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Research Article

Semigroup Method on a $M^X/G/1$ Queueing Model

Alim Mijit

Xinjiang Radio & TV University, Urumqi 830049, China

Correspondence should be addressed to Alim Mijit; alimjanmijit@yahoo.com.cn

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By using the Hille-Yosida theorem, Phillips theorem, and Fattorini theorem in functional analysis we prove that the $M^X/G/1$ queueing model with vacation times has a unique nonnegative time-dependent solution.

1. Introduction

The queueing system when the server become idle is not new. Miller [1] was the first to study such a model, where the server is unavailable during some random length of time for the M/G/1 queueing system. The M/G/1 queueing models of similar nature have also been reported by a number of authors, since Levy and Yechiali [2] included several types of generalizations of the classical M/G/1 queueing system. These generalizations are useful in model building in many real life situations such as digital communication, computer network, and production/inventory system [3–5].

At present, however, most studies are devoted to batch arrival queues with vacation because of its interdisciplinary character. Considerable efforts have been devoted to study these models by Baba [6], Lee and Srinivasan [7], Lee et al. [8, 9], Borthakur and Choudhury [10], and Choudhury [11, 12] among others. However, the recent progress of $M^X/G/1$ type queueing models of this nature has been served by Chae and Lee [13] and Medhi [14].

In 2002, Choudhury [15] studied the $M^X/G/1$ queueing model with vacation times. By using the supplementary variable technique [16] he established the corresponding queueing model and obtained the queue size distribution at a stationary (random) as well as a departure point of time under multiple vacation policy based on the following hypothesis. "The time-dependent solution of the model converges to a nonzero steady-state solution." By reading the paper we find that the previous hypothesis, in fact, implies the following two hypothesis.

Hypothesis 1. The model has a nonnegative time-dependent solution.

Hypothesis 2. The time-dependent solution of the model converges to a nonzero steady-state solution.

In this paper we investigate Hypothesis 1. By using the Hille-Yosida theorem, Phillips theorem, and Fattorini theorem we prove that the model has a unique nonnegative time-dependent solution, and therefore we obtain Hypothesis 1.

According to Choudhury [15], the $M^X/G/1$ queueing system with vacation times can be described by the following system of equations:

$$\frac{dQ(t)}{dt} = -\lambda Q(t) + \int_0^\infty v(x) P_{0,0}(x,t)\,dx + \int_0^\infty b(x) P_{1,1}(x,t)\,dx,$$

$$\frac{\partial P_{0,0}(x,t)}{\partial t} + \frac{\partial P_{0,0}(x,t)}{\partial x} = -[\lambda + v(x)] P_{0,0}(x,t),$$

$$\frac{\partial P_{0,n}(x,t)}{\partial t} + \frac{\partial P_{0,n}(x,t)}{\partial x} = -[\lambda + v(x)] P_{0,n}(x,t) + \lambda \sum_{k=1}^{n} c_k P_{0,n-k}(x,t), \quad n \geq 1,$$

$$\frac{\partial P_{1,n}(x,t)}{\partial t} + \frac{\partial P_{1,n}(x,t)}{\partial x}$$

$$
\begin{aligned}
&= -[\lambda + b(x)]\, P_{1,n}(x,t) \\
&\quad + \lambda \sum_{k=1}^{n} c_k P_{1,n-k+1}(x,t), \quad n \geq 1, \\
&P_{0,0}(t) = \lambda Q(t), \qquad P_{0,n}(0,t) = 0, \quad n \geq 1, \\
&P_{1,n}(0,t) = \int_0^\infty v(x)\, P_{0,n}(x,t)\, dx \\
&\qquad + \int_0^\infty b(x)\, P_{1,n+1}(x,t)\, dx, \quad n \geq 1, \\
&Q(0) = 1, \qquad P_{0,n}(x,0) = 0, \quad n \geq 0, \\
&P_{1,n}(x,0) = 0, \quad n \geq 1,
\end{aligned} \tag{1}
$$

where $(x,t) \in [0,\infty)\times[0,\infty)$; $Q(t)$ represents the probability that there is no customer in the system and the server is idle at time t; $P_{0,n}(x,t)dx\,(n \geq 0)$ represents the probability that at time t there are n customers in the system and the server is on a vacation with elapsed vacation time of the server lying in $[x,\ x+dx)$. $P_{1,n}(x,t)dx\,(n \geq 1)$ represents the probability that at time t there are n customers in the system with elapsed service time of the customer undergoing service lying in $[x,\ x+dx)$. λ is batch arrival rate of customers. $c_k\,(k \geq 1)$ represents the probability that at every arrival epoch a batch of k external customers arrives and satisfies $\sum_{k=1}^{\infty} c_k = 1$. $v(x)$ is the vacation rate of the server, which satisfies

$$
v(x) \geq 0, \qquad \int_0^\infty v(x)\, dx = \infty. \tag{2}
$$

$b(x)$ is the service rate of the server satisfying

$$
b(x) \geq 0, \qquad \int_0^\infty b(x)\, dx = \infty. \tag{3}
$$

2. Problem Formulation

We first formulate the system (1) as an abstract Cauchy problem on a suitable state space. For convenience we take some notations as follows:

$$
\Gamma_1 = \begin{pmatrix} e^{-x} & 0 & 0 & \cdots \\ \lambda e^{-x} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad
\Gamma_2 = \begin{pmatrix} 0 & 0 & v(x) & 0 & \cdots \\ 0 & 0 & 0 & v(x) & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},
$$

$$
\Gamma_3 = \begin{pmatrix} 0 & b(x) & 0 & \cdots \\ 0 & 0 & b(x) & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4}
$$

