic function such that the above integral exists as a subfractional Itô integral for all $0 \leq t \leq T$.

Proposition 10. Assume that $\varphi:[0, T] \rightarrow \mathbb{R}$ is continuous for $1 / 2 \leq H<1$, and $\lambda$-Hölder continuous with $\lambda>1 / 2-H$ for $0<H<1 / 2$. Then the indefinite subfractional Wiener integral $\int_{0}^{t} \varphi(s) d S_{s}^{H}$ exists, and

$$
\begin{equation*}
\int_{0}^{t} \varphi(s) d S_{s}^{H}=\frac{1}{\sqrt{2}} I\left(M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right) \tag{45}
\end{equation*}
$$

Proof. We should prove that $M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0} \in L^{2}(\mathbb{R})$ and $\mathcal{S}\left(\int_{0}^{t} \varphi(s) d S_{s}^{H}\right)(\eta)$ exists.

For $1 / 2 \leq H<1$, since $\varphi$ is continuous on $[0, T]$, by Hardy-Littlwood theorem, it is obvious that $M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0} \in$ $L^{2}(\mathbb{R})$.

For $0<H<1 / 2$, similar to the argument in Proposition 5.1 in [35], there exists a function $g \in L^{2}(\mathbb{R})$, such that

$$
\begin{equation*}
1_{(0, t)} \varphi=I_{-}^{1 / 2-H} g . \tag{46}
\end{equation*}
$$

Hence, $M_{-}^{H}\left(1_{(0, t)} \varphi\right) \in L^{2}(\mathbb{R})$, and so is $M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0} \in L^{2}(\mathbb{R})$. $\varphi$ is a deterministic function implies that $\delta\left(\int_{0}^{t} \varphi(s) d S_{s}^{H}\right)(\eta)$ exists.

Next, consider the $\mathcal{S}$-transform of the right-hand side in (45), then by (19), we obtain that

$$
\begin{align*}
\mathcal{S} & \left(\frac{1}{\sqrt{2}} I\left(M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right)\right)(\eta) \\
& =\frac{1}{\sqrt{2}} \int_{\mathbb{R}} M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0} \eta d s \\
& =\frac{1}{\sqrt{2}} \int_{\mathbb{R}}\left(1_{(0, t)} \varphi\right)^{0} M_{+}^{H}(\eta) d s \\
& =\frac{1}{\sqrt{2}} \int_{0}^{t} \varphi(s)\left(M_{+}^{H} \eta\right)(s) d s+\frac{1}{\sqrt{2}} \int_{0}^{-t} \varphi(-s)\left(M_{+}^{H} \eta\right)(s) d s \\
& =\frac{1}{\sqrt{2}} \int_{0}^{t} \varphi(s)\left(M_{+}^{H} \eta\right)(s) d s-\frac{1}{\sqrt{2}} \int_{0}^{t} \varphi(s)\left(M_{+}^{H} \eta\right)(-s) d s \\
& =\delta\left(\int_{0}^{t} \varphi(s) d S_{s}^{H}\right)(\eta) . \tag{47}
\end{align*}
$$

This completes the proof.

The following lemma is essential to the proof of our Itô's formula.

Lemma 11. Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be continuous and $H>1 / 2$. Then one has

$$
\begin{align*}
& \left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2} \\
& \quad=8 \alpha H \int_{0}^{t} \int_{0}^{\tau} \varphi(s) \varphi(\tau)\left[(\tau-s)^{2 \alpha-1}+(s+\tau)^{2 \alpha-1}\right] d s d \tau . \tag{48}
\end{align*}
$$

