# On a Periodic Solution of the 4-Body Problems 

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#### Abstract

We study the necessary and sufficient conditions on the masses for the periodic solution of planar 4-body problems, where three particles locate at the vertices of an equilateral triangle and rotate with constant angular velocity about a resting particle. We prove that the above periodic motion is a solution of Newtonian 4-body problems if and only if the resting particle is at the origin and the masses of the other three particles are equal and their angular velocity satisfies a special condition.


## 1. Introduction and Main Results

Celestial mechanics is related to the study of bodies (mass points) moving under the influence of their mutual gravitational attractions. The many-body problems, that is, N body problems, are the most important problems in celestial mechanics. $N$-body problems are nonlinear systems of ordinary differential equations which describe the dynamical law for the motion of N -bodies. The motion equations of the Newtonian $N$-body problems [1-3] are

$$
\begin{align*}
m_{k} \ddot{q}_{k} & =\frac{\partial U}{\partial q_{k}} \\
& =\sum_{j \neq k} G m_{k} m_{j} \frac{q_{j}-q_{k}}{\left|q_{j}-q_{k}\right|^{3}}  \tag{1}\\
& k=1,2 \cdots N
\end{align*}
$$

where $q_{k}=\left(x_{k}, y_{k}, z_{k}\right) \in R^{3}$ is the position of the $k$ th body with mass $m_{k}, U(q)$ is the Newtonian potential function

$$
\begin{equation*}
U(q)=\sum_{1 \leq k<j \leq N} \frac{G m_{k} m_{j}}{\left|q_{j}-q_{k}\right|} \tag{2}
\end{equation*}
$$

and $G$ is the gravitational constant which can be taken as one by choosing suitable units.

Definition 1 (see [1-3]). The $N$-bodies form a central configuration at time $t$ if there exists a scalar $\lambda \in R$ such that

$$
\begin{array}{r}
\lambda m_{k} q_{k}+\frac{\partial U}{\partial q_{k}}=\lambda m_{k} q_{k}+m_{k} \ddot{q}_{k}=0,  \tag{3}\\
i=1,2 \cdots N
\end{array}
$$

where $\lambda=U / I$ and

$$
\begin{equation*}
\sum_{k=1}^{N} m_{k} q_{k}=0 \tag{4}
\end{equation*}
$$

$q_{i} \neq q_{j}$ for all $i \neq j$.
A configuration $q=\left(q_{1}, \ldots, q_{N}\right)$ is a central configuration if and only if

$$
\begin{equation*}
\nabla \sqrt{I} U=D \sqrt{I} U=0, \tag{5}
\end{equation*}
$$

where

$$
\begin{equation*}
I=\sum_{k=1}^{N} m_{k} q_{k}^{2}, \quad U=\sum_{1 \leq k<j \leq N} \frac{m_{k} m_{j}}{\left|q_{j}-q_{k}\right|} . \tag{6}
\end{equation*}
$$

It is well-known that for $N \geq 3$, the general solution of Newtonian $N$-body problems cannot be given until now, so great importance has been attached to searching for a
particular solution from the very beginning. Central configurations have something to do with periodic solutions and collapse orbits and parabolic orbits [1-3]. So finding central configurations becomes very important. In 1772, Lagrange showed that for three masses at the vertices of an equilateral triangle,the orbit rotating about the center of masses with an appropriate angular velocity is a periodic solution of the three-body problems. In 1985, Perko and Walter [4] showed that for $N \geq 4, N$ masses at the vertices of a regular polygon rotating about their common center of masses with an appropriate angular velocity describe a periodic solution of the $N$-body problems if and only if the masses are equal. In 1995 and in 2002, Moeckel and Simo [5] and Zhang and Zhou [6] studied the sufficient and necessary conditions for planar nested 2 N -body problems. In this paper, we study the following problems.

Let three particles locate at the vertices of a unit equilateral triangle; the orbits, describing their rotations with angular velocity $\omega$ about the 4th particle which is resting at $q_{4}$, are given by

$$
\begin{array}{r}
q_{k}(t)-q_{4}=\left(\rho_{k}-q_{4}\right) \exp (i \omega t)  \tag{7}\\
k=1,2,3,
\end{array}
$$

where $\rho_{k}$ is the $k$ th complex roots of the unit; that is.

$$
\begin{equation*}
\rho_{k}=\exp \left(\frac{i 2 k \pi}{3}\right), \quad i=\sqrt{-1}, k=1,2,3 . \tag{8}
\end{equation*}
$$

Theorem 2. If $\dot{q}_{4}=0$ and $q_{k}(t)=\left(\rho_{k}-q_{4}\right) \exp (i \omega t)+q_{4}$ is a solution of Newtonian $N$-body problems (1), then $q_{4}=0$ and the center of masses is at the origin. That is, $q=\left(q_{1}, q_{2}, q_{3}, q_{4}\right)$ form a central configuration at any time $t$.

