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Research Article

On a System of Two High-Order Nonlinear Difference Equations

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This paper is concerned with dynamics of the solution to the system of two high-order nonlinear difference equations $x_{n+1} = x_{n-k}/(q + \prod_{i=0}^{k} y_{n-i})$, $y_{n+1} = y_{n-k}/(p + \prod_{i=0}^{k} x_{n-i})$, $k \in N^{+}$, $n = 0, 1, \ldots$, where $p, q \in (0, \infty)$, $x_{-i} \in (0, \infty)$, $y_{-i} \in (0, \infty)$ and $i = 0, 1, \ldots, k$. Moreover the rate of convergence of a solution that converges to the equilibrium $(0, 0)$ of the system is discussed. Finally, some numerical examples are considered to show the results obtained.

1. Introduction

Difference equations or discrete dynamical systems are diverse fields which impact almost every branch of pure and applied mathematics. Every dynamical system $x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k})$ determines a difference equation and vice versa. Recently, there has been great interest in studying the system of difference equations. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economic, probability theory, genetics psychology, and so forth. The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering, and economics. It is very interesting to investigate the behavior of solutions of a system of nonlinear difference equations and to discuss the local asymptotic stability of their equilibrium points.

The study of properties of rational difference equations and systems of rational difference equations has been an area of interest in recent years. There are many papers in which systems of difference equations have been studied.

Çinar et al. [1] have obtained the positive solution of the difference equation system:

$$x_{n+1} = \frac{m}{y_n}, \qquad y_{n+1} = \frac{p y_n}{x_{n-1} y_{n-1}}. \tag{1}$$

Çinar [2] has obtained the positive solution of the difference equation system:

$$x_{n+1} = \frac{1}{y_n}, \qquad y_{n+1} = \frac{y_n}{x_{n-1} y_{n-1}}. \tag{2}$$

Also, Çinar and Yalçinkaya [3] have obtained the positive solution of the difference equation system:

$$x_{n+1} = \frac{1}{z_n}, \qquad y_{n+1} = \frac{x_n}{x_{n-1}}, \qquad z_{n+1} = \frac{1}{x_{n-1}}. \tag{3}$$

Özban [4] has investigated the positive solutions of the system of rational difference equations:

$$x_{n+1} = \frac{1}{y_{n-k}}, \qquad y_{n+1} = \frac{y_n}{x_{n-m} y_{n-m+k}}. \tag{4}$$

Papaschinopoulos and Schinas [5] investigated the global behavior for a system of the following two nonlinear difference equations:

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, \quad y_{n+1} = A + \frac{x_n}{y_{n-q}}, \quad n = 0, 1, \ldots, \tag{5}$$

where A is a positive real number, p, q are positive integers, and $x_{-p}, \ldots, x_0, y_{-q}, \ldots, y_0$ are positive real numbers.

Clark et al. [6, 7] investigated the system of rational difference equations:

$$x_{n+1} = \frac{x_n}{a + cy_n}, \quad y_{n+1} = \frac{y_n}{b + dx_n}, \quad n = 0, 1, \ldots, \tag{6}$$

where $a, b, c, d \in (0, \infty)$ and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers.

In 2012, Zhang et al. [8] investigated the global behavior for a system of the following third order nonlinear difference equations:

$$x_{n+1} = \frac{x_{n-2}}{B + y_{n-2}y_{n-1}y_n}, \qquad y_{n+1} = \frac{y_{n-2}}{A + x_{n-2}x_{n-1}x_n}, \tag{7}$$

where $A, B \in (0, \infty)$, and the initial values $x_{-i}, y_{-i} \in (0, \infty)$, $i = 0, 1, 2$.

Ibrahim [9] has obtained the positive solution of the difference equation system in the modeling competitive populations:

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1}y_n + \alpha}, \qquad y_{n+1} = \frac{y_{n-1}}{y_{n-1}x_n + \beta}. \tag{8}$$

Din et al. [10] studied the global behavior of positive solution to the fourth-order rational difference equations:

$$\begin{aligned} x_{n+1} &= \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}}, \\ y_{n+1} &= \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}, \end{aligned} \tag{9}$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ and the initial conditions x_{-i}, y_{-i}, $i = 0, 1, 2, 3$ are positive real numbers.