In particular,
(1) for all $t>0,\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2} \leq 3 \max _{s \in[0, t]}|\varphi(s)|^{2} t^{2 H}$;
(2) $\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2}$ is differentiable in $t$, and for all $t \geq 0$, one has

$$
\begin{align*}
\left.\frac{d}{d t} \right\rvert\, & \left.M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2} ^{2} \\
& =8 \alpha H \varphi(t) \int_{0}^{t} \varphi(s)\left[(t-s)^{2 \alpha-1}+(s+t)^{2 \alpha-1}\right] d s  \tag{49}\\
& \leq 4 H \max _{s \in[0, t]}|\varphi(s)|^{2} t^{2 \alpha} .
\end{align*}
$$

Proof. For $H>1 / 2$, the following identity holds:

$$
\begin{equation*}
\left|M_{-}^{H} \varphi\right|_{2}^{2}=2 \alpha H \iint_{\mathbb{R}} \varphi(s) \varphi(\tau)|s-\tau|^{2 \alpha-1} d s d \tau \tag{50}
\end{equation*}
$$

Then,

$$
\begin{align*}
& \left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2} \\
& =2 \alpha H \iint_{\mathbb{R}}\left(1_{(0, t)} \varphi\right)^{0}(s)\left(1_{(0, t)} \varphi\right)^{0}(\tau)|s-\tau|^{2 \alpha-1} d s d \tau \\
& =4 \alpha H \iint_{0}^{t} \varphi(s) \varphi(\tau)|s-\tau|^{2 \alpha-1} d s d \tau \\
& \quad+4 \alpha H \iint_{0}^{t} \varphi(s) \varphi(\tau)|s+\tau|^{2 \alpha-1} d s d \tau \\
& =8 \alpha H \int_{0}^{t} \int_{0}^{\tau} \varphi(s) \varphi(\tau)|s-\tau|^{2 \alpha-1} d s d \tau \\
& \quad+8 \alpha H \int_{0}^{t} \int_{0}^{\tau} \varphi(s) \varphi(\tau)|s+\tau|^{2 \alpha-1} d s d \tau \tag{51}
\end{align*}
$$

Equation (48) easily follows and the other assertions are trivial.

Remark 12. Since the right of (48) is not hold when $H<$ $1 / 2$, there is a lack of a result similar to the above Lemma. Hence, we only consider the case of constant $\varphi$, and we have $\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2}=\left(2-2^{2 H-1}\right) \varphi^{2} t^{2 H}$.

Now we give the following Itô formula.
Theorem 13. Let $T>0$, such that
$\left(B_{1}\right) X$ be an indefinite subfractional Wiener integral; that is, for all $0 \leq t \leq T, X_{t}=\int_{0}^{t} \varphi(s) d S_{s}^{H}$, where $\varphi$ is continuous when $H \geq 1 / 2$, constant when $H<1 / 2$;
$\left(B_{2}\right) F \in \mathscr{C}^{1,2}([0, T] \times \mathbb{R}) ;$
$\left(B_{3}\right)$ there exists constants $C \geq 0$ and $\lambda \leq\left[2 \sqrt{3} T^{H}\right.$. $\left.\max _{s \in[0, T]}|\varphi(s)|\right]^{-2}$ such that
$\max \left\{|F(t, x)|,\left|\frac{\partial}{\partial t} F(t, x)\right|,\left|\frac{\partial}{\partial x} F(t, x)\right|,\left|\frac{\partial^{2}}{\partial x^{2}} F(t, x)\right|\right\}$

$$
\begin{equation*}
\leq C e^{\lambda x^{2}} \tag{52}
\end{equation*}
$$

Then the following equality holds in $\left(L^{2}\right)$ :

$$
\begin{align*}
\int_{0}^{T} & \varphi(t) \frac{\partial}{\partial x} F\left(t, X_{t}\right) d S_{t}^{H} \\
= & F\left(T, X_{T}\right)-F(0,0)-\int_{0}^{T} \frac{\partial}{\partial t} F\left(t, X_{t}\right) d t  \tag{53}\\
& -\frac{1}{2} \int_{0}^{T} \frac{d}{d t}\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{o}\right|_{2}^{2} \frac{\partial^{2}}{\partial x^{2}} F\left(t, X_{t}\right) d t
\end{align*}
$$

