Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2014, Article ID 708603, 8 pages
http://dx.doi.org/10.1155/2014/708603

Research Article

New Neumann System Associated with a 3×3 Matrix Spectral Problem

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Received 7 April 2014; Revised 21 June 2014; Accepted 13 July 2014; Published 24 July 2014

Academic Editor: Manuel De León

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The nonlinearization approach of Lax pair is applied to the case of the Neumann constraint associated with a 3×3 matrix spectral problem, from which a new Neumann system is deduced and proved to be completely integrable in the Liouville sense. As an application, solutions of the first nontrivial equation related to the 3×3 matrix spectral problem are decomposed into solving two compatible Hamiltonian systems of ordinary differential equations.

1. Introduction

Soliton equations are nonlinear partial differential equations described by infinite-dimensional integrable systems and have various beautiful algebraic and geometric properties [1–5]. It has been shown that the nonlinearization of spectral problems (NSPs) approach is a powerful tool to study soliton equations. According to this method which was first introduced by Cao [6], each $(1+1)$-dimensional soliton equation is decomposed into two ordinary differential equations: one is spatial and the other is temporal. The resulting decomposition not only inherits many integrable properties from soliton equations such as possessing Lax pairs, but also provides an effective way to derive explicit solutions of soliton equations.

During the 1990s, the method of NSPs has attracted great interest in the soliton field and has been applied to a large number of soliton equations associated with 2×2 matrix spectral problems [7–14]. Furthermore, based on the Cao's nonlinearization technique, the work by Zhou and Qiao is the first time to develop the nonlinearization approach to find the algebrogeometric solutions for integrable systems in both continuous and discrete cases [15–20]. However, due to the complexity of higher-order matrix spectral problems, there is not much research on NSPs for soliton equations associated with higher-order matrix spectral problems [21–27]. In addition, because of the limitation of the Riemann theory, the algebrogeometric solutions of the soliton equations associated with 3×3 matrix spectral problems cannot be obtained with the aid of the nonlinearization technique.

In this paper, we will study the following soliton equation with the help of the method of NSPs:

$$\begin{aligned} u_t &= u_{xx} - u_x v + 2w_x, \\ v_t &= 2u_x, \\ w_t &= -w_{xx} - (vw)_x. \end{aligned} \tag{1}$$

Equation (1) is first proposed in [28] and associated with the 3×3 matrix spectral problem

$$\Psi_x = U\Psi, \qquad U = \begin{pmatrix} 0 & 1 & 0 \\ u-\lambda & v & 1 \\ w & 0 & 0 \end{pmatrix}. \tag{2}$$

In [28], Geng and Du have obtained some explicit solutions, which include soliton and periodic solutions. If $w = 0$, (1) can be reduced to a couple of equations in u and v, which can be presented as

$$u_t = u_{xx} - u_x v, \qquad v_t = 2u_x. \tag{3}$$

The corresponding Lax pair for the reduced system is as follows:

$$\Phi_x = \widehat{U}\Phi, \qquad \widehat{U} = \begin{pmatrix} 0 & 1 \\ u-\lambda & v \end{pmatrix};$$
$$\Phi_t = \widehat{V}\Phi, \qquad \widehat{V} = \begin{pmatrix} u-\lambda & 0 \\ u_x & u-\lambda \end{pmatrix}. \tag{4}$$

The aim of the present paper is to derive the corresponding finite-dimensional Hamiltonian system associated with the 3×3 matrix spectral problem, which is proved to be completely integrable in the Liouville sense. As an application, solutions of (1) are decomposed into solving two compatible Hamiltonian systems of ordinary differential equations.

The paper is organized as follows. In Section 2, we will introduce the Neumann constraint between the potentials and eigenfunctions of the 3×3 matrix spectral problem (2). Under this constraint, we obtain a new Neumann system and a generating function of integrals of motion. In Section 3, the generating function approach is used to calculate the involutivity of integrals of motion, by which the Neumann system is further proved to be completely integrable in the Liouville sense. In Section 4, we will reveal the relation between the Neumann system and (1). Solutions of (1) are decomposed into solving two compatible Hamiltonian systems of ordinary differential equations.

2. A New Neumann System

In this section, we first consider the stationary zero-curvature equation of the spectral problem (2) and its auxiliary problem; that is,

$$V_x = [U, V], \qquad V = \left(V_{ij}\right)_{3\times 3}, \tag{5}$$

where

$$\begin{aligned}
V_{11} &= (v-\partial)A - 2B + \left(-\partial^2 + v\partial + u\right)C - \lambda C,\\
V_{12} &= A, \qquad V_{13} = C,\\
V_{21} &= \left(-\partial^2 + \partial v + u\right)A - 2\partial B\\
&\quad + \left(-\partial^3 + \partial v\partial + \partial u + w\right)C - \lambda A - \lambda\partial C,\\
V_{22} &= 2vA - 2B + \left(-\partial^2 + v\partial + u\right)C - \lambda C,\\
V_{23} &= A + \partial C,\\
V_{31} &= \left(\partial^3 + \partial^2 v + v\partial^2 + v\partial v + w\right)A\\
&\quad - 2\left(\partial^2 + v\partial\right)B - (\partial w + wv)C,\\
V_{32} &= -\left(\partial^2 + \partial v\right)A + 2\partial B + wC,\\
V_{33} &= (\partial + v)A - 2B.
\end{aligned} \tag{6}$$