Although difference equations are sometimes very simple in their forms, they are extremely difficult to understand thoroughly the behavior of their solutions. In [11], Kocić and Ladas have studied global behavior of nonlinear difference equations of higher order. Similar nonlinear systems of rational difference equations were investigated (see [12, 13]). Other related results reader can refer to [14–22].

Our aim in this paper is to investigate the solutions, stability character, and asymptotic behavior of the system of difference equations:

$$x_{n+1} = \frac{x_{n-k}}{q + \prod_{i=0}^{k} y_{n-i}}, \qquad y_{n+1} = \frac{y_{n-k}}{p + \prod_{i=0}^{k} x_{n-i}}, \qquad n = 0, 1, \ldots, k \in N^+. \tag{10}$$

where $p, q \in (0, \infty)$ and initial conditions $x_i, y_i \in (0, \infty)$, $i = -k, -k+1, \ldots, 0$. This paper is natural extension of [8–10, 14].

2. Preliminaries

Let I, J be some intervals of real number and let $f : I^{k+1} \times J^{k+1} \to I$, $g : I^{k+1} \times J^{k+1} \to J$ be continuously differentiable functions. Then for every initial conditions $(x_i, y_j) \in I \times J$ ($i = -k, -k+1, \ldots, 0$; $j = -k, -k+1, \ldots, 0$), the system of difference equations

$$\begin{aligned} x_{n+1} &= f(x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}), \\ y_{n+1} &= g(x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}), \\ & n = 0, 1, 2, \ldots \end{aligned} \tag{11}$$

has a unique solution $\{(x_n, y_n)\}_{n=-k}^{\infty}$. A point $(\overline{x}, \overline{y}) \in I \times J$ is called an equilibrium point of (11) if $\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}, \overline{y}, \overline{y}, \ldots, \overline{y})$, $\overline{y} = g(\overline{x}, \overline{x}, \ldots, \overline{x}, \overline{y}, \overline{y}, \ldots, \overline{y})$; that is, $(x_n, y_n) = (\overline{x}, \overline{y})$ for all $n \geq 0$.

Definition 1. Assume that $(\overline{x}, \overline{y})$ is an equilibrium point of (11). Then one has the following

(i) $(\overline{x}, \overline{y})$ is said to be stable relative to $I \times J$, if, for every $\varepsilon > 0$, and any initial conditions $(x_i, y_i) \in I \times J$, $i \in \{-k, -k+1, \ldots, -1, 0\}$, there exists $\delta > 0$ such that $\sum_{i=-k}^{0} |x_i - \overline{x}| < \delta$, $\sum_{i=-k}^{0} |y_i - \overline{y}| < \delta$, implies $|x_n - \overline{x}| < \varepsilon$, $|y_n - \overline{y}| < \varepsilon$.

(ii) $(\overline{x}, \overline{y})$ is called an attractor relative to $I \times J$ if for all $(x_i, y_i) \in I \times J$, $i \in \{-k, -k+1, \ldots, -1, 0\}$, $\lim_{n\to\infty} x_n = \overline{x}$, $\lim_{n\to\infty} y_n = \overline{y}$.

(iii) $(\overline{x}, \overline{y})$ is called asymptotically stable relative to $I \times J$ if it is stable and an attractor.

(iv) $(\overline{x}, \overline{y})$ is unstable if it is not stable.

Definition 2. Let $(\overline{x}, \overline{y})$ be an equilibrium point of a map $F = (f, x_n, x_{n-1}, \ldots, x_{n-k}, g, y_n, y_{n-1}, \ldots, y_{n-k})$, where f and g are continuously differentiable functions at $(\overline{x}, \overline{y})$. The linearized system of (11) about the equilibrium point $(\overline{x}, \overline{y})$ is

$$X_{n+1} = F(X_n) = J_F X_n, \tag{12}$$

where $X_n = (x_n, \ldots, x_{n-k}, y_n, \ldots, y_{n-k})^T$, and J_F is Jacobian matrix of system (11) about the equilibrium point $(\overline{x}, \overline{y})$.