Proof. It suffices to show that both sides have the same $\mathcal{S}$ transform. Indeed, by Definition 4, the integral of the lefthand side has the $\mathcal{S}$-transform given by

$$
\begin{align*}
& \mathcal{S}\left(\int_{0}^{T} \varphi(t) \frac{\partial}{\partial x} F\left(t, X_{t}\right) d S_{t}^{H}\right)(\eta) \\
&= \frac{1}{\sqrt{2}} \int_{0}^{T} \varphi(t)\left[\left(M_{+}^{H} \eta\right)(t)-\left(M_{+}^{H} \eta\right)(-t)\right] \mathcal{S}  \tag{54}\\
& \quad \times\left(\frac{\partial}{\partial x} F\left(t, X_{t}\right)\right)(\eta) d t
\end{align*}
$$

Henceforth, we just need to show the right-hand side has the same result. Firstly, we show the integrals of the right-hand side exist in $\left(L^{2}\right)$. Without loss of generality, denote $G=F$, $(\partial / \partial t) F(t, x),(\partial / \partial x) F(t, x),\left(\partial^{2} / \partial x^{2}\right) F(t, x)$, and $0 \leq t \leq T$. By the growth condition (52), we obtain

$$
\begin{equation*}
\left\|G\left(t, X_{t}\right)\right\|_{2}^{2} \leq C^{2}\left(1-4 \lambda \mid\left(\left.M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2} ^{2}\right)^{(-1 / 2)} \leq\right. \text { const. } \tag{55}
\end{equation*}
$$

Consequently, $\int_{0}^{T}(\partial / \partial t) F\left(t, X_{t}\right) d t$ exists. For the last one, by Lemma 11 and Remark 12, we have

$$
\begin{align*}
& \int_{0}^{T}\left\|\frac{d}{d t}\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2} \frac{\partial}{\partial x^{2}} F\left(t, X_{t}\right)\right\|_{2} d t \\
& \left.\quad \leq\left.\int_{0}^{T}\left|\frac{d}{d t}\right| M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2} ^{2} \right\rvert\,\left\|\frac{\partial}{\partial x^{2}} F\left(t, X_{t}\right)\right\|_{2} d t  \tag{56}\\
& \quad \leq \text { const. } \int_{0}^{T} t^{2 H} d t<\infty
\end{align*}
$$

Hence, the last integral exists as a Pettis integral in the $\left(L^{2}\right)$ sense.

On the other hand, denote the heat kernel as follows:

$$
\begin{equation*}
g(t, x):=\frac{1}{\sqrt{2 \pi t}} \exp \left\{\frac{-x^{2}}{2 t}\right\} \tag{57}
\end{equation*}
$$

Thanks to the classical Girsanov theorem, for arbitrary $\eta \in$ $\delta(\mathbb{R})$, under the measure $Q_{\eta}$, we can easily calculate that $X_{t}$ is a Gaussian random variable with mean $(1 / \sqrt{2}) \int_{0}^{t} \varphi(s)$ $\left[\left(M_{+}^{H} \eta\right)(s)-\left(M_{+}^{H}\right) \eta(-s)\right] d s$ and variance $\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2}$. Thus, we obtain

$$
\begin{align*}
& \mathcal{S}\left(F\left(t, X_{t}\right)\right)(\eta) \\
& =E^{\mathrm{Q}_{\eta}}\left[F\left(t, X_{t}\right)\right] \\
& =\int_{\mathbb{R}} F\left(t, u+\frac{1}{\sqrt{2}} \int_{0}^{t} \varphi(s)\left[\left(M_{+}^{H} \eta\right)(s)-\left(M_{+}^{H} \eta\right)(-s)\right] d s\right) \\
& \quad \times g\left(\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2}, u\right) d u \tag{58}
\end{align*}
$$