Theorem 3. Let $\dot{q}_{4}=0$ and $q_{k}(t)=\left(\rho_{k}-q_{4}\right) \exp (i \omega t)+q_{4}$ be a solution of Newtonian $N$-body problems (1) if and only if masses $\left(m_{1}, m_{2}, m_{3}\right)$ of the three particles $\left(q_{1}, q_{2}, q_{3}\right)$ are equal, and the angular velocity $\omega$ is

$$
\begin{equation*}
\omega=\sqrt{m_{4}+\frac{m_{1}}{\sqrt{3}}} . \tag{9}
\end{equation*}
$$

## 2. The Proof of Theorem 2

Differentiating twice in both sides of (7) shows that

$$
\begin{equation*}
\ddot{q}_{k}=-\omega^{2}\left(\rho_{k}-q_{4}\right) \exp (i \omega t), \quad k=1,2,3 . \tag{10}
\end{equation*}
$$

Multiplying both sides in (10) by $m_{k}$ and summing these equations over all $k=1,2,3$, we have

$$
\begin{equation*}
\sum_{k=1}^{3} m_{k} \ddot{q}_{k}=-\omega^{2} \sum_{k=1}^{3} m_{k}\left(\rho_{k}-q_{4}\right) \exp (i \omega t) \tag{11}
\end{equation*}
$$

If (7) is a solution of (1), then the left side of the above equation must be zero since the total force of the conservative system is zero, which is shown by (7) as

$$
\begin{equation*}
-\omega^{2} \sum_{k=1}^{3} m_{k}\left(\rho_{k}-q_{4}\right) \exp (i \omega t)=-\omega^{2} \sum_{k=1}^{3} m_{k}\left(q_{k}-q_{4}\right)=0 ; \tag{12}
\end{equation*}
$$

that is,

$$
\begin{equation*}
\sum_{k=1}^{3} m_{k}\left(\rho_{k}-q_{4}\right)=\sum_{k=1}^{3} m_{k}\left(q_{k}-q_{4}\right)=0 \tag{13}
\end{equation*}
$$

hence we have

$$
\begin{array}{r}
q_{4}=\frac{1}{M} \sum_{k=1}^{3} m_{k} q_{k}=\frac{1}{M} \sum_{k=1}^{3} m_{k} \rho_{k}  \tag{14}\\
M=m_{1}+m_{2}+m_{3}
\end{array}
$$

Because of

$$
\begin{equation*}
\rho_{k}=\exp \left(\frac{i 2 k \pi}{3}\right), \quad i=\sqrt{-1}, k=1,2,3, \tag{15}
\end{equation*}
$$

we have

$$
\begin{align*}
q_{4}=\frac{1}{M} & {\left[\left(-\frac{1}{2} m_{1}-\frac{1}{2} m_{2}+m_{3}\right)\right.} \\
& \left.+i\left(\frac{\sqrt{3}}{2} m_{1}-\frac{\sqrt{3}}{2} m_{2}\right)\right] . \tag{16}
\end{align*}
$$

Since the 4th particle is resting, which means that the force exerted on $q_{4}$ by the other particles is zero, then we have

$$
\begin{align*}
& \frac{m_{1} m_{4}\left(q_{1}(t)-q_{4}\right)}{\left|q_{1}(t)-q_{4}\right|^{3}}+\frac{m_{2} m_{4}\left(q_{2}(t)-q_{4}\right)}{\left|q_{2}(t)-q_{4}\right|^{3}} \\
& \quad+\frac{m_{3} m_{4}\left(q_{3}(t)-q_{4}\right)}{\left|q_{3}(t)-q_{4}\right|^{3}}=0 . \tag{17}
\end{align*}
$$

Substituting (7) into the above equation and letting

$$
\begin{align*}
d_{k}=\frac{1}{\left|q_{k}(t)-q_{4}\right|^{3}}= & \frac{1}{\left|\rho_{k}-q_{4}\right|^{3}}  \tag{18}\\
& k=1,2,3
\end{align*}
$$

then (17) is equivalent to the following equation:

$$
\begin{align*}
& m_{1}\left(\rho_{1}-q_{4}\right) d_{1}+m_{2}\left(\rho_{2}-q_{4}\right) d_{2} \\
& \quad+m_{3}\left(\rho_{3}-q_{4}\right) d_{3}=0 \tag{19}
\end{align*}
$$