Theorem 3 (see [11]). *Assume that $X_{n+1} = F(X_n)$, $n = 0, 1, \ldots$, is a system of difference equations and $\overline{X}$ is the equilibrium point of this system; that is, $F(\overline{X}) = \overline{X}$. If all eigenvalues of the Jacobian matrix J_F, evaluated at $\overline{X}$, lie inside the open unit disk $|\lambda| < 1$, then $\overline{X}$ is locally asymptotically stable. If one of them has modulus greater than one, then $\overline{X}$ is unstable.*

Theorem 4 (see [12]). *Assume that $X_{n+1} = F(X_n)$, $n = 0, 1, \ldots$, is a system of difference equations and $\overline{X}$ is the equilibrium point of this system, and the characteristic polynomial* *of this system about the equilibrium point* $\overline{X}$ *is* $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$, *with real coefficients and* $a_0 > 0$. *Then all roots of the polynomial* $p(\lambda)$ *lie inside the open unit disk* $|\lambda| < 1$ *if and only if*

$$\Delta_k > 0 \quad \text{for } k = 1, 2, \ldots, n, \tag{13}$$

where Δ_k *is the principal minor of order* k *of the* $n \times n$ *matrix:*

$$\Delta_n = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}. \tag{14}$$

3. Main Results

The equilibrium points of system (10) are $(0,0)$ and $(\sqrt[k+1]{1-p}, \sqrt[k+1]{1-q})$, for $p < 1$ and $q < 1$. In addition, if $q = 1$, then every point on the x-axis is an equilibrium point, and if $p = 1$, then every point on the y-axis is an equilibrium point. Finally, if $p > 1$ and $q > 1$, $(0, 0)$ is the unique equilibrium point.

We summarize the local stability of the equilibria of (10) as follows.

Theorem 5. *For the equilibrium point* $(0,0)$ *of system* (10)*, the following results hold.*

(i) *If* $p > 1$ *and* $q > 1$, *then the unique equilibrium point* $(0,0)$ *of system* (10) *is locally asymptotically stable.*

(ii) *If* $p < 1$ *or* $q < 1$, *then the equilibrium point* $(0,0)$ *of system* (10) *is unstable.*

Proof. (i) The linearized equation of system (10) about $(0,0)$ is

$$X_{n+1} = J_F(0,0)\, X_n, \tag{15}$$

where $X_n = (x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots y_{n-k})^T$, and

$$J_F(0,0) = \left(d_{ij}\right)_{(2k+2)\times(2k+2)}$$

$$= \begin{pmatrix} 0 & \cdots & 0 & \frac{1}{q} & 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{1}{p} \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{16}$$

The characteristic equation of (15) is

$$\lambda^{2k}\left(\lambda - (-1)^{k+2}\frac{1}{q}\right)\left(\lambda - (-1)^{k+2}\frac{1}{p}\right) = 0. \tag{17}$$

This shows that all the roots of characteristic equation lie inside unit disk. So the unique equilibrium $(0,0)$ is locally asymptotically stable.

(ii) It is easy to see that if $p < 1$ or $q < 1$, then there exists at least one root λ of (17) such that $|\lambda| > 1$. Hence by Theorem 3 if $p < 1$ or $q < 1$, then $(0,0)$ is unstable. The proof is complete. □

Theorem 6. *If* $p < 1$ *and* $q < 1$, *then the positive equilibrium point* $P_1(\overline{x}, \overline{y}) = (\sqrt[k+1]{1-p}, \sqrt[k+1]{1-q})$ *of* (10) *is unstable.*

Proof. The linearized system of (10) about the equilibrium point P_1 is given by

$$X_{n+1} = J_F(P_1)\, X_n, \tag{18}$$

where $X_n = (x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k})^T$, and

$$J_F(P_1) = B_{(2k+2)\times(2k+2)} = \begin{pmatrix} 0 & \cdots & 0 & 1 & -\overline{y}^k\,\overline{x} & \cdots & -\overline{y}^k\,\overline{x} & -\overline{y}^k\,\overline{x} \\ 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ -\overline{x}^k\,\overline{y} & \cdots & -\overline{x}^k\,\overline{y} & -\overline{x}^k\,\overline{y} & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{19}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_{2k+2}$ denote the eigenvalues of matrix B, and let $D = \operatorname{diag}(d_1, d_2, \ldots, d_{2k+2})$ be a diagonal matrix, where $d_1 = d_{k+2} = 1$, $d_i = d_{k+1+i} = 1 - i\varepsilon$ $(i = 2, 3, \ldots, k+1)$, for $0 < \varepsilon < 1$.