Substituting (6) into (5) yields the Lenard equation:

$$(K - \lambda J)G = 0, \qquad G = (A, B, C)^T, \tag{7}$$

where K and J are two skew-symmetric operators defined by

$$\begin{aligned}
K &= \begin{pmatrix} -2\partial^3 - 2v\partial v + \partial u + u\partial & 4v\partial & K_{13} \\ 4\partial v & -6\partial & -2\partial^3 + 2\partial v\partial + 2\partial u \\ K_{31} & -2\partial^3 - 2\partial v\partial + 2u\partial & w\partial^2 - \partial^2 w - \partial v w - v w\partial \end{pmatrix},\\
J &= \begin{pmatrix} 2\partial & 0 & \partial^2 - v\partial \\ 0 & 0 & 2\partial \\ -\partial^2 - \partial v & 2\partial & 0 \end{pmatrix},\\
K_{13} &= -\partial^4 + v\partial^3 + \partial^2 v\partial - v\partial v\partial\\
&\quad + \partial^2 u - v\partial u + 2\partial w + w\partial,\\
K_{31} &= \partial^4 + \partial^3 v + \partial v\partial^2 + \partial v\partial v - u\partial^2\\
&\quad - u\partial v + 2w\partial + \partial w,
\end{aligned} \tag{8}$$

in which we denote ∂ by ∂_x for convenience. Expanding entries A, B, and C as the Laurent expansions in λ

$$A = \sum_{j\geq 0} A_j \lambda^{-j}, \qquad B = \sum_{j\geq 0} B_j \lambda^{-j}, \qquad C = \sum_{j\geq 0} C_j \lambda^{-j}, \tag{9}$$

then (7) leads to

$$Kg_{j-1} = Jg_j, \quad j \geq 0, \quad Jg_{-1} = 0, \tag{10}$$

where $g_{j-1} = (A_j, B_j, C_j)^T$. We can choose the first two members as

$$g_{-1} = (0,0,1)^T, \qquad g_0 = (w,0,u)^T. \tag{11}$$

In order to calculate the functional gradient of the eigenvalue with regard to the potentials, we introduce the spectral problem

$$\begin{pmatrix}\Psi\\ \Phi\end{pmatrix}_x = \begin{pmatrix}U & 0\\ 0 & -U^T\end{pmatrix}\begin{pmatrix}\Psi\\ \Phi\end{pmatrix}, \tag{12}$$

where $\Psi = (\Psi^1, \Psi^2, \Psi^3)^T$, $\Phi = (\Phi^1, \Phi^2, \Phi^3)^T$. Let $\lambda_1, \ldots, \lambda_N$ be N distinct nonzero eigenvalues; then the systems associated with (12) can be written in the form

$$\begin{aligned}\left(q_j^1, q_j^2, q_j^3\right)_x &= \left(q_j^1, q_j^2, q_j^3\right) U\left(\lambda_j\right)^T,\\ \left(p_j^1, p_j^2, p_j^3\right)_x &= -\left(p_j^1, p_j^2, p_j^3\right) U\left(\lambda_j\right),\end{aligned} \tag{13}$$

where $q_j^i = \Psi^i(\lambda_j)$, $p_j^i = \Phi^i(\lambda_j)$, $1 \le i \le 3$, $1 \le j \le N$, are eigenfunctions. A direct calculation gives rise to the functional gradient of the eigenvalue λ_j with regard to the potentials u, v, and w:

$$\nabla\lambda_j = \begin{pmatrix}\dfrac{\delta\lambda_j}{\delta u}\\ \dfrac{\delta\lambda_j}{\delta v}\\ \dfrac{\delta\lambda_j}{\delta w}\end{pmatrix} = \begin{pmatrix}q_j^1 p_j^2\\ q_j^2 p_j^2\\ q_j^1 p_j^3\end{pmatrix}. \tag{14}$$

Now we consider the Neumann constraint

$$\sum_{j=1}^{N} \nabla\lambda_j = g_0, \tag{15}$$

which can be written as

$$u = \left\langle q^1, p^3\right\rangle, \qquad w = \left\langle q^1, p^2\right\rangle, \qquad f_1 = \left\langle q^2, p^2\right\rangle = 0, \tag{16}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner-product in $\mathbb{R}^N$, $q^i = (q_1^i, \ldots, q_N^i)^T$, $p^i = (p_1^i, \ldots, p_N^i)^T$. From (13) and the third expression of (16), it is easy to see that

$$\begin{aligned}f_2 &= \left\langle q^1, p^3\right\rangle\left\langle q^1, p^2\right\rangle - \left\langle \Lambda q^1, p^2\right\rangle\\ &\quad + \left\langle q^3, p^2\right\rangle - \left\langle q^2, p^1\right\rangle = 0,\\ v &= \left\langle q^2, p^1\right\rangle^{-1}\left(\left\langle q^1, p^2\right\rangle\left\langle q^2, p^3\right\rangle\right.\\ &\qquad - \left\langle q^1, p^3\right\rangle\left\langle q^1, p^1\right\rangle + \left\langle \Lambda q^1, p^1\right\rangle\\ &\qquad \left. - \left\langle \Lambda q^2, p^2\right\rangle - \left\langle q^3, p^1\right\rangle\right),\end{aligned} \tag{17}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$. Substituting (16) and (17) into (13), we obtain the following new Neumann system on a $(6N-2)$-dimensional manifold M:

$$\begin{aligned}q_x^1 &= q^2,\\ q_x^2 &= \left\langle q^1, p^3\right\rangle q^1 - \Lambda q^1 + \left\langle q^2, p^1\right\rangle^{-1}\\ &\quad \times\left(\left\langle q^1, p^2\right\rangle\left\langle q^2, p^3\right\rangle - \left\langle q^1, p^3\right\rangle\left\langle q^1, p^1\right\rangle\right.\\ &\qquad \left. + \left\langle \Lambda q^1, p^1\right\rangle - \left\langle \Lambda q^2, p^2\right\rangle - \left\langle q^3, p^1\right\rangle\right) q^2 + q^3,\\ q_x^3 &= \left\langle q^1, p^2\right\rangle q^1,\\ p_x^1 &= -\left\langle q^1, p^3\right\rangle p^2 + \Lambda p^2 - \left\langle q^1, p^2\right\rangle p^3,\\ p_x^2 &= -p^1 - \left\langle q^2, p^1\right\rangle^{-1}\left(\left\langle q^1, p^2\right\rangle\left\langle q^2, p^3\right\rangle\right.\\ &\qquad - \left\langle q^1, p^3\right\rangle\left\langle q^1, p^1\right\rangle + \left\langle \Lambda q^1, p^1\right\rangle\\ &\qquad \left. - \left\langle \Lambda q^2, p^2\right\rangle - \left\langle q^3, p^1\right\rangle\right) p^2,\\ p_x^3 &= -p^2\end{aligned} \tag{18}$$

with the manifold M being defined by

$$\begin{aligned}M = \Big\{ &(p,q) \in \mathbb{R}^{6N} \mid f_1 = f_2 = 0,\\ &q = \left(q^1, q^2, q^3\right)^T, p = \left(p^1, p^2, p^3\right)^T\Big\}.\end{aligned} \tag{19}$$