Moreover, by $\left(B_{3}\right)$, integration and differentiation can be interchanged. Since the heat kernel fulfills $(\partial / \partial t) g=$ $(1 / 2)\left(\partial^{2} / \partial x^{2}\right) g$, we have

$$
\begin{align*}
& \frac{d}{d t} \mathcal{S}\left(F\left(t, X_{t}\right)\right)(\eta) \\
& =\mathcal{S}\left(\frac{\partial}{\partial t} F\left(t, X_{t}\right)\right)(\eta) \\
& \quad+\frac{1}{\sqrt{2}}\left[\left(M_{+}^{H} \eta\right)(t)-\left(M_{+}^{H} \eta\right)(-t)\right] \varphi(t) \mathcal{S}\left(\frac{\partial}{\partial x} F\left(t, X_{t}\right)\right)(\eta) \\
& \quad+\frac{1}{2} \frac{d}{d t}\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2} \cdot \mathcal{S}\left(\frac{\partial^{2}}{\partial x^{2}} F\left(t, X_{t}\right)\right)(\eta) \tag{59}
\end{align*}
$$

Consequently,

$$
\begin{align*}
\mathcal{S}( & \left.F\left(T, X_{T}\right)-F(0,0)\right)(\eta) \\
= & \lim _{\varepsilon \rightarrow 0} \mathcal{S}\left(F\left(T, X_{T}\right)-F\left(\varepsilon, X_{\varepsilon}\right)\right)(\eta) \\
= & \int_{0}^{T} \mathcal{S}\left(\frac{\partial}{\partial t} F\left(t, X_{t}\right)\right)(\eta) d t \\
& +\frac{1}{\sqrt{2}} \int_{0}^{T}\left[\left(M_{+}^{H} \eta\right)(t)-\left(M_{+}^{H} \eta\right)(-t)\right] \varphi(t) \\
& \times \mathcal{S}\left(\frac{\partial}{\partial x} F\left(t, X_{t}\right)\right)(\eta) d t \\
& +\frac{1}{2} \int_{0}^{T} \frac{d}{d t}\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2} \cdot \mathcal{S}\left(\frac{\partial^{2}}{\partial x^{2}} F\left(t, X_{t}\right)\right)(\eta) d t . \tag{60}
\end{align*}
$$

Compared with (54), the proof can be completed.

The objective of this part is to define the geometric subfBm and establish an Itô formula with respect to it.

Definition 14. Let $H \in(0,1), x_{0}>0$, and $\varphi, r:[0, \infty) \rightarrow \mathbb{R}$, Then one calls

$$
\begin{equation*}
P_{t}:=x_{0} \exp \left\{\int_{0}^{t} r(s) d s-\frac{1}{2}\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2}+\int_{0}^{t} \varphi(s) d S_{s}^{H}\right\} \tag{61}
\end{equation*}
$$

a geometric sub- fBm with coefficients $H, x_{0}, \varphi, r$, provided the right-hand side exists as an element of $\left(L^{2}\right)$ for all $0 \leq t<$ $\infty$.

Theorem 15. Let $T>0$, such that
(i) $P$ be a geometric sub- $f B m$ with continuous coefficients $\varphi, r$, and let $\varphi$ be a constant when $H<1 / 2$;
(ii) $\left(B_{2}\right),\left(B_{3}\right)$ hold.

Then the following equality holds in $\left(L^{2}\right)$ :

$$
\begin{align*}
\int_{0}^{T} \varphi(t) & P_{t} \frac{\partial}{\partial x} F\left(t, P_{t}\right) d S_{t}^{H} \\
& =F\left(T, P_{T}\right)-F\left(0, x_{0}\right)-\int_{0}^{T} \frac{\partial}{\partial t} F\left(t, P_{t}\right) d t \\
& -\int_{0}^{T} r(t) P_{t} \frac{\partial}{\partial x} F\left(t, P_{t}\right) d t  \tag{62}\\
& -\frac{1}{2} \int_{0}^{T} \frac{d}{d t}\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2} P_{t}^{2} \frac{\partial^{2}}{\partial x^{2}} F\left(t, P_{t}\right) d t
\end{align*}
$$

Proof. Let

$$
\begin{equation*}
g(t, x):=x_{0} \exp \left\{\int_{0}^{t} r(s) d s-\frac{1}{2}\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2}+x\right\} . \tag{63}
\end{equation*}
$$

Then, apply Theorem 13 to $F(t, g(t, x))$, and the result is obvious.