If we take state space

$$
X = \left\{ (P_0, P_1) \;\middle|\; \begin{array}{l} P_0 = (Q, P_{0,0}, P_{0,1}, \ldots) \\ \quad \in \mathbb{R} \times L^1[0,\infty) \times L^1[0,\infty) \times \cdots \\ P_1 = (P_{1,1}, P_{1,2}, P_{1,3}, \ldots) \\ \quad \in L^1[0,\infty) \times L^1[0,\infty) \\ \qquad \times L^1[0,\infty) \times \cdots \\ \|(P_0, P_1)\| = |Q| + \sum_{i=0}^{\infty} \|P_{0,i}\|_{L^1[0,\infty)} \\ \qquad + \sum_{i=1}^{\infty} \|P_{1,i}\|_{L^1[0,\infty)} < \infty \end{array} \right\}, \tag{5}
$$

then it is obvious that X is a Banach space. In the following we define operators and their domains;

$$
A(P_0, P_1) = \left(\begin{pmatrix} -\lambda & 0 & 0 & \cdots \\ 0 & -\frac{d}{dx} & 0 & \cdots \\ 0 & 0 & -\frac{d}{dx} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q \\ P_{0,0}(x) \\ P_{0,1}(x) \\ \vdots \end{pmatrix}, \begin{pmatrix} -\frac{d}{dx} & 0 & 0 & \cdots \\ 0 & -\frac{d}{dx} & 0 & \cdots \\ 0 & 0 & -\frac{d}{dx} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} P_{1,1}(x) \\ P_{1,2}(x) \\ P_{1,3}(x) \\ \vdots \end{pmatrix} \right),
$$

$$
D(A) = \left\{ (P_0, P_1) \in X \;\middle|\; \begin{array}{l} \frac{dP_{0,n}(x)}{dx}, \frac{dP_{1,k}(x)}{dx} \in L^1[0,\infty), \\ P_{0,n}(x), P_{1,k}(x)\,(n \geq 0,\ k \geq 1) \\ \text{are absolutely continuous} \\ \text{functions and satisfy} \\ P_0(0) = \int_0^\infty \Gamma_1 P_0(x)\, dx, \\ P_1(0) = \int_0^\infty \Gamma_2 P_0(x)\, dx \\ \qquad + \int_0^\infty \Gamma_3 P_1(x)\, dx \end{array} \right\},
$$

$$
U(P_0, P_1) = \left(\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & \tilde{v} & 0 & 0 & \cdots \\ 0 & \lambda c_1 & \tilde{v} & 0 & \cdots \\ 0 & \lambda c_2 & \lambda c_1 & \tilde{v} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q \\ P_{0,0}(x) \\ P_{0,1}(x) \\ P_{0,2}(x) \\ \vdots \end{pmatrix}, \right.
$$

$$\left(\begin{pmatrix} \tilde{b} & 0 & 0 & 0 & \cdots \\ \lambda c_2 & \tilde{b} & 0 & 0 & \cdots \\ \lambda c_3 & \lambda c_2 & \tilde{b} & 0 & \cdots \\ \lambda c_4 & \lambda c_3 & \lambda c_2 & \tilde{b} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} P_{1,1}(x) \\ P_{1,2}(x) \\ P_{1,3}(x) \\ P_{1,4}(x) \\ \vdots \end{pmatrix}\right), \tag{6}$$

where $\tilde{v} = -[\lambda + v(x)]$, $\tilde{b} = -[\lambda + b(x)] + \lambda c_1$,

$$E(P_0, P_1) = \left(\begin{pmatrix} \int_0^\infty v(x) P_{0,0}(x)\,dx + \int_0^\infty b(x) P_{1,1}(x)\,dx \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}\right),$$

$$D(U) = D(E) = X. \tag{7}$$

Then the previous system of equations (1) can be rewritten as an abstract Cauchy problem in the Banach space X:

$$\frac{d(P_0, P_1)(t)}{dt} = (A + U + E)(P_0, P_1)(t), \quad \forall t \in [0, \infty),$$

$$(P_0, P_1)(0) = \left(\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}\right). \tag{8}$$

3. Well-Posedness of The System (8)

Theorem 1. *If $v = \sup_{x\in[0,\infty)} v(x) < \infty$, $b = \sup_{x\in[0,\infty)} b(x) < \infty$, then $A + U + E$ generates a positive contraction C_0-semigroup $T(t)$.*

Proof. We split the proof of the theorem into four steps. Firstly, we prove that $(\gamma I - A)^{-1}$ exists and is bounded for some γ. Secondly, we show that $D(A)$ is dense in X. Thirdly, we verify that U and E are bounded linear operators. Thus by using the Hille-Yosida theorem and the perturbation theorem of C_0-semigroup we deduce that $A + U + E$ generates a C_0-semigroup $T(t)$. Finally, we check that $A+U+E$ is dispersive, and therefore we obtain the desired result.

For any given $(y_0, y_1) \in X$, we consider the equation $(\gamma I - A)(P_0, P_1) = (y_0, y_1)$; that is,

$$(\gamma + \lambda) Q = y_Q, \tag{9}$$

$$\frac{dP_{0,n}(x)}{dx} = -\gamma P_{0,n}(x) + y_{0,n}(x), \quad n \geq 0, \tag{10}$$

$$\frac{dP_{1,n}(x)}{dx} = -\gamma P_{1,n}(x) + y_{1,n}(x), \quad n \geq 1, \tag{11}$$

$$P_{0,0}(0) = \lambda Q, \tag{12}$$

$$P_{0,n}(0) = 0, \quad n \geq 1, \tag{13}$$

$$P_{1,n}(0) = \int_0^\infty v(x) P_{0,n}(x)\,dx + \int_0^\infty b(x) P_{1,n+1}(x)\,dx, \quad n \geq 1. \tag{14}$$

Through solving (9)–(11), we have

$$Q = \frac{1}{\gamma + \lambda} y_Q, \tag{15}$$

$$P_{0,n}(x) = a_{0,n} e^{-\gamma x} + e^{-\gamma x} \int_0^x y_{0,n}(\tau) e^{\gamma\tau} d\tau, \quad n \geq 0, \tag{16}$$

$$P_{1,n}(x) = a_{1,n} e^{-\gamma x} + e^{-\gamma x} \int_0^x y_{1,n}(\tau) e^{\gamma\tau} d\tau, \quad n \geq 1. \tag{17}$$

Combining (16) with (12) and (13), we obtain

$$a_{0,0} = P_{0,0}(0) = \lambda Q = \frac{\lambda}{\gamma + \lambda} y_Q, \tag{18}$$

$$a_{0,n} = P_{0,n}(0) = 0, \quad n \geq 1. \tag{19}$$

Substituting (19) into (16), it follows that

$$P_{0,n}(x) = e^{-\gamma x} \int_0^x y_{0,n}(\tau) e^{\gamma\tau} d\tau, \quad n \geq 1. \tag{20}$$