Consider the following function H:

$$H = \left\langle q^1, p^3\right\rangle\left\langle q^1, p^2\right\rangle - \left\langle \Lambda q^1, p^2\right\rangle + \left\langle q^3, p^2\right\rangle + \left\langle q^2, p^1\right\rangle. \tag{20}$$

Through a direct calculation we have

$$\begin{aligned}&\{f_1, f_2\}_M = 2\left\langle q^2, p^1\right\rangle, \qquad \{H, f_1\}_M = 0,\\ &\{H, f_2\}_M = -2\left\langle q^1, p^2\right\rangle\left\langle q^2, p^3\right\rangle + 2\left\langle q^1, p^3\right\rangle\left\langle q^1, p^1\right\rangle\\ &\qquad - 2\left\langle \Lambda q^1, p^1\right\rangle + 2\left\langle \Lambda q^2, p^2\right\rangle + 2\left\langle q^3, p^1\right\rangle,\end{aligned} \tag{21}$$

in which the poisson bracket of two functions is defined as

$$\begin{aligned}\{f, g\} &= \sum_{j=1}^{N}\sum_{i=1}^{3}\left(\frac{\partial f}{\partial q_j^i}\frac{\partial g}{\partial p_j^i} - \frac{\partial g}{\partial q_j^i}\frac{\partial f}{\partial p_j^i}\right)\\ &= \sum_{i=1}^{3}\left(\left\langle \frac{\partial f}{\partial q^i}, \frac{\partial g}{\partial p^i}\right\rangle - \left\langle \frac{\partial g}{\partial q^i}, \frac{\partial f}{\partial p^i}\right\rangle\right).\end{aligned} \tag{22}$$

Introduce a modified function $\widehat{H}$:

$$\widehat{H} = H + af_1 + bf_2, \tag{23}$$

where a and b are two Lagrangian multipliers:

$$\begin{aligned} a &= -\frac{\{H, f_2\}}{\{f_1, f_2\}} \\ &= \frac{1}{\langle q^2, p^1\rangle}\left(\langle q^1, p^2\rangle\langle q^2, p^3\rangle - \langle q^1, p^3\rangle\langle q^1, p^1\rangle \right. \\ &\qquad \left. + \langle \Lambda q^1, p^1\rangle - \langle \Lambda q^2, p^2\rangle - \langle q^3, p^1\rangle\right), \\ b &= \frac{\{H, f_1\}}{\{f_1, f_2\}} = 0. \end{aligned} \tag{24}$$

This means that $\widehat{H}$ is tangent to the manifold M. Therefore, (18) can be represented as the standard canonical equation on M:

$$\begin{pmatrix} q \\ p \end{pmatrix}_x = \begin{pmatrix} \dfrac{\partial \widehat{H}}{\partial p} \\ -\dfrac{\partial \widehat{H}}{\partial q} \end{pmatrix}. \tag{25}$$

On the other hand, through tedious calculations we obtain

$$\left(K - \lambda_j J\right)\widehat{\nabla}\lambda_j = 0, \tag{26}$$

where

$$\widehat{\nabla}\lambda_j = \left(q_j^1 p_j^2, \frac{1}{2}\left(q_j^2 p_j^2 - q_j^1 p_j^1 - q_j^3 p_j^3\right), q_j^1 p_j^3\right)^T. \tag{27}$$

Then the solution of the Lenard equation $KG = \lambda JG$ with parameter λ can be written as

$$\begin{aligned} G_\lambda &= g_{-1} + \sum_{j=1}^{N} \frac{\widehat{\nabla}\lambda_j}{\lambda - \lambda_j} \\ &= \begin{pmatrix} Q_\lambda^{12} \\ \dfrac{1}{2}\left(Q_\lambda^{22} - Q_\lambda^{11} - Q_\lambda^{33}\right) \\ Q_\lambda^{13} + 1 \end{pmatrix}, \end{aligned} \tag{28}$$

where $Q_\lambda^{ij} = \sum_{k=1}^{N}(q_k^i p_k^j/(\lambda - \lambda_k))$, and G_λ satisfies $(K - \lambda J)G_\lambda = 0$ under the Neumann constraint (16).

3. The Liouville Integrability

Now we introduce a Lax matrix by

$$\mathscr{V}_\lambda = \begin{pmatrix} Q_\lambda^{11} - \lambda & Q_\lambda^{12} & Q_\lambda^{13} + 1 \\ Q_\lambda^{21} & Q_\lambda^{22} - \lambda & Q_\lambda^{23} \\ Q_\lambda^{31} + \langle q^1, p^1\rangle & Q_\lambda^{32} + \langle q^1, p^2\rangle & Q_\lambda^{33} \end{pmatrix}. \tag{29}$$

Through a direct calculation we can prove that $\mathscr{V}_\lambda$ and $\zeta I - \mathscr{V}_\lambda$ are two solutions of $V_x = [U, V]$, where I is a 3×3 unit matrix and ζ is a parameter. Then $\mathscr{F}_\lambda^{(2)} = \det \mathscr{V}_\lambda$ and $\mathscr{F}_{\zeta\lambda} = \det(\zeta I - \mathscr{V}_\lambda)$ are independent of x. It is easy to see that

$$\mathscr{F}_{\zeta\lambda} = \zeta^3 - \mathscr{F}_\lambda^{(0)}\zeta^2 + \mathscr{F}_\lambda^{(1)}\zeta - \mathscr{F}_\lambda^{(2)}, \tag{30}$$

where

$$\begin{aligned} \mathscr{F}_\lambda^{(0)} &= Q_\lambda^{11} + Q_\lambda^{22} + Q_\lambda^{33} - 2\lambda, \\ \mathscr{F}_\lambda^{(1)} &= \sum_{1\le i<j\le 3} \begin{vmatrix} \mathscr{V}_\lambda^{ii} & \mathscr{V}_\lambda^{ij} \\ \mathscr{V}_\lambda^{ji} & \mathscr{V}_\lambda^{jj} \end{vmatrix}. \end{aligned} \tag{31}$$