Clearly, D is invertible. In computing matrix DBD^{-1}, we obtain that

$$DBD^{-1} = \begin{pmatrix} 0 & \cdots & 0 & d_1 d_{k+1}^{-1} & -\overline{y}^k \overline{x} d_1 d_{k+2}^{-1} & \cdots & -\overline{y}^k \overline{x} d_1 d_{2k+1}^{-1} & -\overline{y}^k \overline{x} d_1 d_{2k+2}^{-1} \\ d_2 d_1^{-1} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & d_{k+1} d_k^{-1} & 0 & 0 & \cdots & 0 & 0 \\ -\overline{x}^k \overline{y} d_{k+2} d_1^{-1} & \cdots & -\overline{x}^k \overline{y} d_{k+2} d_k^{-1} & -\overline{x}^k \overline{y} d_{k+2} d_{k+1}^{-1} & 0 & \cdots & 0 & d_{k+2} d_{2k+2}^{-1} \\ 0 & \cdots & 0 & 0 & d_{k+3} d_{k+2}^{-1} & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & d_{2k+2} d_{2k+1}^{-1} & 0 \end{pmatrix}. \tag{20}$$

From $d_1 > d_2 > \cdots > d_{k+1} > 0$ and $d_{k+2} > d_{k+3} > \cdots > d_{2k+2} > 0$ it implies that

$$\begin{gathered} d_2 d_1^{-1} < 1, \; d_3 d_2^{-1} < 1, \ldots, \; d_{k+1} d_k^{-1} < 1, \\ d_{k+3} d_{k+2}^{-1} < 1, \ldots, \; d_{2k+2} d_{2k+1}^{-1} < 1. \end{gathered} \tag{21}$$

On the other hand

$$\begin{aligned} &\frac{1}{1-(k+1)\varepsilon} + \overline{y}^k \overline{x} + \overline{y}^k \overline{x} \frac{1}{1-2\varepsilon} + \cdots + \overline{y}^k \overline{x} \frac{1}{1-(k+1)\varepsilon} \\ &\quad > 1 \\ &\overline{x}^k \overline{y} + \overline{x}^k \overline{y} \frac{1}{1-2\varepsilon} + \cdots + \overline{x}^k \overline{y} \frac{1}{1-(k+1)\varepsilon} + \frac{1}{1-(k+1)\varepsilon} \\ &\quad > 1. \end{aligned} \tag{22}$$

It is well known that B has the same eigenvalues as DBD^{-1}, and we have that

$$\begin{aligned} &\max_{1 \le i \le 2k+2} |\lambda_i| \\ &\quad \le \left\| DBD^{-1} \right\|_\infty \\ &\quad = \max \Big\{ d_2 d_1^{-1}, \ldots, d_{k+1} d_k^{-1}, d_{k+3} d_{k+2}^{-1}, \ldots, d_{2k+2} d_{2k+1}^{-1}, \\ &\qquad \frac{1}{1-(k+1)\varepsilon} + \overline{y}^k \overline{x} + \overline{y}^k \overline{x} \frac{1}{1-2\varepsilon} \\ &\qquad + \cdots + \overline{y}^k \overline{x} \frac{1}{1-(k+1)\varepsilon}, \overline{x}^k \overline{y} + \overline{x}^k \overline{y} \frac{1}{1-2\varepsilon} \\ &\qquad + \cdots + \overline{x}^k \overline{y} \frac{1}{1-(k+1)\varepsilon} + \frac{1}{1-(k+1)\varepsilon} \Big\} \\ &\quad > 1. \end{aligned} \tag{23}$$

This implies that the equilibrium $(\overline{x}, \overline{y})$ of (10) is unstable. □

The following theorem is similar to Theorem 3.4 of [8].