The special case $F(t, x)=x$ yields the following.
Corollary 16. Let $P$ be a geometric sub- $f B m$ as in Theorem 15; then for all $t \geq 0$,

$$
\begin{equation*}
P_{t}=x_{0}+\int_{0}^{t} r(s) P_{s} d s+\int_{0}^{t} \varphi(s) P_{s} d S_{s}^{H} \tag{64}
\end{equation*}
$$

For this reason, one calls it "geometric sub-fBm".

## 5. Explicit Solution of a Class of Linear Subfractional BSDEs

General BSDEs driven by a Brownian motion are usually of the form

$$
\begin{gather*}
-d Y_{t}=f\left(t, Y_{t}, Z_{t}\right) d t-Z_{t} d B_{t}, \quad t \in[0, T] \\
Y_{T}=\xi \tag{65}
\end{gather*}
$$

where $f, \xi$ are given. The generator $f(t, y, z)$ is a $\mathscr{G}_{t}$-adapted process for every pair $(y, z) \in \mathbb{R}^{2}$, the terminal value $\xi$ is a $\mathscr{G}_{T}$-measureable random variable, and $\mathscr{G}_{t}$ denotes the filtration generated by $B_{t}$. We say a pair $(Y, Z)$ is a solution of this equation, if the processes $Y, Z$ which are $\mathscr{G}_{t}$-adapted and satisfy a suitable integrability condition solve the equation $P$ almost surely.

After these preparations, we now turn to the problems to solve the BSDEs driven by a sub-fBm of the form

$$
\begin{equation*}
Y_{t}=\xi-\int_{t}^{T} f\left(s, Y_{t}, Z_{t}\right) d s-\int_{t}^{T} Z_{t} d S_{s}^{H} \tag{66}
\end{equation*}
$$

where $f, \xi=Y_{T}$ are given. The generator $f(t, y, z)$ is a $\mathscr{G}_{t^{-}}$ adapted process for every pair $(y, z) \in \mathbb{R}^{2}$, the terminal value $\xi$ is a $\mathscr{G}_{T}$-measureable random variable, and $\mathscr{G}_{t}$ denotes the filtration generated by $S_{t}^{H}$. We say a pair $(Y, Z)$ is a solution of this equation, if the processes $Y, Z$ which are $\mathscr{G}_{t}$-adapted and
satisfy a suitable integrability condition solve the equation $P$ almost surely.

Let us recall a result about the following PDE, which is a parabolic partial differential equation solved by the heat equation (see Theorem 9 in [18]). Let the following conditions be satisfied:
$\left(C_{1}\right) S \in \mathscr{C}^{1}((0, T), \mathbb{R}) \cap \mathscr{C}([0, T], \mathbb{R})$ and $S$ is strictly increasing with $S(0)=0$ and $S^{\prime} \in L^{1}([0, T], \mathbb{R})$;
$\left(C_{1}\right) r, A, f \in \mathscr{C}((0, T], \mathbb{R}) \cap L^{1}([0, T], \mathbb{R}) ;$
$\left(C_{1}\right) \phi \in \mathscr{C}(\mathbb{R}, \mathbb{R})$ and there exists constant $C \geq 0$ and $\lambda<(8 S(T))^{-1}$ such that for all $(t, x) \in[0, T] \times \mathbb{R}$, $|\phi(t, x)| \leq C e^{\lambda x^{2}}$.