By combining (15), (16), (17), and (20) with (14), we deduce

$$\begin{aligned} a_{1,n} &= P_{1,n}(0) \\ &= \int_0^\infty v(x) e^{-\gamma x} \int_0^x y_{0,n}(\tau) e^{\gamma\tau} d\tau\,dx \\ &\quad + a_{1,n+1} \int_0^\infty b(x) e^{-\gamma x} dx \\ &\quad + \int_0^\infty b(x) e^{-\gamma x} \int_0^x y_{1,n+1}(\tau) e^{\gamma\tau} d\tau\,dx, \quad n \geq 1, \end{aligned} \tag{21}$$

$$\Longrightarrow$$

$$\begin{aligned} a_{1,n} - a_{1,n+1} &\int_0^\infty b(x) e^{-\gamma x} dx \\ &= \int_0^\infty v(x) e^{-\gamma x} \int_0^x y_{0,n}(\tau) e^{\gamma\tau} d\tau\,dx \\ &\quad + \int_0^\infty b(x) e^{-\gamma x} \int_0^x y_{1,n+1}(\tau) e^{\gamma\tau} d\tau\,dx. \end{aligned} \tag{22}$$

If we set

$$\mathscr{C} = \begin{pmatrix} 1 & -\int_0^\infty b(x)\,e^{-\gamma x}dx & 0 & 0 & \cdots \\ 0 & 1 & -\int_0^\infty b(x)\,e^{-\gamma x}dx & 0 & \cdots \\ 0 & 0 & 1 & -\int_0^\infty b(x)\,e^{-\gamma x}dx & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$
$$\vec{a}_1 = \begin{pmatrix} a_{1,1} \\ a_{1,2} \\ a_{1,3} \\ \vdots \end{pmatrix}, \tag{23}$$

then (21) can be rewritten as follows:

$$\mathscr{C}\vec{a}_1 = \begin{pmatrix} \int_0^\infty v(x)\,e^{-\gamma x}\int_0^x y_{0,1}(\tau)\,e^{\gamma\tau}d\tau\,dx + \int_0^\infty b(x)\,e^{-\gamma x}\int_0^x y_{1,2}(\tau)\,e^{\gamma\tau}d\tau\,dx \\ \int_0^\infty v(x)\,e^{-\gamma x}\int_0^x y_{0,2}(\tau)\,e^{\gamma\tau}d\tau\,dx + \int_0^\infty b(x)\,e^{-\gamma x}\int_0^x y_{1,3}(\tau)\,e^{\gamma\tau}d\tau\,dx \\ \vdots \end{pmatrix}. \tag{24}$$

It is easy to calculate

$$\mathscr{C}^{-1} = \begin{pmatrix} 1 & \int_0^\infty b(x)\,e^{-\gamma x}dx & \left(\int_0^\infty b(x)e^{-\gamma x}dx\right)^2 & \left(\int_0^\infty b(x)e^{-\gamma x}dx\right)^3 & \cdots \\ 0 & 1 & \int_0^\infty b(x)\,e^{-\gamma x}dx & \left(\int_0^\infty b(x)e^{-\gamma x}dx\right)^2 & \cdots \\ 0 & 0 & 1 & \int_0^\infty b(x)\,e^{-\gamma x}dx & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{25}$$

From which together with (24), we derive

$$\begin{aligned} a_{1,n} = {} & \sum_{k=0}^\infty \left(\int_0^\infty b(x)\,e^{-\gamma x}dx\right)^k \\ & \times \left\{\int_0^\infty v(x)\,e^{-\gamma x}\int_0^x y_{0,k+n}(\tau)\,e^{\gamma\tau}d\tau\,dx \right. \\ & \left. + \int_0^\infty b(x)\,e^{-\gamma x}\int_0^x y_{1,k+n+1}(\tau)\,e^{\gamma\tau}d\tau\,dx\right\}, \quad n \ge 1. \end{aligned} \tag{26}$$

By using Fubini theorem we estimate (16) and (17) as follows (assume that $\gamma > v + b$):

$$\begin{aligned} & \|P_{0,n}\|_{L^1[0,\infty)} \\ & \le \int_0^\infty |a_{0,n}|\,e^{-\gamma x}dx + \int_0^\infty e^{-\gamma x}\int_0^x |y_{0,n}(\tau)|\,e^{\gamma\tau}d\tau\,dx \\ & = \frac{1}{\gamma}|a_{0,n}| + \int_0^\infty |y_{0,n}(\tau)|\,e^{\gamma\tau}\int_\tau^\infty e^{-\gamma x}dx\,d\tau \\ & = \frac{1}{\gamma}|a_{0,n}| + \frac{1}{\gamma}\|y_{0,n}\|_{L^1[0,\infty)}, \quad n \ge 0, \\ & \|P_{1,n}\|_{L^1[0,\infty)} \le \frac{1}{\gamma}|a_{1,n}| + \frac{1}{\gamma}\|y_{1,n}\|_{L^1[0,\infty)}, \quad n \ge 1. \end{aligned} \tag{27}$$

From (26) and Fubini theorem, we deduce

$$\begin{aligned} \sum_{n=1}^\infty |a_{1,n}| \le {} & \sum_{n=1}^\infty\sum_{k=0}^\infty \left(\frac{b}{\gamma}\right)^k \\ & \times \left\{v\int_0^\infty |y_{0,k+n}(\tau)|\,e^{\gamma\tau}\int_\tau^\infty e^{-\gamma x}dx\,d\tau \right. \\ & \left. + b\int_0^\infty |y_{1,k+n+1}(\tau)|\,e^{\gamma\tau}\int_\tau^\infty e^{-\gamma x}dx\,d\tau\right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\upsilon}{\gamma}\sum_{k=0}^{\infty}\left(\frac{b}{\gamma}\right)^{k}\sum_{n=1}^{\infty}\left\|y_{0,k+n}\right\|_{L^1[0,\infty)} \\
&\quad + \sum_{k=0}^{\infty}\left(\frac{b}{\gamma}\right)^{k+1}\sum_{n=1}^{\infty}\left\|y_{1,k+n+1}\right\|_{L^1[0,\infty)} \\
&\le \frac{\upsilon}{\gamma}\sum_{k=0}^{\infty}\left(\frac{b}{\gamma}\right)^{k}\sum_{n=0}^{\infty}\left\|y_{0,n}\right\|_{L^1[0,\infty)} \\
&\quad + \sum_{k=0}^{\infty}\left(\frac{b}{\gamma}\right)^{k+1}\sum_{n=1}^{\infty}\left\|y_{1,n}\right\|_{L^1[0,\infty)} \\
&= \frac{\upsilon}{\gamma-b}\sum_{n=0}^{\infty}\left\|y_{0,n}\right\|_{L^1[0,\infty)} + \frac{b}{\gamma-b}\sum_{n=1}^{\infty}\left\|y_{1,n}\right\|_{L^1[0,\infty)}.
\end{aligned}
\tag{28}
$$

