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Research Article

Conservative Difference Scheme for Generalized Rosenau-KdV Equation

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A conservative Crank-Nicolson finite difference scheme for the initial-boundary value problem of generalized Rosenau-KdV equation is proposed. The difference scheme shows a discrete analogue of the main conservation law associated to the equation. On the other hand the scheme is implicit and stable with second order convergence. Numerical experiments verify the theoretical results.

1. Introduction

In this paper, we consider the following initial-boundary value problem of the generalized Rosenau-KdV equation:

$$u_t + u_{xxxxt} + u_x + (u^p)_x + u_{xxx} = 0, \tag{1}$$

with an initial condition

$$u(x,0) = u_0(x), \quad x \in [x_L, x_R], \tag{2}$$

and boundary conditions

$$\begin{aligned} &u(x_L,t) = u(x_R,t) = 0, \\ &u_x(x_L,t) = u_x(x_R,t) = 0, \\ &u_{xx}(x_L,t) = u_{xx}(x_R,t) = 0, \\ &\qquad t \in [0,T], \end{aligned} \tag{3}$$

where $p \geq 2$ is a integer and $u_0(x)$ is a known smooth function. When $p = 2$, (1) is called usual Rosenau-KdV equation:

$$u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} = 0. \tag{4}$$

Zuo [1] discussed the solitary wave solutions and periodic solutions for Rosenau-KdV equation. In [2], a conservative nonlinear finite difference scheme for an initial-boundary value problem of Rosenau-Kdv equation is considered.

In [3, 4] the solitary solution and invariant for generalized Rosenau-KdV equation are given. In [4] the singular 1-soliton solution is derived by the ansatz method, and the adiabatic parameter dynamics of the water waves is obtained by perturbation theory. In [5, 6], the ansatz method is applied to obtain the topological soliton solution of the generalized Rosenau-KdV equation. The method as well as the exp-function method is also applied to extract a few more solutions to this equation. In [7], Zheng and Zhou give an average linear scheme for the generalized Rosenau-KdV equation. In this paper, we propose a conservative Crank-Nicolson finite difference scheme for an initial-boundary value problem of the generalized Rosenau-Kdv equation.

The initial-boundary value problem (1)–(3) possesses the following conservative property [3, 4]:

$$E(t) = \|u\|_{L_2}^2 + \|u_{xx}\|_{L_2}^2 = E(0). \tag{5}$$

When $-x_L \gg 0$, $x_R \gg 0$, the initial-boundary value problem (1)–(3) and the Cauchy problem (1) are consistent, so that the boundary conditions (3) are reasonable.

It is known that the conservative scheme is better than the nonconservative ones. The nonconservative scheme may easily show nonlinear blow-up. A lot of numerical experiments show that the conservative scheme can possess some invariant properties of the original differential equation [7–18]. The conservative scheme is more suitable for long-time calculations. In [18] Pan and Zhang said "... in some areas, the ability to preserve some invariant properties of the original differential equation is a criterion to judge the success of a numerical simulation".

The rest of this paper is organized as follows. In Section 2, we propose a Crank-Nicolson implicit nonlinear finite difference scheme for the generalized Rosenau-KdV equation and discuss the property of its solution. In Section 3, we prove that the finite difference scheme is of second order convergence. Finally, some numerical tests are given in Section 4 to verify our theoretical analysis.

2. Finite Difference Scheme and Its Property

Let $h = (x_R - x_L)/J$ and let τ be the uniform step size in the spatial and temporal direction, respectively. Denote $x_j = x_L + jh$ $(j = -1, 0, 1, 2, \ldots, J, J+1)$, $t_n = n\tau$ $(n = 0, 1, 2, \ldots, N, N = [T/\tau])$, $u_j^n \approx u(x_j, t_n)$, and $Z_h^0 = \{u = (u_j) \mid u_{-1} = u_0 = u_J = u_{J+1} = 0, j = -1, 0, 1, 2, \ldots, J, J+1\}$. Throughout this paper, we denote C as a generic positive constant independent of h and τ, which may have different values in different occurrences. We introduce the following notations:

$$\left(u_j^n\right)_x = \frac{u_{j+1}^n - u_j^n}{h}, \qquad \left(u_j^n\right)_{\bar{x}} = \frac{u_j^n - u_{j-1}^n}{h},$$
$$\left(u_j^n\right)_{\hat{x}} = \frac{u_{j+1}^n - u_{j-1}^n}{2h}, \qquad \left(u_j^n\right)_t = \frac{u_j^{n+1} - u_j^n}{\tau},$$
$$u_j^{n+(1/2)} = \frac{u_j^{n+1} + u_j^n}{2}, \tag{6}$$
$$\left\langle u^n, v^n \right\rangle = h\sum_{j=1}^{J-1} u_j^n v_j^n,$$
$$\left\|u^n\right\|^2 = \left\langle u^n, u^n \right\rangle, \qquad \left\|u^n\right\|_\infty = \max_{1\le j\le J-1}\left\|u_j^n\right\|.$$

We propose a conservative Crank-Nicolson finite difference scheme for the solution of (1)–(3):

$$\begin{aligned}&\left(u_j^n\right)_t + \left(u_j^n\right)_{xx\bar{x}\bar{x}t} + \left(u_j^{n+(1/2)}\right)_{\hat{x}} + \left(u_j^{n+(1/2)}\right)_{x\bar{x}\hat{x}} \\ &\quad + \frac{p}{p+1}\left(u_j^{n+(1/2)}\right)^{p-1}\left(u_j^{n+(1/2)}\right)_{\hat{x}} + \left[\left(u_j^{n+(1/2)}\right)^p\right]_{\hat{x}},\end{aligned} \tag{7}$$

$$j = 1, 2, 3, \ldots, J-1; \quad n = 1, 2, 3, \ldots, N-1, \tag{8}$$

$$u_j^0 = u_0\left(x_j\right), \quad j = 0, 1, 2, 3, \ldots, J, \tag{9}$$

$$\begin{gathered}\left(u_0^n\right)_{\hat{x}} = \left(u_J^n\right)_{\hat{x}} = 0 \qquad \left(u_0^n\right)_{x\bar{x}} = \left(u_J^n\right)_{x\bar{x}} = 0, \\ u^n \in Z_h^0, \quad n = 1, 2, 3, \ldots, N.\end{gathered} \tag{10}$$

From the boundary conditions (3), we know that (10) is reasonable.

Lemma 1. *It follows from summation by parts that, for any two mesh functions $u, v \in Z_h^0$,*

$$\begin{gathered}\left\langle u_x, v \right\rangle = -\left\langle u, v_x \right\rangle, \qquad \left\langle u_{\hat{x}}, v \right\rangle = -\left\langle u, v_{\hat{x}} \right\rangle, \\ \left\langle u_{x\bar{x}}, v \right\rangle = -\left\langle u_x, v_x \right\rangle.\end{gathered} \tag{11}$$

Then one has

$$\left\langle u_{x\bar{x}}, u \right\rangle = -\left\langle u_x, u_x \right\rangle = -\left\|u_x\right\|^2. \tag{12}$$

Furthermore, if $(u_0^n)_{x\bar{x}} = (u_J^n)_{x\bar{x}} = 0$, then

$$\left\langle u_{xx\bar{x}\bar{x}}, u \right\rangle = \left\|u_{xx}\right\|^2. \tag{13}$$

To show the existence of the solution for (7)–(10), the following Brouwer fixed point theorem should be introduced. For the proof, see [19].

Lemma 2. *Let H be a finite dimensional inner product space, let $\|\cdot\|$ be the associated norm, and let $g : H \to H$ be continuous. Assume, moreover, that there exists an $\alpha > 0$, for all $x \in H$ and $\|x\| = \alpha$, $\langle g(x), x\rangle > 0$. Then there exists $x^* \in H$ such that $g(x^*) = 0$ and $\|x^*\| \le \alpha$.*

Then one has the following theorem.