Then the PDE

$$
\begin{gather*}
\partial_{t} u(t, x)=-\frac{1}{2} S^{\prime}(t) \partial_{x x} u(t, x)+r(t) \partial_{x} u(t, x) \\
+A(t) u(t, x)+f(t),  \tag{67}\\
u(T, x)=\phi(x)
\end{gather*}
$$

has a classical solution given by

$$
\begin{align*}
u(t, x):= & -\int_{t}^{T} f(s) e^{\int_{s}^{t} A(u) d u} d s \\
& +\frac{e^{-\int_{t}^{T} A(s) d s}}{\sqrt{2 \pi(S(T)-S(t))}} \\
& \times \int_{\mathbb{R}} \phi(y) \exp \left\{\frac{-\left(x-y-\int_{t}^{T} r(s) d s\right)^{2}}{2(S(T)-S(t))}\right\} d y . \tag{68}
\end{align*}
$$

Next we give the main result of this paper.
Theorem 17. Let $\Phi_{t}=x_{0}+b(t)+\int_{0}^{t} \varphi(s) d S_{s}^{H}$ and $T>0$. Suppose the following conditions are satisfied:
$\left(D_{1}\right) \varphi:[0, T] \rightarrow \mathbb{R}^{+}$is continuous when $H \geq 1 / 2$, constant when $H<1 / 2$, and there exist constants $0<K_{1} \leq$ $K_{2}$, such that $K_{1} \leq \varphi(t) \leq K_{2}, t \in[0, T] ;$
$\left(D_{2}\right) x_{0} \in \mathbb{R}, b \in \mathscr{C}^{1}((0, T), \mathbb{R}) \cap \mathscr{C}([0, T], \mathbb{R}) ;$
$\left(D_{3}\right)\left(C_{2}\right)$ holds with $r(t)=\varphi(t) h(t)-b^{\prime}(t)$ and $h:[0, T] \rightarrow$ $\mathbb{R}$ with $\varphi$ h bounded on $[0, T]$;
$\left(D_{4}\right) \phi \in \mathscr{C}(\mathbb{R}, \mathbb{R})$ and there exist constants $C \geq 0$, and $\lambda \leq$ $1 /\left(12 T^{2 H} \max _{s \in[0, T]}|f(s)|^{2}\right)$ such that for all $(t, x) \in$ $[0, T] \times \mathbb{R},|\phi(t, x)| \leq C e^{\lambda x^{2}}$.

Then the BSDEs,

$$
\begin{align*}
Y_{t}= & \phi\left(\Phi_{T}\right)-\int_{t}^{T}\left[f(s)+A(s) Y_{s}+h(s) Z_{s}\right] d s  \tag{69}\\
& -\int_{t}^{T} Z_{s} d S_{s}^{H}
\end{align*}
$$

have a solution $(Y, Z)$ of the form

$$
\begin{align*}
& Y(t):=v\left(t, \Phi_{t}\right), \quad Z_{t}:=\varphi(t) \partial_{x} v\left(t, \Phi_{t}\right), \\
& v(t, x) \\
& :=-\int_{t}^{T} f(s) e^{\int_{s}^{t} A(u) d u} d s \\
& +\frac{e^{-\int_{t}^{T} A(s) d s}}{\sqrt{2 \pi\left(\left|M_{-}^{H}\left(1_{(0, T)} \varphi\right)^{0}\right|_{2}^{2}-\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2}\right)}} \\
& \quad \times \int_{\mathbb{R}} \phi(y) \exp \left\{\frac{-\left(x-y-\int_{t}^{T}\left(\varphi(s) h(s)-b^{\prime}(s)\right) d s\right)^{2}}{2\left|M_{-}^{H}\left(1_{(0, T)} \varphi\right)^{0}\right|_{2}^{2}-2\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2}}\right\} d y . \tag{70}
\end{align*}
$$