By inserting (18), (19), and (28) into (27) and using inequality $(\gamma+\upsilon-b)/\gamma(\gamma-b) < 1/(\gamma-\upsilon-b)$, we estimate

$$
\begin{aligned}
\left\|(P_0,P_1)\right\| &\le \frac{|y_Q|}{\gamma+\lambda} + \frac{1}{\gamma}\sum_{n=0}^{\infty}|a_{0,n}| + \frac{1}{\gamma}\sum_{n=0}^{\infty}\left\|y_{0,n}\right\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\gamma}\sum_{n=1}^{\infty}|a_{1,n}| + \frac{1}{\gamma}\sum_{n=1}^{\infty}\left\|y_{1,n}\right\|_{L^1[0,\infty)} \\
&\le \frac{1}{\gamma}|y_Q| + \frac{\gamma+\upsilon-b}{\gamma(\gamma-b)}\sum_{n=0}^{\infty}\left\|y_{0,n}\right\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\gamma-b}\sum_{n=1}^{\infty}\left\|y_{1,n}\right\|_{L^1[0,\infty)} \\
&\le \frac{1}{\gamma-\upsilon-b}|y_Q| + \frac{1}{\gamma-\upsilon-b}\sum_{n=0}^{\infty}\left\|y_{0,n}\right\|_{L^1[0,\infty)} \\
&\quad + \frac{1}{\gamma-\upsilon-b}\sum_{n=1}^{\infty}\left\|y_{1,n}\right\|_{L^1[0,\infty)} \\
&= \frac{1}{\gamma-\upsilon-b}\left\|(y_0,y_1)\right\|.
\end{aligned}
\tag{29}
$$

Equation (29) shows that $(\gamma I - A)^{-1}$ exists for $\gamma > \upsilon + b$ and

$$
(\gamma I - A)^{-1} : X \longrightarrow D(A), \quad \left\|(\gamma I - A)^{-1}\right\| \le \frac{1}{\gamma-\upsilon-b}.
\tag{30}
$$

As far as the second step is concerned, from $|Q| + \sum_{n=0}^{\infty}\|P_{0,n}\|_{L^1[0,\infty)} + \sum_{n=1}^{\infty}\|P_{1,n}\|_{L^1[0,\infty)} < \infty$ for $(P_0,P_1) \in X$ it follows that, for any $\epsilon > 0$, there exists a positive integer N such that $\sum_{n=N}^{\infty}\|P_{0,n}\|_{L^1[0,\infty)} < \epsilon$, $\sum_{n=N}^{\infty}\|P_{1,n}\|_{L^1[0,\infty)} < \infty$. Let

$$
\mathbb{L} = \left\{ (P_0,P_1) \;\middle|\; \begin{array}{l}
P_0(x) = (Q, P_{0,0}(x), P_{0,1}(x), \ldots, \\
\qquad P_{0,N}(x), 0, \ldots), \\
P_1(x) = (P_{1,1}(x), P_{1,2}(x), \ldots, \\
\qquad P_{1,N}(x), 0, \ldots), \\
Q \in \mathbb{R}, P_{0,i}(x), \\
P_{1,j}(x) \in L^1[0,\infty), \; i = 0,1,\ldots,N; \\
j = 1,2,\ldots,N; \\
N \text{ is a finite positive integer}
\end{array} \right\},
\tag{31}
$$

then $\mathbb{L}$ is dense in X. If we set

$$
\mathbb{Z} = \left\{ (P_0,P_1) \;\middle|\; \begin{array}{l}
P_0(x) = (Q, P_{0,0}(x), P_{0,1}(x), \ldots, \\
\qquad P_{0,l}(x), 0, \ldots), \\
P_1(x) = (P_{1,1}(x), P_{1,2}(x), \ldots, \\
\qquad P_{1,l}(x), 0, \ldots), \\
P_{0,i}(x), \; P_{1,j}(x) \in C_0^{\infty}[0,\infty), \\
\text{and there exists positive} \\
\text{numbers } c_{0,i} > 0, \; c_{1,j} > 0, \\
\text{such that } P_{0,i}(x) = 0, x \in [0, c_{0,i}], \\
P_{1,j}(x) = 0, \; x \in [0, c_{1,j}]; \\
i = 0,1,2,\ldots,l; \; j = 1,2,\ldots,l
\end{array} \right\},
\tag{32}
$$

then by the relationship $C_0^{\infty}[0,\infty)$ and $L^1[0,\infty)$ in Adams [17], we know that $\mathbb{Z}$ is dense in $\mathbb{L}$. Hence in order to prove denseness of $D(A)$, it suffices to prove that $D(A)$ is dense in $\mathbb{Z}$. Take any $(P_0, P_1) \in \mathbb{Z}$, then there are a finite positive integer l and positive numbers $c_{0,i} > 0$, $c_{1,j} > 0$ $(i = 0,1,\ldots,l;\; j = 1,2,\ldots,l)$ such that

$$
\begin{aligned}
P_0(x) &= (Q, P_{0,0}(x), P_{0,1}(x), \ldots, P_{0,l}(x), 0, \ldots), \\
P_1(x) &= (P_{1,1}(x), P_{1,2}(x), \ldots, P_{1,l}(x), 0, \ldots), \\
P_{0,i}(x) &= 0, \quad \text{for } x \in [0, c_{0,i}], \; i = 0,1,2,\ldots,l, \\
P_{1,j}(x) &= 0, \quad \text{for } x \in [0, c_{1,j}], \; j = 1,2,\ldots,l,
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
P_{0,i}(x) &= 0, \quad x \in [0,2s], \; i = 0,1,\ldots,l; \\
P_{1,j}(x) &= 0, \quad x \in [0,2s], \; j = 1,\ldots,l,
\end{aligned}
\tag{34}
$$

where $0 < 2s < \min\{c_{0,0}, c_{0,1}, \ldots, c_{0,l}, c_{1,1}, \ldots, c_{1,l}\}$. Define

$$
\begin{aligned}
f_0^s(0) &= \left(Q, f_{0,0}^s(0), f_{0,1}^s(0), \ldots, f_{0,l}^s(0), 0, \ldots\right) \\
&= (Q, \lambda Q, 0, \ldots, 0, 0, \ldots), \\
f_1^s(0) &= \left(f_{1,1}^s(0), f_{1,2}^s(0), \ldots, f_{1,l}^s(0), 0, \ldots\right),
\end{aligned}
\tag{35}
$$

where

$$f_{1,j}^s(0) = \int_{2s}^{\infty} v(x) P_{0,j}(x)\,dx + \int_{2s}^{\infty} b(x) P_{1,j+1}(x)\,dx, \quad j = 1, 2, \ldots, l-1,$$