In order to generate the Hamiltonians, we take the following notations

$$\begin{aligned} \widehat{\mathscr{F}}_\lambda^{(0)} &= \mathscr{F}_\lambda^{(0)} + 2\lambda = Q_\lambda^{11} + Q_\lambda^{22} + Q_\lambda^{33}, \\ \widehat{\mathscr{F}}_\lambda^{(1)} &= \mathscr{F}_\lambda^{(1)} + \lambda\mathscr{F}_\lambda^{(0)} + \lambda^2 \\ &= \sum_{1\le i<j\le 3} \begin{vmatrix} Q_\lambda^{ii} & Q_\lambda^{ij} \\ Q_\lambda^{ji} & Q_\lambda^{jj} \end{vmatrix} - \langle q^1, p^1\rangle\left(Q_\lambda^{13} + 1\right) \\ &\qquad - \langle q^1, p^2\rangle Q_\lambda^{23} - Q_\lambda^{31} - \lambda Q_\lambda^{33}, \\ \widehat{\mathscr{F}}_\lambda^{(2)} &= \mathscr{F}_\lambda^{(2)} + \lambda\mathscr{F}_\lambda^{(1)} + \lambda^2\mathscr{F}_\lambda^{(0)} + \lambda^3 \\ &= \begin{vmatrix} Q_\lambda^{11} & Q_\lambda^{12} & Q_\lambda^{13} \\ Q_\lambda^{21} & Q_\lambda^{22} & Q_\lambda^{23} \\ Q_\lambda^{31} & Q_\lambda^{32} & Q_\lambda^{33} \end{vmatrix} + \langle q^1, p^1\rangle \begin{vmatrix} Q_\lambda^{12} & Q_\lambda^{13} \\ Q_\lambda^{22} & Q_\lambda^{23} \end{vmatrix} \\ &\qquad - \langle q^1, p^2\rangle \begin{vmatrix} Q_\lambda^{11} & Q_\lambda^{13} \\ Q_\lambda^{21} & Q_\lambda^{23} \end{vmatrix} + \lambda \begin{vmatrix} Q_\lambda^{11} & Q_\lambda^{12} \\ Q_\lambda^{21} & Q_\lambda^{22} \end{vmatrix} \\ &\qquad + \begin{vmatrix} Q_\lambda^{21} & Q_\lambda^{22} \\ Q_\lambda^{31} & Q_\lambda^{32} \end{vmatrix} - \langle q^1, p^1\rangle Q_\lambda^{22} + \langle q^1, p^2\rangle Q_\lambda^{21}. \end{aligned} \tag{32}$$

Substituting the Laurent expansion of Q_λ^{ik} into (32) we have

$$\begin{aligned} \widehat{\mathscr{F}}_\lambda^{(0)} &= \sum_{m\ge 0} F_m^{(0)}\lambda^{-m-1}, \qquad \widehat{\mathscr{F}}_\lambda^{(1)} = \sum_{m\ge 0} F_m^{(1)}\lambda^{-m}, \\ \widehat{\mathscr{F}}_\lambda^{(2)} &= \sum_{m\ge 0} F_m^{(2)}\lambda^{-m-1}, \end{aligned} \tag{33}$$

where

$$\begin{aligned} F_m^{(0)} &= \langle \Lambda^m q^1, p^1\rangle + \langle \Lambda^m q^2, p^2\rangle \\ &\qquad + \langle \Lambda^m q^3, p^3\rangle, \quad m \ge 0, \\ F_0^{(1)} &= -\langle q^1, p^1\rangle - \langle q^3, p^3\rangle, \end{aligned}$$

$$
\begin{aligned}
F_m^{(1)} = & \sum_{1\le i<j\le 3} \sum_{\substack{l+k=m-2\\ l,k\ge 0}} \begin{vmatrix} \langle \Lambda^l q^i, p^i\rangle & \langle \Lambda^k q^i, p^j\rangle \\ \langle \Lambda^l q^j, p^i\rangle & \langle \Lambda^k q^j, p^j\rangle \end{vmatrix} \\
& - \langle q^1, p^1\rangle \langle \Lambda^{m-1} q^1, p^3\rangle \\
& - \langle q^1, p^2\rangle \langle \Lambda^{m-1} q^2, p^3\rangle \\
& - \langle \Lambda^{m-1} q^3, p^1\rangle - \langle \Lambda^m q^3, p^3\rangle, \\
& \qquad m \ge 1, \\
F_0^{(2)} = & -\langle q^1, p^1\rangle \langle q^2, p^2\rangle + \langle q^1, p^2\rangle \langle q^2, p^1\rangle \\
& + \langle q^1, p^1\rangle \langle q^2, p^2\rangle - \langle q^1, p^2\rangle \langle q^2, p^1\rangle = 0, \\
F_m^{(2)} = & \sum_{\substack{l+k+s=m-2\\ l,k,s\ge 0}} \begin{vmatrix} \langle \Lambda^l q^1, p^1\rangle & \langle \Lambda^k q^1, p^2\rangle & \langle \Lambda^s q^1, p^3\rangle \\ \langle \Lambda^l q^2, p^1\rangle & \langle \Lambda^k q^2, p^2\rangle & \langle \Lambda^s q^2, p^3\rangle \\ \langle \Lambda^l q^3, p^1\rangle & \langle \Lambda^k q^3, p^2\rangle & \langle \Lambda^s q^3, p^3\rangle \end{vmatrix} \\
& + \langle q^1, p^1\rangle \sum_{\substack{l+k=m-1\\ l,k\ge 0}} \begin{vmatrix} \langle \Lambda^l q^1, p^2\rangle & \langle \Lambda^k q^1, p^3\rangle \\ \langle \Lambda^l q^2, p^2\rangle & \langle \Lambda^k q^2, p^3\rangle \end{vmatrix} \\
& - \langle q^1, p^2\rangle \sum_{\substack{l+k=m-1\\ l,k\ge 0}} \begin{vmatrix} \langle \Lambda^l q^1, p^1\rangle & \langle \Lambda^k q^1, p^3\rangle \\ \langle \Lambda^l q^2, p^1\rangle & \langle \Lambda^k q^2, p^3\rangle \end{vmatrix} \\
& + \sum_{\substack{l+k=m-1\\ l,k\ge 0}} \begin{vmatrix} \langle \Lambda^l q^2, p^1\rangle & \langle \Lambda^k q^2, p^2\rangle \\ \langle \Lambda^l q^3, p^1\rangle & \langle \Lambda^k q^3, p^2\rangle \end{vmatrix} \\
& + \sum_{\substack{l+k=m\\ l,k\ge 0}} \begin{vmatrix} \langle \Lambda^l q^1, p^1\rangle & \langle \Lambda^k q^1, p^2\rangle \\ \langle \Lambda^l q^2, p^1\rangle & \langle \Lambda^k q^2, p^2\rangle \end{vmatrix} \\
& - \langle q^1, p^1\rangle \langle \Lambda^m q^2, p^2\rangle + \langle q^1, p^2\rangle \langle \Lambda^m q^2, p^1\rangle, \\
& \qquad m \ge 1.
\end{aligned} \tag{34}
$$