Theorem 7. *Let $p < 1$ and $q < 1$, (x_n, y_n) is a solution of system (10), and then, for $m = -k, -k+1, \ldots, 0$, the following statements are true.*

(i) *If $(x_m, y_m) \in (0, \sqrt[k+1]{1-p}) \times (\sqrt[k+1]{1-q}, +\infty)$, then $(x_n, y_n) \in (0, \sqrt[k+1]{1-p}) \times (\sqrt[k+1]{1-q}, +\infty)$.*

(ii) *If $(x_m, y_m) \in (\sqrt[k+1]{1-p}, +\infty) \times (0, \sqrt[k+1]{1-q})$, then $(x_n, y_n) \in (\sqrt[k+1]{1-p}, +\infty) \times (0, \sqrt[k+1]{1-q})$.*

Theorem 8. *Let (x_n, y_n) be positive solution of system (10), then for $m \ge 0$ the following results hold:*

$$0 \le x_n \le \begin{cases} \left(\dfrac{1}{q}\right)^{m+1} x_{-k}, & \text{if } n = (k+1)m + 1, \\ \left(\dfrac{1}{q}\right)^{m+1} x_{-k+1}, & \text{if } n = (k+1)m + 2, \\ \vdots & \\ \left(\dfrac{1}{q}\right)^{m+1} x_0, & \text{if } n = (k+1)m + k + 1, \end{cases}$$

$$0 \le y_n \le \begin{cases} \left(\dfrac{1}{p}\right)^{m+1} y_{-k}, & \text{if } n = (k+1)m + 1, \\ \left(\dfrac{1}{p}\right)^{m+1} y_{-k+1}, & \text{if } n = (k+1)m + 2, \\ \vdots & \\ \left(\dfrac{1}{p}\right)^{m+1} y_0, & \text{if } n = (k+1)m + k + 1. \end{cases} \tag{24}$$

Proof. It is true for $m = 0$. Suppose that results are true for $m = h \ge 1$, namely,

$$0 \le x_n \le \begin{cases} \left(\dfrac{1}{q}\right)^{h+1} x_{-k}, & \text{if } n = (k+1)h + 1, \\ \left(\dfrac{1}{q}\right)^{h+1} x_{-k+1}, & \text{if } n = (k+1)h + 2, \\ \vdots & \\ \left(\dfrac{1}{q}\right)^{h+1} x_0, & \text{if } n = (k+1)h + k + 1, \end{cases}$$

$$0 \le y_n \le \begin{cases} \left(\dfrac{1}{p}\right)^{h+1} y_{-k}, & \text{if } n = (k+1)h+1, \\ \left(\dfrac{1}{p}\right)^{h+1} y_{-k+1}, & \text{if } n = (k+1)h+2, \\ \vdots & \\ \left(\dfrac{1}{p}\right)^{h+1} y_0, & \text{if } n = (k+1)h+k+1. \end{cases} \quad (25)$$

Now, for $m = h + 1$, by virtue of system (10), we have

$$\begin{aligned} 0 \le x_{(k+1)h+k+2} &= \frac{x_{(k+1)h+1}}{q + \prod_{i=0}^{k-1} y_{(k+1)h+k+2-i}} \\ &\le \frac{x_{(k+1)h+1}}{q} \le \left(\frac{1}{q}\right)^{h+2} x_{-k}, \\ 0 \le x_{(k+1)h+k+3} &= \frac{x_{(k+1)h+2}}{q + \prod_{i=0}^{k-1} y_{(k+1)h+k+3-i}} \\ &\le \frac{x_{(k+1)h+2}}{q} \le \left(\frac{1}{q}\right)^{h+2} x_{-k+1}, \end{aligned} \quad (26)$$

and similarly,

$$\begin{aligned} 0 \le x_{(k+1)h+2k+2} &= \frac{x_{(k+1)h+k+1}}{q + \prod_{i=0}^{k-1} y_{(k+1)h+2k+2-i}} \\ &\le \frac{x_{(k+1)h+k+1}}{q} \le \left(\frac{1}{q}\right)^{h+2} x_0, \\ 0 \le y_{(k+1)h+k+2} &= \frac{y_{(k+1)h+1}}{p + \prod_{i=0}^{k-1} x_{(k+1)h+k+2-i}} \\ &\le \frac{y_{(k+1)h+1}}{p} \le \left(\frac{1}{p}\right)^{h+2} y_{-k}, \\ 0 \le y_{(k+1)h+k+3} &= \frac{y_{(k+1)h+2}}{p + \prod_{i=0}^{k-1} x_{(k+1)h+k+3-i}} \\ &\le \frac{y_{(k+1)h+2}}{p} \le \left(\frac{1}{p}\right)^{h+2} y_{-k+1}, \end{aligned} \quad (27)$$