$$f_{1,l}^s(0) = \int_{2s}^{\infty} v(x) P_{0,l}(x)\,dx,$$

$$f_0^s(x) = \left(Q, f_{0,0}^s(x), f_{0,1}^s(x), \ldots, f_{0,l}^s(x), 0, \ldots\right),$$

$$f_1^s(x) = \left(f_{1,1}^s(x), f_{1,2}^s(x), \ldots, f_{1,l}^s(x), 0, \ldots\right), \tag{36}$$

where

$$f_{0,i}^s(x) = \begin{cases} f_{0,i}^s(0)\left(1-\dfrac{x}{s}\right)^2, & x \in [0,s), \\ -\mu_{0,i}(x-s)^2(x-2s)^2, & x \in [s,2s), \\ P_{0,i}(x), & x \in [2s,\infty), \end{cases}$$

$$f_{1,j}^s(x) = \begin{cases} f_{1,j}^s(0)\left(1-\dfrac{x}{s}\right)^2, & x \in [0,s), \\ -\mu_{1,j}(x-s)^2(x-2s)^2, & x \in [s,2s), \\ P_{1,j}(x), & x \in [2s,\infty), \end{cases} \tag{37}$$

$$\mu_{0,i} = \frac{\int_0^s v(x) f_{0,i}^s(0)(1-(x/s))^2 dx}{\int_s^{2s} v(x)(x-s)^2(x-2s)^2 dx},$$

$$\mu_{1,j} = \frac{\int_0^s b(x) f_{1,j}^s(0)(1-(x/s))^2 dx}{\int_s^{2s} b(x)(x-s)^2(x-2s)^2 dx},$$

$$i = 0, 1, \ldots, l; \quad j = 1, \ldots, l.$$

Then it is not difficult to verify that $(f_0^s, f_1^s) \in D(A)$. Moreover,

$$\begin{aligned}
&\left\|(P_0, P_1) - (f_0^s, f_1^s)\right\| \\
&= \sum_{i=0}^{l} \int_0^{\infty} \left|P_{0,i}(x) - f_{0,i}^s(x)\right| dx \\
&\quad + \sum_{j=1}^{l} \int_0^{\infty} \left|P_{1,j}(x) - f_{1,j}^s(x)\right| dx \\
&= \sum_{i=0}^{l} \int_0^{s} \left|f_{0,i}^s(0)\right| \left(1-\frac{x}{s}\right)^2 dx \\
&\quad + \sum_{i=0}^{l} \left|\mu_{0,i}\right| \int_s^{2s} (x-s)^2(x-2s)^2 dx \\
&\quad + \sum_{j=1}^{l} \int_0^{s} \left|f_{1,j}^s(0)\right| \left(1-\frac{x}{s}\right)^2 dx \\
&\quad + \sum_{j=1}^{l} \left|\mu_{1,j}\right| \int_s^{2s} (x-s)^2(x-2s)^2 dx \\
&= \sum_{i=0}^{l} \left|f_{0,i}^s(0)\right| \frac{s}{3} + \sum_{i=0}^{l} \left|\mu_{0,i}\right| \frac{s^5}{30} + \sum_{j=1}^{l} \left|f_{1,j}^s(0)\right| \frac{s}{3} \\
&\quad + \sum_{j=1}^{l} \left|\mu_{1,j}\right| \frac{s^5}{30} \longrightarrow 0 \quad \text{as } s \longrightarrow 0,
\end{aligned} \tag{38}$$

which shows that $D(A)$ is dense in $\mathbb{Z}$. Hence, $D(A)$ is dense in X. From the first step, the second step, and the Hille-Yosida theorem [18] we know that A generates a C_0-semigroup.

Next we will verify that U and E are bounded linear operators. From the definition of U and E and $\sum_{k=1}^{\infty} c_k = 1$ we have

$$\begin{aligned}
\left\|U(P_0, P_1)\right\| &\le \sum_{n=0}^{\infty} \int_0^{\infty} [\lambda + v(x)] \left|P_{0,n}(x)\right| dx \\
&\quad + \lambda \sum_{k=1}^{\infty} c_k \sum_{n=0}^{\infty} \int_0^{\infty} \left|P_{0,n}(x)\right| dx \\
&\quad + \sum_{n=1}^{\infty} \int_0^{\infty} [\lambda + b(x)] \left|P_{1,n}(x)\right| dx \\
&\quad + \lambda \sum_{k=1}^{\infty} c_k \sum_{n=1}^{\infty} \int_0^{\infty} \left|P_{1,n}(x)\right| dx \\
&\le (2\lambda + v) \sum_{n=0}^{\infty} \int_0^{\infty} \left|P_{0,n}(x)\right| dx \\
&\quad + (2\lambda + b) \sum_{n=1}^{\infty} \int_0^{\infty} \left|P_{1,n}(x)\right| dx \\
&\le \max\{(2\lambda + v), (2\lambda + b)\} \left\|(P_0, P_1)\right\|, \\
\left\|E(P_0, P_1)\right\| &\le \int_0^{\infty} v(x) \left|P_{0,0}(x)\right| dx \\
&\quad + \int_0^{\infty} b(x) \left|P_{1,1}(x)\right| dx \\
&\le v\left\|P_{0,0}\right\|_{L^1[0,\infty)} + b\left\|P_{1,1}\right\|_{L^1[0,\infty)} \\
&\le \max\{v, b\} \left\|(P_0, P_1)\right\|.
\end{aligned} \tag{39}$$

The previous two formulas show that U and E are bounded operators. It is easy to check that U and E are linear operators. Hence from the perturbation theorem of C_0-semigroup [18], we obtain that $A + U + E$ generates a C_0-semigroup $T(t)$.