Theorem 3. *There exists $u^n \in Z_h^0$ which satisfies the difference scheme (7)–(10) $(1 \le n \le N)$.*

Proof. In order to prove the theorem by the mathematical induction, we assume that $u^0, u^1, \ldots, u^n$ which satisfy (7)–(10) exist for $n \le N-1$. Next prove that there also exists u^{n+1} which satisfies (7)–(10).

We define g on Z_h^0 as follows:

$$\begin{aligned}g(v) &= 2v - 2u^n + 2v_{xx\bar{x}\bar{x}} - 2u^n_{xx\bar{x}\bar{x}} + \tau v_{\hat{x}} \\ &\quad + \tau v_{x\bar{x}\hat{x}} + \frac{\tau p}{1+p}\left[v^{p-1}v_{\hat{x}} + \left(v^p\right)_{\hat{x}}\right].\end{aligned} \tag{14}$$

Taking an inner product of (14) with v and considering

$$\left\langle v_{\hat{x}}, v \right\rangle = 0, \qquad \left\langle v_{x\bar{x}\hat{x}}, v \right\rangle = 0, \qquad \left\langle v^{p-1}v_{\hat{x}} + \left(v^p\right)_{\hat{x}}, v \right\rangle = 0, \tag{15}$$

we have

$$\begin{aligned}\left\langle g(v), v \right\rangle &= 2\|v\|^2 - 2\left\langle u^n, v \right\rangle + 2\left\|v_{xx}\right\|^2 - 2\left\langle u^n_{xx}, v_{xx} \right\rangle \\ &\ge 2\|v\|^2 - 2\left\|u^n\right\| \cdot \|v\| + 2\left\|v_{xx}\right\|^2 \\ &\quad - 2\left\|u^n_{xx}\right\| \cdot \left\|v_{xx}\right\| \\ &\ge 2\|v\|^2 - \left(\left\|u^n\right\|^2 + \|v\|^2\right) + 2\left\|v_{xx}\right\|^2 \\ &\quad - \left(\left\|u^n_{xx}\right\|^2 + \left\|v_{xx}\right\|^2\right) \\ &\ge \|v\|^2 - \left(\left\|u^n\right\|^2 + \left\|u^n_{xx}\right\|^2\right) + \left\|v_{xx}\right\|^2 \\ &\ge \|v\|^2 - \left(\left\|u^n\right\|^2 + \left\|u^n_{xx}\right\|^2\right).\end{aligned} \tag{16}$$

Hence, it is obvious that $\langle g(v), v\rangle > 0$ for all $\forall v \in Z_h^0$ with $\|v\|^2 = \|u^n\|^2 + \|u_{xx}^n\|^2 + 1$. It follows from Lemma 2 that there exists $v^* \in Z_h^0$ such that $g(v^*) = 0$. Let $u^{n+1} = 2v^* - u^n$; then u^{n+1} satisfies (7). □

The difference scheme (7)–(10) simulates the conservation property of the problem (1)–(3) as follows.

Theorem 4. *Suppose that $u_0 \in H_0^2[x_L, x_R]$, then the difference scheme* (7)–(10) *is conservative:*

$$E^n = \|u^n\|^2 + \|u_{xx}^n\|^2 = E^{n-1} = \cdots = E^0. \quad (17)$$

Proof. Taking an inner product of (7) with $2u^{n+(1/2)}$(i.e., $u^{n+1} + u^n$), according to the boundary condition (10) and Lemma 1, we obtain