and similarly,

$$\begin{aligned} 0 \le y_{(k+1)h+2k+2} &= \frac{y_{(k+1)h+k+1}}{p + \prod_{i=0}^{k-1} x_{(k+1)h+2k+2-i}} \\ &\le \frac{y_{(k+1)h+k+1}}{p} \le \left(\frac{1}{p}\right)^{h+2} y_0. \end{aligned} \quad (28)$$

Hence, for $\forall m \ge 0$, the results are true. □

Theorem 9. *If $p > 1$ and $q > 1$, then the unique equilibrium point $(0, 0)$ of system (10) is globally asymptotically stable.*

Proof. From (i) of Theorem 5, we obtain that the unique equilibrium point $(0, 0)$ of system (10) is locally asymptotically stable. By virtue of Theorem 7, it is clear that every positive solution (x_n, y_n) is bounded. That is, $0 \le x_n \le \alpha$ and $0 \le y_n \le \beta$, where $\alpha = \max\{x_{-k}, x_{-k+1}, \dots, x_0\}$, $\beta = \max\{y_{-k}, y_{-k+1}, \dots, y_0\}$.

Now, it is sufficient to prove that (x_n, y_n) is decreasing. From system (10) one has

$$x_{n+1} = \frac{x_{n-k}}{q + \prod_{i=0}^{k} y_{n-i}} \le \frac{x_{n-k}}{q} \le x_{n-k} \quad (29)$$

This implies that $x_{(k+1)n+1} \le x_{(k+1)n-k}$ and $x_{(k+1)n+k+2} \le x_{(k+1)n+1}$; hence, the subsequences $\{x_{(k+1)n+1}\}$, $\{x_{(k+1)n+2}\}$, $\dots$, $\{x_{(k+1)n+k+1}\}$ are decreasing. So sequence $\{x_n\}$ is decreasing. Similarly, it is easy to prove that sequence $\{y_n\}$ is also decreasing. Hence $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. Therefore the equilibrium point $(0, 0)$ is globally asymptotically stable. □

4. Rate of Convergence

In this section we will determine the rate of convergence of a solution that converges to the equilibrium point $(0, 0)$ of the system (10). The following result gives the rate of convergence of solution of a system of difference equations:

$$X_{n+1} = [A + B(n)] X_n, \quad (30)$$

where X_n is an m-dimensional vector, $A \in C^{m\times m}$ is a constant matrix, and $B : Z^+ \to C^{m\times m}$ is a matrix function satisfying

$$\|B(n)\| \longrightarrow 0, \quad \text{when} \quad n \longrightarrow \infty, \quad (31)$$

where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm.

Theorem 10 (see [23]). *Assume that condition (31) holds, if X_n is a solution of (30), then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \sqrt[n]{\|X_n\|} \quad (32)$$

or

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \quad (33)$$

exists and is equal to the modulus of one the eigenvalues of the matrix A.

Assume that $\lim_{n\to\infty} x_n = \bar{x}$, $\lim_{n\to\infty} y_n = \bar{y}$, we will find a system of limiting equations for the system (10). The error terms are given as

$$\begin{aligned} x_{n+1} - \bar{x} &= \sum_{i=0}^{k} A_i (x_{n-i} - \bar{x}) + \sum_{i=0}^{k} B_i (y_{n-i} - \bar{y}), \\ y_{n+1} - \bar{y} &= \sum_{i=0}^{k} C_i (x_{n-i} - \bar{x}) + \sum_{i=0}^{k} D_i (y_{n-i} - \bar{y}). \end{aligned} \quad (34)$$