Lastly, we will prove that $A+U+E$ is a dispersive operator. For $(P_0, P_1) \in D(A)$ we take (ϕ_0, ϕ_1) as

$$\begin{aligned}
\phi_0(x) &= \left(\frac{[Q]^+}{Q}, \frac{[P_{0,0}(x)]^+}{P_{0,0}(x)}, \frac{[P_{0,1}(x)]^+}{P_{0,1}(x)}, \ldots\right), \\
\phi_1(x) &= \left(\frac{[P_{1,1}(x)]^+}{P_{1,1}(x)}, \frac{[P_{1,2}(x)]^+}{P_{1,2}(x)}, \ldots\right),
\end{aligned} \tag{40}$$

where

$$[Q]^+ = \begin{cases} Q, & \text{if } Q > 0, \\ 0, & \text{if } Q \le 0, \end{cases}$$

$$[P_{0,n}(x)]^+ = \begin{cases} P_{0,n}(x), & \text{if } P_{0,n}(x) > 0, \\ 0, & \text{if } P_{0,n}(x) \le 0, \end{cases} \quad n \ge 0, \tag{41}$$

$$[P_{1,n}(x)]^+ = \begin{cases} P_{1,n}(x), & \text{if } P_{1,n}(x) > 0, \\ 0, & \text{if } P_{1,n}(x) \le 0, \end{cases} \quad n \ge 1.$$

If we define $V_{0,i} = \{x \in [0,\infty) \mid P_{0,i}(x) > 0\}$ and $W_{0,i} = \{x \in [0,\infty) \mid P_{0,i}(x) \le 0\}$ for $i = 0, 1, 2, \ldots$, then by a short argument we calculate

$$\begin{aligned} &\int_0^\infty \frac{dP_{0,i}(x)}{dx} \frac{[P_{0,i}(x)]^+}{P_{0,i}(x)} dx \\ &\quad = \int_{V_{0,i}} \frac{dP_{0,i}(x)}{dx} \frac{[P_{0,i}(x)]^+}{P_{0,i}(x)} dx \\ &\qquad + \int_{W_{0,i}} \frac{dP_{0,i}(x)}{dx} \frac{[P_{0,i}(x)]^+}{P_{0,i}(x)} dx \\ &\quad = \int_{V_{0,i}} \frac{dP_{0,i}(x)}{dx} \frac{[P_{0,i}(x)]^+}{P_{0,i}(x)} dx = \int_{V_{0,i}} \frac{dP_{0,i}(x)}{dx} dx \\ &\quad = \int_0^\infty \frac{d[P_{0,i}(x)]^+}{dx} dx = -[P_{0,i}(0)]^+, \quad i \ge 0. \end{aligned} \tag{42}$$

Similar to (42), we get

$$\int_0^\infty \frac{dP_{1,j}(x)}{dx} \frac{[P_{1,j}(x)]^+}{P_{1,j}(x)} dx = -[P_{1,j}(0)]^+, \quad j \ge 1. \tag{43}$$

By using boundary conditions on $(P_0, P_1) \in D(A)$, (42), (43), and $\sum_{k=1}^\infty c_k = 1$ for such (ϕ_0, ϕ_1), we derive

$$\begin{aligned} &\langle (A+U+E)(P_0,P_1), (\phi_0,\phi_1) \rangle \\ &= \left\{ -\lambda Q + \int_0^\infty v(x) P_{0,0}(x)\, dx \right. \\ &\qquad \left. + \int_0^\infty b(x) P_{1,1}(x)\, dx \right\} \frac{[Q]^+}{Q} \\ &\qquad + \int_0^\infty \left\{ -\frac{dP_{0,0}(x)}{dx} - [\lambda + v(x)] P_{0,0}(x) \right\} \frac{[P_{0,0}(x)]^+}{P_{0,0}(x)} dx \\ &\qquad + \sum_{n=1}^\infty \int_0^\infty \left\{ -\frac{dP_{0,n}(x)}{dx} - [\lambda + v(x)] P_{0,n}(x) \right. \\ &\qquad\qquad \left. + \lambda \sum_{k=1}^n c_k P_{0,n-k}(x) \right\} \frac{[P_{0,n}(x)]^+}{P_{0,n}(x)} dx \\ &\qquad + \sum_{n=1}^\infty \int_0^\infty \left\{ -\frac{dP_{1,n}(x)}{dx} - [\lambda + b(x)] P_{1,n}(x) \right. \\ &\qquad\qquad \left. + \lambda \sum_{k=1}^n c_k P_{1,n-k+1}(x) \right\} \frac{[P_{1,n}(x)]^+}{P_{1,n}(x)} dx \\ &= -\lambda [Q]^+ + \frac{[Q]^+}{Q} \int_0^\infty v(x) P_{0,0}(x)\, dx \\ &\qquad + \frac{[Q]^+}{Q} \int_0^\infty b(x) P_{1,1}(x)\, dx \\ &\qquad + \sum_{n=0}^\infty [P_{0,n}(0)]^+ - \sum_{n=0}^\infty \int_0^\infty [\lambda + v(x)] [P_{0,n}(x)]^+ dx \\ &\qquad + \lambda \sum_{k=1}^\infty c_k \sum_{n=k}^\infty \int_0^\infty P_{0,n-k}(x) \frac{[P_{0,n}(x)]^+}{P_{0,n}(x)} dx \\ &\qquad + \sum_{n=1}^\infty [P_{1,n}(0)]^+ - \sum_{n=1}^\infty \int_0^\infty [\lambda + b(x)] [P_{1,n}(x)]^+ dx \\ &\qquad + \lambda \sum_{k=1}^\infty c_k \sum_{n=k}^\infty \int_0^\infty P_{1,n-k+1}(x) \frac{[P_{1,n}(x)]^+}{P_{1,n}(x)} dx \\ &\le -\lambda [Q]^+ + \frac{[Q]^+}{Q} \int_0^\infty v(x) [P_{0,0}(x)]^+ dx \\ &\qquad + \frac{[Q]^+}{Q} \int_0^\infty b(x) [P_{1,1}(x)]^+ dx \\ &\qquad + \lambda [Q]^+ - \sum_{n=0}^\infty \int_0^\infty [\lambda + v(x)] [P_{0,n}(x)]^+ dx \\ &\qquad + \lambda \sum_{k=1}^\infty c_k \sum_{n=k}^\infty \int_0^\infty [P_{0,n-k}(x)]^+ dx \\ &\qquad + \sum_{n=1}^\infty \int_0^\infty v(x) [P_{0,n}(x)]^+ dx + \sum_{n=2}^\infty \int_0^\infty b(x) [P_{1,n}(x)]^+ dx \\ &\qquad - \sum_{n=1}^\infty \int_0^\infty [\lambda + b(x)] [P_{1,n}(x)]^+ dx \\ &\qquad + \lambda \sum_{k=1}^\infty c_k \sum_{n=k}^\infty \int_0^\infty [P_{1,n-k+1}(x)]^+ dx \\ &\le \left(\frac{[Q]^+}{Q} - 1 \right) \\ &\qquad \times \left(\int_0^\infty v(x) [P_{0,0}(x)]^+ dx + \int_0^\infty b(x) [P_{1,1}(x)]^+ dx \right) \\ &\qquad - \sum_{n=0}^\infty \int_0^\infty \lambda [P_{0,n}(x)]^+ dx + \lambda \sum_{k=1}^\infty c_k \sum_{n=0}^\infty \int_0^\infty [P_{0,n}(x)]^+ dx \\ &\qquad - \sum_{n=1}^\infty \int_0^\infty \lambda [P_{1,n}(x)]^+ dx + \lambda \sum_{k=1}^\infty c_k \sum_{n=1}^\infty \int_0^\infty [P_{1,n}(x)]^+ dx \\ &\le 0. \end{aligned} \tag{44}$$