We can prove the following assertion.

Proposition 1. *Suppose that* $(p, q) \in \mathbb{R}^{6N}$, *then*

$$\left\{\mathscr{F}_{\xi\mu}, \mathscr{F}_{\zeta\lambda}\right\} = 0, \quad \forall \lambda, \zeta, \mu, \xi \in \mathbb{C}; \tag{35}$$

$$\left\{F_m^{(i)}, F_k^{(j)}\right\} = 0, \quad \forall m, k \ge 0,\ 0 \le i,\ j \le 2. \tag{36}$$

Proof. Through tedious calculation we can obtain

$$\left\{\widehat{\mathscr{F}}_\lambda^{(i)}, \widehat{\mathscr{F}}_\mu^{(j)}\right\} = 0, \quad \forall \lambda, \mu \in \mathbb{C},\ 0 \le i,\ j \le 2; \tag{37}$$

then we have

$$\left\{\mathscr{F}_\lambda^{(i)}, \mathscr{F}_\mu^{(j)}\right\} = 0, \quad \forall \lambda, \mu \in \mathbb{C},\ 0 \le i,\ j \le 2. \tag{38}$$

Substituting (30) and (38) into the left hand side of (35) shows that (35) holds. Moreover, relation (36) follows by comparison of power of λ^N in (33) with (37) taken into account. □

In order to guarantee that the Hamiltonians are tangent to the constrained manifold M, we calculate that

$$
\begin{aligned}
& \left\{F_j^{(i)}, f_1\right\} = 0, \quad i = 0, 1, 2, \\
& \left\{F_j^{(0)}, f_2\right\} = 0, \quad j \ge 0, \\
& \left\{F_0^{(1)}, f_2\right\} = 0, \\
& \left\{F_m^{(1)}, f_2\right\}_M = 2\langle q^2, p^1\rangle \langle \Lambda^{m-1} q^1, p^3\rangle, \\
& \left\{F_m^{(2)}, f_2\right\}_M = -2\langle q^2, p^1\rangle \sum_{\substack{l+k=m-1\\ l,k\ge 0}} \begin{vmatrix} \langle \Lambda^l q^1, p^2\rangle & \langle \Lambda^k q^1, p^3\rangle \\ \langle \Lambda^l q^2, p^2\rangle & \langle \Lambda^k q^2, p^3\rangle \end{vmatrix} \\
& \qquad + 2\langle q^2, p^1\rangle \langle \Lambda^m q^2, p^2\rangle, \quad m \ge 1.
\end{aligned} \tag{39}
$$

The Lagrangian multipliers are given by

$$
\begin{aligned}
& \mu_m^{(1)} = \frac{\left\{F_m^{(1)}, f_2\right\}}{\{f_1, f_2\}} = \langle \Lambda^{m-1} q^1, p^3\rangle, \\
& \mu_m^{(2)} = \frac{\left\{F_m^{(2)}, f_2\right\}}{\{f_1, f_2\}} \\
& \qquad = \langle \Lambda^m q^2, p^2\rangle - \sum_{\substack{l+k=m-1\\ l,k\ge 0}} \begin{vmatrix} \langle \Lambda^l q^1, p^2\rangle & \langle \Lambda^k q^1, p^3\rangle \\ \langle \Lambda^l q^2, p^2\rangle & \langle \Lambda^k q^2, p^3\rangle \end{vmatrix}, \\
& \qquad m \ge 1.
\end{aligned} \tag{40}
$$

Thus the modified functions

$$
\begin{aligned}
& \widehat{F}_j^{(0)} = F_j^{(0)}, \qquad \widehat{F}_0^{(1)} = F_0^{(1)}, \qquad \widehat{F}_m^{(i)} = F_m^{(i)} - \mu_m^{(i)} f_1, \\
& \qquad i = 1, 2, \quad j \ge 0, \quad m \ge 1,
\end{aligned} \tag{41}
$$

are tangent to the manifold M and are in involution in pairs on M; that is

$$\left\{\widehat{F}_m^{(i)}, \widehat{F}_l^{(j)}\right\}_M = 0, \quad 0 \le i,\ j \le 2,\ m, l \ge 0. \tag{42}$$

Proposition 2. *The* $3N$ *1-forms* $dF_m^{(0)}$, $dF_k^{(1)}$, $dF_l^{(2)}$ $(0 \le m \le N-1,\ 0 \le k \le N,\ 1 \le l \le N-1)$ *are linearly independent.*

Proof. Assume that there exist $3N$ constants $b_s^{(i)}$, so that

$$\sum_{m=0}^{N-1} b_m^{(0)} \frac{\partial F_m^{(0)}}{\partial p^j} + \sum_{k=0}^{N} b_k^{(1)} \frac{\partial F_k^{(1)}}{\partial p^j} + \sum_{l=1}^{N-1} b_l^{(2)} \frac{\partial F_l^{(2)}}{\partial p^j} = 0, \quad 1 \le j \le 3. \tag{43}$$