Because

$$
\begin{gather*}
\rho_{1}=-\frac{1}{2}+i \frac{\sqrt{3}}{2}, \quad \rho_{2}=-\frac{1}{2}-i \frac{\sqrt{3}}{2}, \quad \rho_{3}=1, \\
q_{4}=\frac{1}{M}\left[\left(-\frac{1}{2} m_{1}-\frac{1}{2} m_{2}+m_{3}\right)+i\left(\frac{\sqrt{3}}{2} m_{1}-\frac{\sqrt{3}}{2} m_{2}\right)\right], \tag{20}
\end{gather*}
$$

then (19) is equivalent to the following linear equations:

$$
\begin{gather*}
m_{1} m_{3} d_{1}+m_{2} m_{3} d_{2}-m_{3}\left(m_{1}+m_{2}\right) d_{3}=0 \\
m_{1}\left(m_{3}+2 m_{2}\right) d_{1}-m_{2}\left(m_{3}+2 m_{1}\right) d_{2}  \tag{21}\\
\quad-m_{3}\left(m_{1}-m_{2}\right) d_{3}=0
\end{gather*}
$$

where $d_{1}, d_{2}, d_{3}$ are regarded as unknowns.

It is easy to prove that the rank of the coefficient matrix of the above linear equations (21) is two, which means all the solutions of the linear equations satisfy

$$
\begin{equation*}
\left(d_{1}, d_{2}, d_{3}\right)^{T}=C\left(a_{1}, a_{2}, a_{3}\right)^{T} \tag{22}
\end{equation*}
$$

where $C$ is any real number and $\left(a_{1}, a_{2}, a_{3}\right)^{T} \neq 0$ is one solution for (21).

Because $(1,1,1)^{T}$ is one solution for the linear equations (21), hence we have

$$
\begin{equation*}
d_{1}=d_{2}=d_{3} \tag{23}
\end{equation*}
$$

which means

$$
\begin{equation*}
\frac{1}{\left|\rho_{1}-q_{4}\right|^{3}}=\frac{1}{\left|\rho_{2}-q_{4}\right|^{3}}=\frac{1}{\left|\rho_{3}-q_{4}\right|^{3}} . \tag{24}
\end{equation*}
$$

Then we get $q_{4}=0$ and the common center of masses

$$
\begin{align*}
m_{1} q_{1} & +m_{2} q_{2}+m_{3} q_{3}+m_{4} q_{4} \\
& =\left(m_{1}+m_{2}+m_{3}+m_{4}\right) q_{4}=0 \tag{25}
\end{align*}
$$

and $d_{1}=d_{2}=d_{3}=1$, and (7) becomes

$$
\begin{equation*}
q_{k}(t)=\rho_{k} \exp (i \omega t) \quad k=1,2,3 \tag{26}
\end{equation*}
$$

so we have

$$
\begin{array}{r}
\ddot{q}_{k}(t)=-\omega^{2} \rho_{k} \exp (i \omega t)=-\omega^{2} q_{k}(t)  \tag{27}\\
k=1,2,3 .
\end{array}
$$

By Definition 1, $q_{1}, q_{2}, q_{3}, q_{4}$ form a central configuration at any time and $\omega^{2}=\lambda=U / I$.

The proof of Theorem 2 is completed.

## 3. The Proof of Theorem 3

By Theorem 2, we have

$$
\begin{equation*}
q_{4}=\frac{1}{M}\left(m_{1} \rho_{1}+m_{2} \rho_{2}+m_{3} \rho_{3}\right)=0 \tag{28}
\end{equation*}
$$

and substituting $\rho_{1}, \rho_{2}, \rho_{3}$ into the above equation, we have

$$
\begin{gather*}
-\frac{1}{2} m_{1}-\frac{1}{2} m_{2}+m_{3}=0 \\
\frac{\sqrt{3}}{2} m_{1}-\frac{\sqrt{3}}{2} m_{2}=0 \tag{29}
\end{gather*}
$$

That is

$$
\begin{equation*}
m_{1}=m_{2}=m_{3} \tag{30}
\end{equation*}
$$

which means that the masses of the particles at the vertices of the equilateral triangle are equal.

By Theorem 2 and (6), we have

$$
\begin{align*}
3 m_{1} \omega^{2} & =I \omega^{2}=U \\
& =\sum_{1 \leq k<j \leq 3} \frac{m_{1}^{2}}{\left|q_{k}-q_{j}\right|}+\sum_{k=1}^{3} \frac{m_{4} m_{1}}{\left|q_{4}-q_{k}\right|} \tag{31}
\end{align*}
$$

and we can easily get

$$
\begin{equation*}
\omega=\sqrt{m_{4}+\frac{m_{1}}{\sqrt{3}}} . \tag{32}
\end{equation*}
$$

The proof of Theorem 3 is completed.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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