$$\begin{aligned} &\frac{1}{\tau}\left(\left\|u^{n+1}\right\|^2 - \left\|u^n\right\|^2\right) + \frac{1}{\tau}\left(\left\|u_{xx}^{n+1}\right\|^2 - \left\|u_{xx}^n\right\|^2\right) \\ &\quad + 2\left\langle u_{\hat{x}}^{n+(1/2)}, u^{n+(1/2)}\right\rangle + 2\left\langle u_{x\bar{x}\hat{x}}^{n+(1/2)}, u^{n+(1/2)}\right\rangle \\ &\quad + 2\left\langle \varphi\left(u^{n+(1/2)}\right), u^{n+(1/2)}\right\rangle = 0, \end{aligned} \quad (18)$$

where $\varphi(u_j^{n+(1/2)}) = (p/(p+1))[(u_j^{n+(1/2)})^{p-1} \cdot (u_j^{n+(1/2)})_{\hat{x}} + ((u_j^{n+(1/2)})^p)_{\hat{x}}]$.

From

$$\begin{aligned} &\left\langle u_{\hat{x}}^{n+(1/2)}, u^{n+(1/2)}\right\rangle = 0, \\ &\left\langle u_{x\bar{x}\hat{x}}^{n+(1/2)}, u^{n+(1/2)}\right\rangle = 0, \end{aligned} \quad (19)$$

$$\begin{aligned} &\left\langle \varphi\left(u^{n+(1/2)}\right), 2u^{n+(1/2)}\right\rangle \\ &= \frac{2p}{p+1} h \sum_{j=1}^{J-1} \left[\left(u_j^{n+(1/2)}\right)^{p-1} \cdot \left(u_j^{n+(1/2)}\right)_{\hat{x}}\right. \\ &\qquad \left. + u_j^{n+(1/2)} + \left(\left(u_j^{n+(1/2)}\right)^p\right)_{\hat{x}}\right] u_j^{n+(1/2)} \\ &= \frac{p}{p+1} h \sum_{j=1}^{J-1} \left(u_j^{n+(1/2)}\right)^{p-1} \cdot \left(u_{j+1}^{n+(1/2)} - u_{j-1}^{n+(1/2)}\right) \\ &\quad + \left[\left(u_{j+1}^{n+(1/2)}\right)^p - \left(u_{j-1}^{n+(1/2)}\right)^p\right] u_j^{n+(1/2)} \\ &= \frac{p}{p+1} \sum_{j=1}^{J-1} \left[\left(u_{j+1}^{n+(1/2)}\right)^{p-1} u_j^{n+(1/2)} + \left(u_j^{n+(1/2)}\right)^p\right] \\ &\quad \cdot u_{j+1}^{n+(1/2)} - \frac{p}{p+1} \sum_{j=1}^{J-1} \left[\left(u_j^{n+(1/2)}\right)^{p-1} u_{j-1}^{n+(1/2)} + \left(u_{j-1}^{n+(1/2)}\right)^p\right] \\ &\quad \cdot u_j^{n+(1/2)} = 0, \end{aligned} \quad (20)$$

we have

$$\left(\left\|u^{n+1}\right\|^2 - \left\|u^n\right\|^2\right) + \left(\left\|u_{xx}^{n+1}\right\|^2 - \left\|u_{xx}^n\right\|^2\right) = 0. \quad (21)$$

Then (17) is gotten from (21). □

In order to prove the bounded quality of the difference solution, we introduce the following lemma.

Lemma 5 (discrete Sobolev's inequality [2]). *There exist two constants C_1 and C_2 such that*

$$\left\|u^n\right\|_\infty \le C_1 \left\|u^n\right\| + C_2 \left\|u_x^n\right\|. \quad (22)$$

Theorem 6. *Suppose $u_0 \in H_0^2[x_L, x_R]$; then the solution of* (7)–(10) *satisfies*

$$\left\|u^n\right\| \le C, \qquad \left\|u_x^n\right\| \le C, \quad (23)$$

which yield

$$\left\|u^n\right\|_\infty \le C, \quad \left\|u_x^n\right\|_\infty \le C \quad (n = 1, 2, \ldots, N). \quad (24)$$