It is easy to see that

$$\frac{\partial F_m^{(0)}}{\partial p^1} = \Lambda^m q^1, \qquad \frac{\partial F_0^{(1)}}{\partial p^1} = -q^1,$$
$$\left.\frac{\partial F_k^{(1)}}{\partial p^1}\right|_{(p^1,p^2,p^3)=0} = -\Lambda^{k-1} q^3, \qquad k \ge 1, \tag{44}$$
$$\left.\frac{\partial F_l^{(2)}}{\partial p^1}\right|_{(p^1,p^2,p^3)=0} = 0.$$

Then we have

$$\sum_{m=0}^{N-1} b_m^{(0)} \Lambda^m q^1 - b_0^{(1)} q^1 - \sum_{k=1}^{N} b_k^{(1)} \Lambda^{k-1} q^3 = 0, \tag{45}$$

which gives rise to

$$b_0^{(0)} - b_0^{(1)} + \sum_{m=1}^{N-1} b_m^{(0)} \left(\lambda_j\right)^m = 0,$$
$$\sum_{k=1}^{N} b_k^{(1)} \left(\lambda_j\right)^{k-1} = 0, \quad 1 \le j \le N, \tag{46}$$

by substituting $q^3 = 0$ and $q^1 = 0$ into (45), respectively. Therefore, we have $b_0^{(0)} = b_0^{(1)}$, $b_m^{(0)} = b_k^{(1)} = 0$, $1 \le m \le N-1$, $1 \le k \le N$, by utilizing that Vandermonde determinant is not zero. Then (43) is transformed to

$$b_0^{(0)} \left(\frac{\partial F_0^{(0)}}{\partial p^j} + \frac{\partial F_0^{(1)}}{\partial p^j} \right) + \sum_{l=1}^{N-1} b_l^{(2)} \frac{\partial F_l^{(2)}}{\partial p^j} = 0, \quad 1 \le j \le 3. \tag{47}$$

According to (34), we obtain

$$\frac{\partial F_0^{(0)}}{\partial p^2} = q^2, \qquad \frac{\partial F_0^{(1)}}{\partial p^2} = 0,$$
$$\left.\frac{\partial F_l^{(2)}}{\partial p^2}\right|_{q^1=0} = \sum_{\substack{m+k=l-1 \\ m,k\ge 0}} \left(\left\langle \Lambda^k q^2, p^1 \right\rangle \Lambda^m q^3 - \left\langle \Lambda^k q^3, p^1 \right\rangle \Lambda^m q^2 \right). \tag{48}$$

Let

$$q_j^3 = \left(p_j^1\right)^{-1} \prod_{k=1,k\ne j}^{N} \left(\lambda_j - \lambda_k\right)^{-1},$$
$$r_k = \sum_{j=1}^{N} \left(\lambda_j\right)^k q_j^2 p_j^1 = \left\langle \Lambda^k q^2, p^1 \right\rangle, \tag{49}$$

and noticing that

$$\sum_{j=1}^{N} \left(\mu_j\right)^k \prod_{k=1,k\ne j}^{N} \left(\mu_j - \mu_k\right)^{-1} = \begin{cases} 0, & 0 \le k \le N-2, \\ \displaystyle\sum_{\substack{m_1+\cdots+m_N=k-N+1 \\ m_1,\ldots,m_N\ge 0}} (\mu_1)^{m_1} \cdots (\mu_N)^{m_N}, & k \ge N-1, \end{cases} \tag{50}$$

then we have

$$\left\langle \Lambda^k q^3, p^1 \right\rangle = \sum_{j=1}^{N} \left(\lambda_j\right)^k q_j^3 p_j^1 = \begin{cases} 0, & 0 \le k \le N-2, \\ 1, & k = N-1. \end{cases} \tag{51}$$

Thus (47) yields

$$Lb = 0, \qquad b = \left(b_1^{(2)}, \ldots, b_{N-1}^{(2)}, b_0^{(0)}\right)^T, \tag{52}$$

where

$$L = \begin{pmatrix} r_0 q_1^3 & (r_0\lambda_1 + r_1) q_1^3 & \cdots & \left(\sum_{k=0}^{N-2} r_k (\lambda_1)^{N-2-k}\right) q_1^3 & q_1^2 \\ r_0 q_2^3 & (r_0\lambda_2 + r_1) q_2^3 & \cdots & \left(\sum_{k=0}^{N-2} r_k (\lambda_2)^{N-2-k}\right) q_2^3 & q_2^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_0 q_N^3 & (r_0\lambda_N + r_1) q_N^3 & \cdots & \left(\sum_{k=0}^{N-2} r_k (\lambda_N)^{N-2-k}\right) q_N^3 & q_N^2 \end{pmatrix}. \tag{53}$$

To arrive at $b = 0$, one needs only to prove $\det L \neq 0$. Therefore we introduce a matrix T:

$$T = \begin{pmatrix} p_1^1 & p_2^1 & \cdots & p_N^1 \\ \lambda_1 p_1^1 & \lambda_2 p_2^1 & \cdots & \lambda_N p_N^1 \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_1)^{N-1} p_1^1 & (\lambda_2)^{N-1} p_2^1 & \cdots & (\lambda_N)^{N-1} p_N^1 \end{pmatrix}. \quad (54)$$

It is easy to verify that $\det T = \prod_{j=1}^N p_j^1 \prod_{i<j}(\lambda_i - \lambda_j) \neq 0$, and

$$TL = \begin{pmatrix} 0 & 0 & \cdots & 0 & r_0 \\ 0 & 0 & \cdots & r_0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & r_0 & \cdots & * & * \\ r_0 & * & \cdots & * & * \end{pmatrix}, \qquad \det(TJ) = (r_0)^N \neq 0, \quad (55)$$

where $*$ represents the entries which may not be zero, and this proves Proposition 2. □

Resorting to the two propositions above and noticing that $f_1 = F_0^{(0)} + F_0^{(1)}$, we obtain the following assertion.