Set $e_n^1 = x_n - \overline{x}$, $e_n^2 = y_n - \overline{y}$; therefore, it follows that

$$\begin{aligned} e_{n+1}^1 &= \sum_{i=0}^{k} A_i e_{n-i}^1 + \sum_{i=0}^{k} B_i e_{n-i}^2 \\ e_{n+1}^2 &= \sum_{i=0}^{k} C_i e_{n-i}^1 + \sum_{i=0}^{k} D_i e_{n-i}^2, \end{aligned} \tag{35}$$

where

$$\begin{aligned} & A_i = 0, \quad i = 0, 1, \ldots, k-1, \quad A_k = \frac{1}{q + \prod_{i=0}^{k} y_{n-i}}, \\ & B_0 = -\frac{x_{n-k}\prod_{i=1}^{k} y_{n-i}}{\left(q + \prod_{i=0}^{k} y_{n-i}\right)^2}, \ldots, B_k = -\frac{x_{n-k}\prod_{i=0}^{k-1} y_{n-i}}{\left(q + \prod_{i=0}^{k} y_{n-i}\right)^2}, \\ & C_0 = -\frac{y_{n-k}\prod_{i=1}^{k} x_{n-i}}{\left(p + \prod_{i=0}^{k} x_{n-i}\right)^2}, \ldots, C_k = -\frac{y_{n-k}\prod_{i=0}^{k-1} x_{n-i}}{\left(p + \prod_{i=0}^{k} x_{n-i}\right)^2}, \\ & D_i = 0, \quad i = 0, 1, \ldots, k-1, \quad D_k = \frac{1}{p + \prod_{i=0}^{k} x_{n-i}}. \end{aligned} \tag{36}$$

Now it is clear that

$$\begin{aligned} & \lim_{n\to\infty} A_i = 0, \quad \text{for } i \in \{0, 1, \ldots, k-1\}, \\ & \lim_{n\to\infty} A_k = \frac{1}{q + \overline{y}^{k+1}}, \\ & \lim_{n\to\infty} B_i = -\frac{\overline{x}\,\overline{y}^k}{\left(q + \overline{y}^{k+1}\right)^2}, \quad i \in \{0, 1, \ldots, k\}. \end{aligned} \tag{37}$$

$$\begin{aligned} & \lim_{n\to\infty} C_i = -\frac{\overline{y}\,\overline{x}^k}{\left(p + \overline{x}^{k+1}\right)^2}, \quad i \in \{0, 1, \ldots, k\}, \\ & \lim_{n\to\infty} D_i = 0, \quad \text{for } i \in \{0, 1, \ldots, k-1\}, \\ & \lim_{n\to\infty} D_k = \frac{1}{p + \overline{x}^{k+1}}. \end{aligned} \tag{38}$$

Hence, the limiting system of error terms at $(0,0)$ can be written as

$$E_{n+1} = G E_n, \tag{39}$$

where $E_n = (e_n^1, e_{n-1}^1, \ldots, e_{n-k}^1, e_n^2, e_{n-1}^2, \ldots, e_{n-k}^2)^T$, and

$$\begin{aligned} G &= J_F(0,0) = \left(d_{ij}\right)_{(2k+2)\times(2k+2)} \\ &= \begin{pmatrix} 0 & \cdots & 0 & \frac{1}{q} & 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \frac{1}{p} \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \end{aligned} \tag{40}$$

Using Theorem 10, we have the following result.

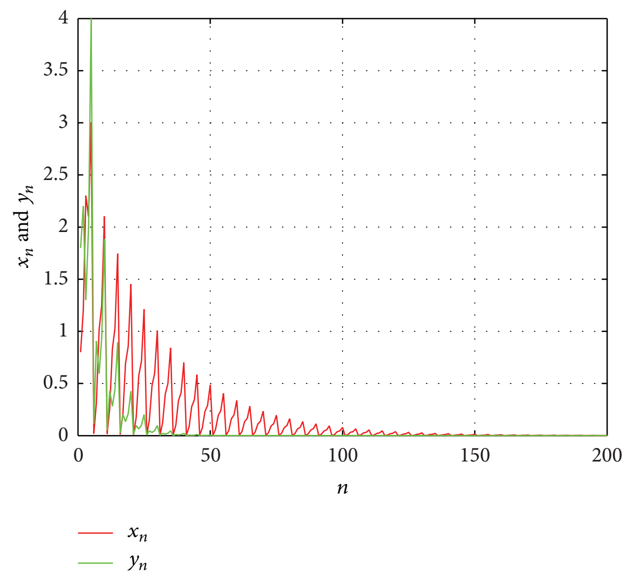

Figure 1: The plot of system (42).