Proof. Let $S(t):=\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2}$; from Lemma 11 and Remark 12, we have $S(t)$ satisfies $\left(C_{1}\right)$. By the growth condition $\left(D_{4}\right),\left(C_{3}\right)$ is yielded, and $\left(C_{2}\right)$ follows from $\left(D_{3}\right)$. Henceforth, $v(t, x)$ is a classical solution of the PDE

$$
\begin{align*}
\partial_{t} v(t, x)= & -\frac{1}{2} \frac{d}{d t}\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2} \partial_{x x} v(t, x) \\
+ & {\left[\varphi(t) h(t)-b^{\prime}(t)\right] \partial_{x} v(t, x)+A(t) v(t, x) } \\
+ & f(t) ; \quad(t, x) \in((0, T), \mathbb{R}), \\
& v(T, x)=\phi(x), \quad x \in \mathbb{R} . \tag{71}
\end{align*}
$$

Moreover, by Lemma 10 and Corollary 11 in [18], suppose that $F(t, x):=v\left(t, x_{0}+b(t)+x\right)$, which fulfills the conditions of Theorem 13 for all $0 \leq t \leq T-\varepsilon$ and $\varepsilon>0$. Consequently,

$$
\begin{align*}
v & \left(t, \Phi_{t}\right) \\
& =v\left(T-\varepsilon, \Phi_{T-\varepsilon}\right)-\int_{t}^{T-\varepsilon} \varphi(s) \partial_{x} v\left(s, \Phi_{s}\right) d S_{s}^{H} \\
& -\int_{t}^{T-\varepsilon} f(s)+A(s) v\left(s, \Phi_{s}\right)+h(s) \varphi(s) \partial_{x} v\left(s, \Phi_{s}\right) d s \tag{72}
\end{align*}
$$

Next, according to Definition 4 and the growth condition, $\int_{t}^{T-\varepsilon} \varphi(s) \partial_{x} v\left(s, \Phi_{s}\right) d S_{s}^{H}$ exists when $\varepsilon$ tends to zero.

On the other hand, similar to (58), we obtain

$$
\begin{align*}
& \mathcal{S}\left(v\left(T-\varepsilon, \Phi_{T-\varepsilon}\right)\right)(\eta) \\
& \begin{aligned}
=\int_{\mathbb{R}} F(T-\varepsilon, x
\end{aligned} \\
& \left.\quad+\frac{1}{\sqrt{2}} \int_{0}^{T-\varepsilon} \varphi(s)\left[\left(M_{+}^{H} \eta\right)(s)-\left(M_{+}^{H} \eta\right)(-s)\right] d s\right) \\
& \quad \times g\left(\left|M_{-}^{H}\left(1_{(0, T-\varepsilon)} \varphi\right)^{0}\right|_{2}^{2}, x\right) d x . \tag{73}
\end{align*}
$$

By the growth and the continuity conditions of $v$, we have $\mathcal{S}\left(v\left(T-\varepsilon, \Phi_{T-\varepsilon}\right)\right)(\eta)$ converges to $\mathcal{S}\left(v\left(T, \Phi_{T}\right)\right)(\eta)$ as $\varepsilon$ tends to zero.

Now it remains to show the existence of the last integral of (72). In fact, there exists a constant $K$, such that

$$
\begin{align*}
& \int_{0}^{T}\left\|f(s)+A(s) v\left(s, \Phi_{s}\right)+h(s) \varphi(s) \partial_{x} v\left(s, \Phi_{s}\right)\right\|_{2} d s \\
& \quad \leq|f|_{L^{1}([0, T])}+K|A|_{L^{1}([0, T])} \\
& \quad+K \int_{0}^{T} \frac{1}{\sqrt{\left|M_{-}^{H}\left(1_{(0, T)} \varphi\right)^{0}\right|_{2}^{2}-\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2}}} d t . \tag{74}
\end{align*}
$$