Proposition 3. *The Neumann system defined by* (18) *is completely integrable in the Liouville sense on M.*

4. The Representation of Solutions

In this section, we will give the representation of solutions for (1). To this end, we denote the variable of $\widehat{H}_1$-flow by t, where

$$\begin{aligned} \widehat{H}_1 &= H_1 + a_1 f_1, \\ H_1 &= -F_1^{(1)} - F_1^{(0)} \\ &= \langle q^1, p^3 \rangle \langle q^1, p^1 \rangle + \langle q^2, p^3 \rangle \langle q^1, p^2 \rangle \\ &\quad - \langle \Lambda q^1, p^1 \rangle - \langle \Lambda q^2, p^2 \rangle + \langle q^3, p^1 \rangle, \end{aligned} \quad (56)$$

and a_1 is the Lagrangian multiplier

$$a_1 = -\frac{\{H_1, f_2\}}{\{f_1, f_2\}} = \langle q^1, p^3 \rangle. \quad (57)$$

Then the canonical equation of the $\widehat{H}_1$-flow on M is

$$\begin{aligned} q_t^1 &= \frac{\partial \widehat{H}_1}{\partial p^1} = \langle q^1, p^3 \rangle q^1 - \Lambda q^1 + q^3, \\ q_t^2 &= \frac{\partial \widehat{H}_1}{\partial p^2} = \langle q^2, p^3 \rangle q^1 + \langle q^1, p^3 \rangle q^2 - \Lambda q^2, \\ q_t^3 &= \frac{\partial \widehat{H}_1}{\partial p^3} = \langle q^1, p^1 \rangle q^1 + \langle q^1, p^2 \rangle q^2, \\ p_t^1 &= -\frac{\partial \widehat{H}_1}{\partial q^1} = \Lambda p^1 - \langle q^1, p^3 \rangle p^1 - \langle q^2, p^3 \rangle p^2 \\ &\quad - \langle q^1, p^1 \rangle p^3, \\ p_t^2 &= -\frac{\partial \widehat{H}_1}{\partial q^2} = \Lambda p^2 - \langle q^1, p^3 \rangle p^2 - \langle q^1, p^2 \rangle p^3, \\ p_t^3 &= -\frac{\partial \widehat{H}_1}{\partial q^3} = -p^1. \end{aligned} \quad (58)$$

Using (58) and the Neumann constraints (16) and (17), we derive that

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \langle q^1, p^3 \rangle^2 - \langle \Lambda q^1, p^3 \rangle + \langle q^3, p^3 \rangle - \langle q^1, p^1 \rangle \\ -2\langle q^1, p^2 \rangle + 2\langle q^2, p^3 \rangle \\ \langle q^3, p^2 \rangle - \langle q^1, p^2 \rangle \langle q^1, p^3 \rangle \end{pmatrix}. \quad (59)$$

On the other hand, combining (18), (16), and (17) we have

$$Jg_0 = \begin{pmatrix} 2\partial w + (\partial^2 - v\partial) u \\ 2\partial u \\ -(\partial^2 + \partial v) w \end{pmatrix} = \begin{pmatrix} \langle q^1, p^3 \rangle^2 - \langle \Lambda q^1, p^3 \rangle + \langle q^3, p^3 \rangle - \langle q^1, p^1 \rangle \\ -2\langle q^1, p^2 \rangle + 2\langle q^2, p^3 \rangle \\ \langle q^3, p^2 \rangle - \langle q^1, p^2 \rangle \langle q^1, p^3 \rangle \end{pmatrix}. \quad (60)$$

Then we arrive at

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = J \begin{pmatrix} \langle q^1, p^2 \rangle \\ 0 \\ \langle q^1, p^3 \rangle \end{pmatrix} = Jg_0, \quad (61)$$

which is (1). Therefore we obtain the following result.

Proposition 4. *Let $(q^i(x,t), p^i(x,t))$, $1 \leq i \leq 3$ be a compatible solution of the Hamiltonian systems* (18) *and* (58)*; then the functions*

$$\begin{aligned} u &= \langle q^1, p^3 \rangle, \qquad w = \langle q^1, p^2 \rangle, \\ v &= \langle q^2, p^1 \rangle^{-1} \big(\langle q^1, p^2 \rangle \langle q^2, p^3 \rangle - \langle q^1, p^3 \rangle \langle q^1, p^1 \rangle \\ &\quad + \langle \Lambda q^1, p^1 \rangle - \langle \Lambda q^2, p^2 \rangle - \langle q^3, p^1 \rangle \big) \end{aligned} \quad (62)$$

solve (1).

Proof. We only need to prove that the Hamiltonian systems (18) and (58) are compatible. In fact, it is not difficult to verify that $\{\widehat{H}, \widehat{H}_1\} = 0$. Hence the proof is completed [29]. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant no. 11301487), the Foundation for the Author of National Excellent Doctoral Dissertation of China (no. 201313), the Research Foundation of Henan University of Technology (Grant no. 2009BS041), and Soft Science Research Project of the Science and Technology Department of Henan Province (no. 142400410274).

References

[1] A. C. Newell, *Solitons in Mathematics and Physics*, SIAM, Philadelphia, Pa, USA, 1985.

[2] C. H. Gu, *Soliton Theory and Its Application*, Springer, Berlin, Germany, 1995.

[3] X. G. Geng and C. W. Cao, "Decomposition of the (2+1)-dimensional Gardner equation and its quasi-periodic solutions," *Nonlinearity*, vol. 14, no. 6, pp. 1433–1452, 2001.

[4] Z. Zhou, "Liouville integrability of the finite dimensional Hamiltonian systems derived from principal chiral field," *Journal of Mathematical Physics*, vol. 43, no. 10, pp. 5002–5012, 2002.

[5] X. G. Geng and F. Li, "A hierarchy of new nonlinear evolution equations associated with a 3×3 matrix spectral problem," *Chinese Physics Letters*, vol. 26, Article ID 050201, 2009.

[6] C. W. Cao, "Nonlinearization of the Lax system for AKNS hierarchy," *Science in China A*, vol. 33, no. 5, pp. 528–536, 1990.

[7] C. W. Cao and X. G. Geng, "C Neumann and Bargmann systems associated with the coupled KdV soliton hierarchy," *Journal of Physics. A: Mathematical and General*, vol. 23, no. 18, pp. 4117–4126, 1990.

[8] X. Geng, "Discrete Bargmann and Neumann systems and finite-dimensional integrable systems," *Physica A*, vol. 212, no. 1-2, pp. 132–142, 1994.