Proof. It follows from (17) that

$$\left\|u^n\right\| \le C, \qquad \left\|u_{xx}^n\right\| \le C. \quad (25)$$

According to (12) and Schwarz inequality, we get

$$\left\|u_x^n\right\|^2 \le \left\|u^n\right\| \cdot \left\|u_{xx}^n\right\| \le \frac{1}{2}\left(\left\|u^n\right\|^2 + \left\|u_{xx}^n\right\|^2\right) \le C. \quad (26)$$

Using Lemma 5, we have

$$\left\|u^n\right\|_\infty \le C, \qquad \left\|u_x^n\right\|_\infty \le C. \quad (27)$$

□

Remark 7. Theorem 6 implies that the solution of difference scheme (7)–(10) is stable in the sense of norm $\|\cdot\|_\infty$.

3. Convergence

In order to prove the convergence of the difference scheme, we need to introduce the lemma as follows:

Lemma 8 (discrete Gronwall inequality [2]). *Suppose $w(k)$ and $\rho(k)$ are nonnegative functions and $\rho(k)$ is nondecreasing. If $C > 0$ and*

$$w(k) \le \rho(k) + C\tau \sum_{l=0}^{k-1} w(l), \quad \forall k, \quad (28)$$

then

$$w(k) \le \rho(k) e^{C\tau k}, \quad \forall k. \quad (29)$$

Theorem 9. *Suppose that $u_0 \in H_0^2[x_L, x_R]$; then the solution u^n of* (1)–(3) *satisfies*

$$\begin{aligned} &\|u\|_{L_2} \le C, \qquad \left\|u_x\right\|_{L_2} \le C, \\ &\|u\|_{L_\infty} \le C, \qquad \left\|u_x\right\|_{L_\infty} \le C. \end{aligned} \quad (30)$$

Proof. It follows from (5) that

$$\|u\|_{L_2} \le C, \qquad \left\|u_{xx}\right\|_{L_2} \le C. \quad (31)$$

By Holder inequality and Schwarz inequality, we get

$$\begin{aligned}\|u_x\|_{L_2}^2 &= \int_{x_L}^{x_R} u_x u_x dx = uu_x\big|_{x_L}^{x_R} - \int_{x_L}^{x_R} uu_{xx}dx \\ &= -\int_{x_L}^{x_R} uu_{xx}dx \\ &\le \|u\|_{L_2}\cdot\|u_{xx}\|_{L_2} \le \frac{1}{2}\left(\|u\|_{L_2}^2 + \|u_{xx}\|_{L_2}^2\right),\end{aligned} \tag{32}$$

which implies

$$\|u_x\|_{L_2} \le C. \tag{33}$$

Using Sobolev inequality, we get

$$\|u\|_{L_\infty} \le C, \qquad \|u_x\|_{L_\infty} \le C. \tag{34}$$

□

Let $v(x,t)$ be the solution of problem (1)–(3), $v_j^n = v(x_j, t_n)$; then the truncation error of the difference scheme (7)–(10) is

$$\begin{aligned} r_j^n = &\left(v_j^n\right)_t + \left(v_j^n\right)_{xx\overline{x}\overline{x}t} + \left(v_j^{n+(1/2)}\right)_{x\hat{x}} \\ &+ \left(v_j^{n+(1/2)}\right)_{x\overline{x}\hat{x}} + \varphi\left(v_j^{n+(1/2)}\right).\end{aligned} \tag{35}$$

Making use of Taylor expansion, we know that $r_j^n = O(\tau^2+h^2)$ holds if $h, \tau \to 0$.