Theorem 11. *Assume that $p > 1, q > 1$, and $\{(x_n, y_n\}$ are a positive solution of the system (10). Then, the error vector E_n of every solution of (10) satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} \sqrt[n]{\|E_n\|} = \left|\lambda J_F(0,0)\right|, \qquad \lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = \left|\lambda J_F(0,0)\right|, \tag{41}$$

where $\lambda J_F(0,0)$ is equal to the modulus of one the eigenvalues of the Jacobian matrix evaluated at the equilibrium $(0,0)$.

5. Numerical Examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions to system of nonlinear difference equations.

Example 1. Consider the system (10) with initial conditions $x_{-4} = 0.8$, $x_{-3} = 1.2$, $x_{-2} = 2.3$, $x_{-1} = 2.1$, $x_0 = 3.0$, $y_{-4} = 1.8$, $y_{-3} = 2.2$, $y_{-2} = 1.3$, $y_{-1} = 2.0$ and $y_0 = 4.0$, moreover, choosing the parameters $p = 2.1$, $q = 1.2$ and $k = 4$. Then system (10) can be written as

$$x_{n+1} = \frac{x_{n-4}}{1.2 + \prod_{i=0}^{4} y_{n-i}}, \qquad y_{n+1} = \frac{y_{n-4}}{2.1 + \prod_{i=0}^{4} x_{n-i}}. \tag{42}$$

The plot of system (42) is shown in Figure 1.

Example 2. Consider the system (10) with initial conditions $x_{-5} = 1.3$, $x_{-4} = 0.8$, $x_{-3} = 0.2$, $x_{-2} = 1.3$, $x_{-1} = 2.1$, $x_0 = 2.6$, $y_{-5} = 0.3$, $y_{-4} = 1.8$, $y_{-3} = 2.2$, $y_{-2} = 0.3$, $y_{-1} = 3.0$ and

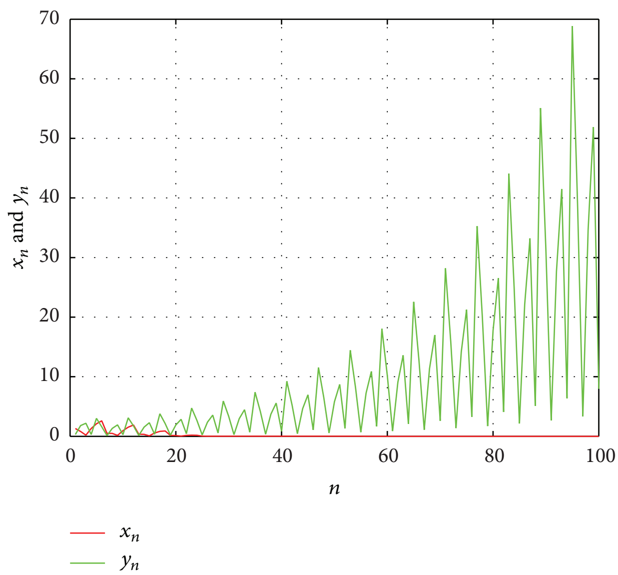

FIGURE 2: The plot of system (43).

$y_0 = 1.6$, moreover, choosing the parameters $p = 0.8$, $q = 0.9$ and $k = 5$. Then system (10) can be written as

$$x_{n+1} = \frac{x_{n-5}}{0.9 + \prod_{i=0}^{5} y_{n-i}}, \qquad y_{n+1} = \frac{y_{n-5}}{0.8 + \prod_{i=0}^{5} x_{n-i}}. \tag{43}$$

The plot of system (43) is shown in Figure 2.

6. Conclusions and Future Work

In this paper, we discussed the dynamics of high-order discrete system which is extension of [8, 10, 14]. We conclude that (i) the equilibrium point $(0, 0)$ is globally asymptotically stable if $p > 1$, $q > 1$, (ii) the equilibrium $(0, 0)$ and $(\sqrt[1+k]{1-p}, \sqrt[1+k]{1-q})$ if $p < 1$ or $q < 1$ is unstable. Some numerical examples are provided to support our theoretical results. It is our future work to study the dynamical behavior of system (10) when $p = 1$ or $q = 1$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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