For $H>1 / 2,(48)$ and $\left(D_{1}\right)$ yield

$$
\begin{align*}
& \int_{0}^{T} \frac{1}{\sqrt{\left|M_{-}^{H}\left(1_{(0, T)} \varphi\right)^{0}\right|_{2}^{2}-\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2}}} d t \\
& =\int_{0}^{T}\left[8 \alpha H \int_{t}^{T} \int_{0}^{\tau} \varphi(s) \varphi(\tau)\left[(\tau-s)^{2 \alpha-1}+(s+\tau)^{2 \alpha-1}\right] d s d \tau\right]^{-1 / 2} d t \\
& \leq \frac{1}{K_{1}} \int_{0}^{T}\left[8 \alpha H \int_{t}^{T} \int_{0}^{\tau}\left[(\tau-s)^{2 \alpha-1}+(s+\tau)^{2 \alpha-1}\right] d s d \tau\right]^{-1 / 2} d t \\
& =\frac{1}{K_{1}} \int_{0}^{T} \frac{1}{\sqrt{T^{2 H}-t^{2 H}}} d t=\frac{T^{1-H}}{H K_{1} 2^{H+1}} \frac{\Gamma(1 / 2) \Gamma(1 / 2 H)}{\Gamma(1 / 2 H+1 / 2)} . \tag{75}
\end{align*}
$$

For $H<1 / 2,\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2}=\left(2-2^{2 H-1}\right) \varphi^{2} t^{2 H}$, we obtain

$$
\begin{gather*}
\int_{0}^{T} \frac{1}{\sqrt{\left|M_{-}^{H}\left(1_{(0, T)} \varphi\right)^{0}\right|_{2}^{2}-\left|M_{-}^{H}\left(1_{(0, t)} \varphi\right)^{0}\right|_{2}^{2}}} d t \\
\quad=\frac{1}{\sqrt{2-2^{2 H-1}} \varphi} \int_{0}^{T} \frac{1}{\sqrt{T^{2 H}-t^{2 H}}} d t  \tag{76}\\
\quad=\frac{1}{\sqrt{2-2^{2 H-1}} f} \frac{T^{1-H} \Gamma(1 / 2) \Gamma(1 / 2 H)}{2 H \Gamma(1 / 2 H+1 / 2)}
\end{gather*}
$$

This means that $\int_{0}^{T} f(s)+A(s) v\left(s, \Phi_{s}\right)+h(s) \varphi(s) \partial_{x} v\left(s, \Phi_{s}\right) d s$ is well defined, which completes the proof.

The above theorem also holds for geometric sub- fBm as described in the following proposition.

Proposition 18. Let a geometric sub-fBm $P_{t}=x_{0} \exp \left\{k S_{t}^{H}+\right.$ $\left.\mu t-(1 / 2)\left(2-2^{2 H-1}\right) k^{2} t^{2 H}\right\}$, and $G$ is continuous and of polynomial growth. Then Theorem 17 holds with the terminal value of the form $G\left(P_{T}\right)$.

Proof. We just need to apply Theorem 17 with $\Phi(t)=\ln x_{0}+$ $k S_{t}^{H}+\mu t-(1 / 2)\left(2-2^{2 H-1}\right) k^{2} t^{2 H}$ and $\phi(x)=G\left(e^{x}\right)$.

The regularity of the obtained solutions is described as follows.

Proposition 19. Let $Y, Z$ as defined in Theorem 17. Then $Y \in$ $L^{2}\left([0, T],\left(L^{2}\right)\right)$. Moreover, $Z \in L^{1 / H}\left([0, T],\left(L^{2}\right)\right)$ when $H>$ $1 / 4$.

It is a straightforward result in view of the growth condition of $v$.

## 6. Conclusion

We have presented the subfractional Itô integral using the method of the $\mathcal{S}$-transform. A Girsanov theorem with respect to the subfractional Itô integral and an Itô formula for functionals of a subfractional Wiener integral has been established. As an application, we obtain explicit solutions for a class of linear BSDEs driven by a sub-fBm with arbitrary Hurst parameter under suitable assumptions.

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