Theorem 10. *Suppose $u_0 \in H_0^2[x_L, x_R]$; then the solution u^n of the difference scheme (7)–(10) converges to the solution $v(x,t)$ of the problem (1)–(3) with order $O(\tau^2 + h^2)$ in norm $\|\cdot\|_\infty$.*

Proof. Subtracting (7) from (35) and letting $e_j^n = v_j^n - u_j^n$, we have

$$\begin{aligned} r_j^n = &\left(e_j^n\right)_t + \left(e_j^n\right)_{xx\overline{x}\overline{x}t} + \left(e_j^{n+(1/2)}\right)_{\hat{x}} + \left(e_j^{n+(1/2)}\right)_{x\overline{x}\hat{x}} \\ &+\varphi\left(v_j^{n+(1/2)}\right) - \varphi\left(u_j^{n+(1/2)}\right).\end{aligned} \tag{36}$$

Computing the inner product of (36) with $2e^{n+(1/2)}$, we obtain

$$\begin{aligned}\left\langle r^n, 2e^{n+(1/2)}\right\rangle = &\frac{1}{\tau}\left(\left\|e^{n+1}\right\|^2 - \left\|e^n\right\|^2\right) \\ &+ \frac{1}{\tau}\left(\left\|e_{xx}^{n+1}\right\|^2 - \left\|e_{xx}^n\right\|^2\right) + \left\langle e_{\hat{x}}^{n+1}, 2e^{n+(1/2)}\right\rangle \\ &+ \left\langle e_{x\overline{x}\hat{x}}^{n+(1/2)}, 2e^{n+(1/2)}\right\rangle \\ &+ \left\langle \varphi\left(v^{n+(1/2)}\right) - \varphi\left(u^{n+(1/2)}\right), 2e^{n+(1/2)}\right\rangle.\end{aligned} \tag{37}$$

Similar to the proof of (19), we have

$$\begin{aligned}\left\langle e_{\hat{x}}^{\mathrm{n}+1}, 2e^{n+(1/2)}\right\rangle &= 0, \\ \left\langle e_{x\overline{x}\hat{x}}^{n+(1/2)}, 2e^{n+(1/2)}\right\rangle &= 0.\end{aligned} \tag{38}$$

This indicates

$$\begin{aligned}&\left(\left\|e^{n+1}\right\|^2 - \left\|e^n\right\|^2\right) + \left(\left\|e_{xx}^{n+1}\right\|^2 - \left\|e_{xx}^n\right\|^2\right) \\ &\quad = \tau\left\langle r^n, 2e^{n+(1/2)}\right\rangle - \tau\left\langle Q_1 + Q_2, 2e^{n+1/2}\right\rangle,\end{aligned} \tag{39}$$

where

$$\begin{aligned} Q_1 = &\frac{ph}{p+1}\sum_{j=1}^{J-1}\left[\left(v_j^{n+(1/2)}\right)^{p-1}\cdot\left(v_j^{n+(1/2)}\right)_{\hat{x}}\right. \\ &\left. - \left(u_j^{n+(1/2)}\right)^{p-1}\cdot\left(u_j^{n+(1/2)}\right)_{\hat{x}}\right], \\ Q_2 = &\frac{ph}{p+1}\sum_{j=1}^{J-1}\left[\left(\left(v_j^{n+(1/2)}\right)^p\right)_{\hat{x}} - \left(\left(u_j^{n+(1/2)}\right)^p\right)_{\hat{x}}\right].\end{aligned} \tag{40}$$