[9] W. X. Ma and W. Strampp, "An explicit symmetry constraint for the LAX pairs and the adjoint LAX pairs of AKNS systems," *Physics Letters A*, vol. 185, no. 3, pp. 277–286, 1994.

[10] D. Du and C. Cao, "The Lie-Poisson representation of the nonlinearized eigenvalue problem of the Kac-van Moerbeke hierarchy," *Physics Letters A*, vol. 278, no. 4, pp. 209–224, 2001.

[11] C. W. Cao, "Parametric representation of the finite-band solution of the Heisenberg equation," *Physics Letters. A*, vol. 184, no. 4-5, pp. 333–338, 1994.

[12] X. G. Geng, Y. T. Wu, and C. W. Cao, "Quasi-periodic solutions of the modified Kadomtsev-Petviashvili equation," *Journal of Physics A: Mathematical and General*, vol. 32, no. 20, pp. 3733–3742, 1999.

[13] W.-X. Ma and Z. Zhou, "Binary symmetry constraints of N-wave interaction equations in $1+1$ and #+1$ dimensions," *Journal of Mathematical Physics*, vol. 42, no. 9, pp. 4345–4382, 2001.

[14] R. G. Zhou, "Nonlinearizations of spectral problems of the nonlinear Schrödinger equation and the real-valued modified Korteweg-de Vries equation," *Journal of Mathematical Physics*, vol. 48, no. 1, Article ID 013510, 2007.

[15] Z. Qiao and R. Zhou, "Discrete and continuous integrable systems possessing the same non-dynamical r-matrix," *Physics Letters A*, vol. 235, no. 1, pp. 35–40, 1997.

[16] R. G. Zhou, "The finite-band solution of the Jaulent-Miodek equation," *Journal of Mathematical Physics*, vol. 38, no. 5, pp. 2535–2546, 1997.

[17] Z. Qiao, "r-matrix and algebraic-geometric solution for integrable symplectic map," *Chinese Science Bulletin*, vol. 44, no. 2, pp. 114–118, 1999.

[18] Z. J. Qiao, "Generalized r-matrix structure and algebro-geometric solution for integrable system," *Reviews in Mathematical Physics*, vol. 13, no. 5, pp. 545–586, 2001.

[19] Z. J. Qiao, "R-matrix and algebro-geometric solution for the AKNS system," *Theoretical and Mathematical Physics*, vol. 127, pp. 827–834, 2001.

[20] Z. Qiao, "The Camassa-Holm hierarchy, N-dimensional integrable systems, and algebro-geometric solution on a symplectic submanifold," *Communications in Mathematical Physics*, vol. 239, no. 1-2, pp. 309–341, 2003.

[21] Y. Wu and X. Geng, "A finite-dimensional integrable system associated with the three-wave interaction equations," *Journal of Mathematical Physics*, vol. 40, no. 7, pp. 3409–3430, 1999.

[22] H. H. Dai and X. G. Geng, "Finite-dimensional integrable systems through the decomposition of a modified Boussinesq equation," *Physics Letters A*, vol. 317, no. 5-6, pp. 389–400, 2003.

[23] D. L. Du, "Complex form, reduction and Lie-Poisson structure for the nonlinearized spectral problem of the Heisenberg hierarchy," *Physica A*, vol. 303, no. 3-4, pp. 439–456, 2002.

[24] R. Zhou, "Dynamical r-matrices for the constrained Harry-Dym flows," *Physics Letters A*, vol. 220, no. 6, pp. 320–330, 1996.

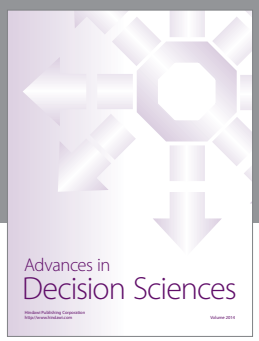
[25] X. G. Geng and H. H. Dai, "Nonlinearization of the 3×3 matrix eigenvalue problem related to coupled nonlinear Schrödinger equations," *Journal of Mathematical Analysis and Applications*, vol. 233, no. 1, pp. 26–55, 1999.

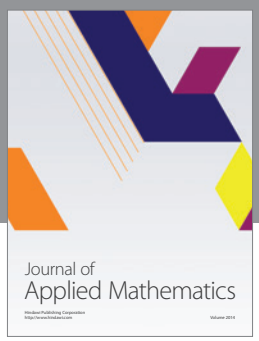
[26] M. De la Sen, "Sufficiency-type stability and stabilization criteria for linear time-invariant systems with constant point delays," *Acta Applicandae Mathematica*, vol. 83, no. 3, pp. 235–256, 2004.

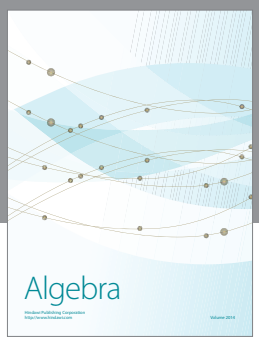
[27] Z. Qiao and S. Li, "A new integrable hierarchy, parametric solutions and traveling wave solutions," *Mathematical Physics, Analysis and Geometry*, vol. 7, no. 4, pp. 289–308, 2004.

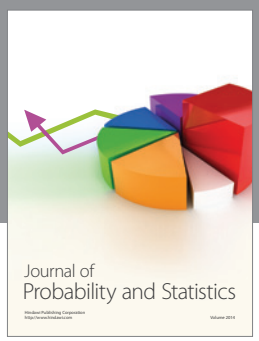
[28] X. G. Geng and D. L. Du, "Two hierarchies of new nonlinear evolution equations associated with 3×3 matrix spectral problems," *Chaos, Solitons and Fractals*, vol. 29, no. 5, pp. 1165–1172, 2006.

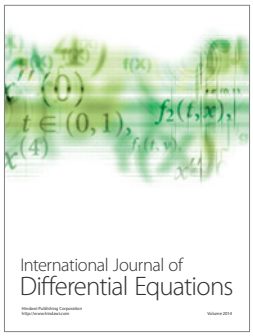
[29] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, 1978.











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