Equation (44) shows that $A+U+E$ is a dispersive operator. From which together with the first step, the second step, and the Phillips theorem, we know that $A+U+E$ generates a positive contraction C_0-semigroup [18]. By the uniqueness of a C_0-semigroup we conclude that this semigroup is just $T(t)$. □

It is not difficult to see that X^*, the dual space of X, is

$$X^* = \left\{ (q_0^*, q_1^*) \left| \begin{array}{l} q_0^* = \left(q^*, q_{0,0}^*, q_{0,1}^*, \dots\right) \\ \quad \in \mathbb{R} \times L^\infty[0,\infty) \times L^\infty[0,\infty) \times \cdots, \\ q_1^* = \left(q_{1,1}^*, q_{1,2}^*, \dots\right) \\ \quad \in L^\infty[0,\infty) \times L^\infty[0,\infty) \times \cdots, \\ \|\|(q_0^*, q_1^*)\|\| = \sup\left\{ |q^*|, \sup\limits_{n\ge 0}\left\|q_{0,n}^*\right\|_{L^\infty[0,\infty)}, \right. \\ \qquad\qquad \left. \sup\limits_{n\ge 1}\left\|q_{1,n}^*\right\|_{L^\infty[0,\infty)} \right\} \end{array} \right. \right\}. \tag{45}$$

It is easy to check that X^* is a Banach space. If we take a set S in X as

$$\begin{aligned} S = \{ &(P_0, P_1) \in X \mid Q \ge 0, P_{0,n}(x) \ge 0, n \ge 0; \\ &P_{1,n}(x) \ge 0, n \ge 1, x \in [0,\infty) \}, \end{aligned} \tag{46}$$

then S is a cone in X. For $(P_0, P_1) \in D(A) \cap S$, we take

$$(q_0^*, q_1^*) = \|(P_0, P_1)\| \left(\begin{pmatrix} 1 \\ 1 \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \end{pmatrix} \right) \in X^*, \tag{47}$$

then we have

$$\begin{aligned} &\langle (P_0, P_1), (q_0^*, q_1^*) \rangle \\ &= \|(P_0, P_1)\| \left(|Q| + \sum_{n=0}^{\infty} P_{0,n}(x)\,dx + \sum_{n=1}^{\infty} P_{1,n}(x)\,dx \right) \\ &= \|(P_0, P_1)\|^2 = \|\|(q_0^*, q_1^*)\|\|^2, \end{aligned} \tag{48}$$

that is

$$\begin{aligned} &(q_0^*, q_1^*) \in \theta((P_0, P_1)) \\ &\quad = \left\{ (q_0^*, q_1^*) \in X^* \mid \langle (P_0, P_1), (q_0^*, q_1^*) \rangle \right. \\ &\qquad \left. = \|(P_0, P_1)\|^2 = \|\|(q_0^*, q_1^*)\|\|^2 \right\}. \end{aligned} \tag{49}$$

For such (q_0^*, q_1^*), by using boundary conditions on $(P_0, P_1) \in D(A) \cap S$ and $\sum_{k=1}^{\infty} c_k = 1$, we have

$$\begin{aligned} &\langle (A+U+E)(P_0, P_1), (q_0^*, q_1^*) \rangle \\ &= \|(P_0, P_1)\| \\ &\quad \times \left\{ -\lambda Q + \int_0^\infty v(x) P_{0,0}(x)\,dx + \int_0^\infty b(x) P_{1,1}(x)\,dx \right\} \\ &\quad + \int_0^\infty \left\{ -\frac{dP_{0,0}(x)}{dx} - [\lambda + v(x)] P_{0,0}(x) \right\} \|(P_0, P_1)\|\,dx \\ &\quad + \sum_{n=1}^{\infty} \int_0^\infty \left\{ -\frac{dP_{0,n}(x)}{dx} - [\lambda + v(x)] P_{0,n}(x) \right. \\ &\qquad \left. + \lambda \sum_{k=1}^{n} c_k P_{0,n-k}(x) \right\} \|(P_0, P_1)\|\,dx \\ &\quad + \sum_{n=1}^{\infty} \int_0^\infty \left\{ -\frac{dP_{1,n}(x)}{dx} - [\lambda + b(x)] P_{1,n}(x) \right. \\ &\qquad \left. + \lambda \sum_{k=1}^{n} c_k P_{1,n-k+1}(x) \right\} \|(P_0, P_1)\|\,dx \\ &= \|(P_0, P_1)\| \\ &\quad \times \left\{ -\lambda Q + \int_0^\infty v(x) P_{0,0}(x)\,dx + \int_0^\infty b(x) P_{1,1}(x)\,dx \right\} \\ &\quad - \|(P_0, P_1)\| \\ &\quad \times \left\{ \sum_{n=0}^{\infty} \int_0^\infty -\frac{dP_{0,n}(x)}{dx}\,dx + \sum_{n=0}^{\infty} \int_0^\infty [\lambda + v(x)] P_{0,n}(x)\,dx \right. \\ &\qquad - \lambda \sum_{k=1}^{\infty} c_k \sum_{n=0}^{\infty} \int_0^\infty P_{0,n}(x)\,dx + \sum_{n=1}^{\infty} \int_0^\infty \frac{dP_{1,n}(x)}{dx}\,dx \\ &\qquad + \sum_{n=1}^{\infty} \int_0^\infty [\lambda + b(x)] P_{1,n}(x)\,dx \\ &\qquad \left. - \lambda \sum_{k=1}^{\infty} c_k \sum_{n=1}^{\infty} \int_0^\infty P_{1,n}(x)\,dx \right\} \\ &= \|(P_0, P_1)\| \\ &\quad \times \left\{ -\lambda Q + \int_0^\infty v(x) P_{0,0}(x)\,dx + \int_0^\infty b(x) P_{1,1}(x)\,dx \right\} \\ &\quad - \|(P_0, P_1)\| \\ &\quad \times \left\{ -\lambda Q + \sum_{n=0}^{\infty} \int_0^\infty v(x) P_{0,n}(x)\,dx \right. \\ &\qquad - \sum_{n=1}^{\infty} \int_0^\infty v(x) P_{0,n}(x)\,dx \\ &\qquad \left. - \sum_{n=2}^{\infty} \int_0^\infty b(x) P_{1,n}(x)\,dx + \sum_{n=1}^{\infty} \int_0^\infty b(x) P_{1,n}(x)\,dx \right\} \\ &= 0. \end{aligned} \tag{50}$$

Which shows that $A+U+E$ is a conservative operator. So we can use the Fattorini theorem [19] and state it as follows.

Theorem 2. *$T(t)$ is isometric for the initial value of the system (8); that is,*

$$\|T(t)(P_0, P_1)(0)\| = \|(P_0, P_1)(0)\|, \quad \forall t \in [0,\infty). \tag{51}$$

Proof. Since $A+U+E$ is conservative with respect to θ and $(P_0,P_1)(0)\in D(A^2)\cap S$, from the Taylor expansion of $T(t+h)$ for $t,h\geq 0$ we have

$$\begin{aligned}
&T(t+h)(P_0,P_1)(0)\\
&\quad=T(t)(P_0,P_1)(0)+h(A+U+E)T(t)(P_0,P_1)(0)\\
&\qquad+\int_t^{t+h}(t+h-s)T(s)(A+U+E)^2(P_0,P_1)(0)\,ds\\
&\quad=T(t)(P_0,P_1)(0)+h(A+U+E)T(t)(P_0,P_1)(0)\\
&\qquad+h\rho(t,h),
\end{aligned}\tag{52}$$

where $\|\rho(t,h)\|\to 0$ as $h\to 0$, uniformly in $t\geq 0$. Then

$$\begin{aligned}
&\|T(t+h)(P_0,P_1)(0)\|\,\|T(t)(P_0,P_1)(0)\|\\
&\quad\geq\left|\left\langle T(t+h)(P_0,P_1)(0),(T(t)(P_0,P_1)(0))^*\right\rangle\right|\\
&\quad\geq\operatorname{Re}\left\langle T(t+h)(P_0,P_1)(0),(T(t)(P_0,P_1)(0))^*\right\rangle\\
&\quad=\|T(t)(P_0,P_1)(0)\|^2\\
&\qquad+h\operatorname{Re}\left\langle\rho(t,h),(T(t)(P_0,P_1)(0))^*\right\rangle.
\end{aligned}\tag{53}$$

In view of (52) and the fact that $A+U+E$ is conservative with respect to θ, consider the set

$$\Omega=\{t\in[0,\infty)\mid\|T(t)(P_0,P_1)(0)\|=\|(P_0,P_1)(0)\|\}.\tag{54}$$

Since $0\in\Omega$, Ω is nonempty, moreover Ω is obviously a closed interval because of continuity of $T(t)$. If $\Omega\neq[0,\infty)$, then let t_0 be the right end point of Ω and $\eta>0$ is so small that $\|T(t)(P_0,P_1)(0)\|$ is bounded away from zero in $t_0\leq t\leq t_0+\eta$. For any such t we divide (53) by $\|T(t)(P_0,P_1)(0)\|$ and get

$$\begin{aligned}
&\|T(t+h)(P_0,P_1)(0)\|\\
&\quad\geq\|T(t)(P_0,P_1)(0)\|+\frac{h}{\|T(t)(P_0,P_1)(0)\|}\\
&\qquad\times\operatorname{Re}\left\langle\rho(t,h),(T(t)(P_0,P_1)(0))^*\right\rangle\\
&\quad\geq\|T(t)(P_0,P_1)(0)\|-h\beta(t,h),
\end{aligned}\tag{55}$$

where $\beta(t,h)$ is positive and $\beta(t,h)\to 0$ when $h\to 0$, uniformly in $t_0\leq t\leq t_0+\eta$.

Let now ϵ be are small positive number and $\delta>0$ such that $\|\beta(t,h)\|\leq\epsilon$ for $0\leq h\leq\delta$ and $t_0\leq t\leq t_0+\eta$. Let $t_0<t_1<t_2<\cdots<t_m=t_0+\eta$ be partition of the interval $t_0\leq t\leq t_0+\eta$ such that $t_j-t_{j-1}\leq\delta$ $(1\leq j\leq m)$. Then by (55), one has

$$\begin{aligned}
0&\leq\|T(t_0)(P_0,P_1)(0)\|-\|T(t_0+\eta)(P_0,P_1)(0)\|\\
&\leq\sum_{j=1}^{m}\left(\|T(t_{j-1})(P_0,P_1)(0)\|-\|T(t_j)(P_0,P_1)(0)\|\right)\\
&\leq\sum_{j=1}^{m}(t_j-t_{j-1})\beta(t_j,t_j-t_{j-1})\leq\eta\epsilon.
\end{aligned}\tag{56}$$

Since ϵ is arbitrary, it follows that $\|T(t_0+\eta)(P_0,P_1)(0)\|=\|T(t_0)(P_0,P_1)(0)\|$, which contradicts the fact that t_0 is the right endpoint of Ω. Hence $\Omega=[0,\infty)$. That is, $\|T(t)(P_0,P_1)(0)\|=\|(P_0,P_1)(0)\|$ for $t\in[0,\infty)$. The proof of the theorem is complete. □

From Theorems 1 and 2 we obtain the main result in this paper.

Theorem 3. *If $v=\sup_{x\in[0,\infty)}v(x)<\infty$, $b=\sup_{x\in[0,\infty)}b(x)<\infty$, then the system (8) has a unique nonnegative time-dependent solution $(P_0,P_1)(x,t)$, which satisfies*

$$\|(P_0,P_1)(\cdot,t)\|=1,\quad\forall t\in[0,\infty).\tag{57}$$

Proof. Since $(P_0,P_1)(0)\in D(A^2)\cap S$, by Theorem 1 and Theorem 11 in Gupur et al. [18], we know that the system (8) has a unique nonnegative time-dependent solution $(P_0,P_1)(x,t)$ which can be expressed as

$$(P_0,P_1)(x,t)=T(t)(P_0,P_1)(0),\quad\forall t\in[0,\infty).\tag{58}$$

From which together with Theorem 2 (i.e., (51)) we have

$$\begin{aligned}
\|(P_0,P_1)(\cdot,t)\|=\|T(t)(P_0,P_1)(0)\|=\|(P_0,P_1)(0)\|=1,\\
\forall t\in[0,\infty),
\end{aligned}\tag{59}$$

this just reflects the physical background of the problem. □

4. Concluding Remarks

If we know the spectrum of $A+U+E$ on the imaginary axis, then by Theorem 1 and Theorem 14 in Gupur et al. [18], we obtain the asymptotic behavior of the time-dependent solution of the system (8), which describes Hypothesis 2. It is our next research work.

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