Noting that

$$\begin{aligned}&\left\langle Q_1, 2e^{n+(1/2)}\right\rangle \\ &= -\frac{2p}{1+p}h\sum_{j=0}^{J-1}\left[\left(v_j^{n+(1/2)}\right)^{p-1}\cdot\left(v_j^{n+(1/2)}\right)_{\hat{x}}\right. \\ &\qquad\left. -\left(u_j^{n+(1/2)}\right)^{p-1}\cdot\left(u_j^{n+(1/2)}\right)_{\hat{x}}\right]e_j^{n+(1/2)} \\ &= -\frac{2p}{1+p}h\sum_{j=0}^{J-1}\left[\left(v_j^{n+(1/2)}\right)^{p-1}\cdot\left(e_j^{n+(1/2)}\right)_{\hat{x}}e_j^{n+(1/2)}\right] \\ &\quad - \frac{2p}{1+p}h\sum_{j=0}^{J-1}\left[\left(v_j^{n+(1/2)}\right)^{p-1} - \left(u_j^{n+(1/2)}\right)^{p-1}\right] \\ &\quad \times\left(u_j^{n+(1/2)}\right)_{\hat{x}}e_j^{n+(1/2)} \\ &= -\frac{2p}{1+p}h\sum_{j=0}^{J-1}\left(v_j^{n+(1/2)}\right)^{p-1}\cdot\left(e_j^{n+(1/2)}\right)_{\hat{x}} \\ &\qquad \times e_j^{n+(1/2)} - \frac{2p}{1+p}h\sum_{j=0}^{J-1}e_j^{n+(1/2)} \\ &\qquad \times\sum_{k=0}^{p-2}\left[\left(v_j^{n+(1/2)}\right)^{p-2-k}\left(u_j^{n+(1/2)}\right)^k\right] \\ &\qquad \times\left(u_j^{n+(1/2)}\right)_{\hat{x}}e_j^{n+(1/2)} \\ &\le Ch\sum_{j=0}^{J-1}\left|\left(e_j^{n+(1/2)}\right)_{\hat{x}}\right|\cdot\left|e_j^{n+(1/2)}\right| + Ch\sum_{j=0}^{J-1}\left|\left(e_j^{n+(1/2)}\right)^2\right| \\ &\le C\left(\left\|e_x^{n+(1/2)}\right\|^2 + \left\|e^{n+(1/2)}\right\|^2\right) \\ &\le C\left(\left\|e_x^{n+1}\right\|^2 + \left\|e_x^n\right\|^2 + \left\|e^{n+1}\right\|^2 + \left\|e^n\right\|^2\right), \\ &\left\langle Q_2, 2e^{n+(1/2)}\right\rangle \\ &= -\frac{2p}{1+p}h\sum_{j=0}^{J-1}\left[\left(\left(v_j^{n+(1/2)}\right)^p\right)_{\hat{x}} - \left(\left(u_j^{n+(1/2)}\right)^p\right)_{\hat{x}}\right]e_j^{n+(1/2)}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2p}{1+p} h \sum_{j=0}^{J-1} \left[\left(v_j^{n+(1/2)} \right)^p - \left(u_j^{n+(1/2)} \right)^p \right] \left(e_j^{n+(1/2)} \right)_{\hat{x}} \\
&= \frac{2p}{1+p} h \sum_{j=0}^{J-1} \left[\sum_{k=0}^{p-1} \left(v_j^{n+(1/2)} \right)^{p-1-k} \left(u_j^{n+(1/2)} \right)^k \right] \\
&\quad \times \left(u_j^{n+(1/2)} \right)_{\hat{x}} e_j^{n+(1/2)} \\
&\le C \left(\left\| e_x^{n+(1/2)} \right\|^2 + \left\| e^{n+(1/2)} \right\|^2 \right) \\
&\le C \left(\left\| e_x^{n+1} \right\|^2 + \left\| e_x^n \right\|^2 + \left\| e^{n+1} \right\|^2 + \left\| e^n \right\|^2 \right)
\end{aligned} \tag{41}
$$

and with

$$
\begin{aligned}
\left\langle r^n, 2e^{n+(1/2)} \right\rangle &= \left\langle r^n, e^{n+1} + e^n \right\rangle \\
&\le \left\| r^n \right\|^2 + \frac{1}{2} \left(\left\| e^{n+1} \right\|^2 + \left\| e^n \right\|^2 \right),
\end{aligned} \tag{42}
$$

we have

$$
\begin{aligned}
&\left(\left\| e^{n+1} \right\|^2 - \left\| e^n \right\|^2 \right) + \left(\left\| e_{xx}^{n+1} \right\|^2 - \left\| e_{xx}^n \right\|^2 \right) \\
&\quad \le C\tau \left(\left\| e^{n+1} \right\|^2 + \left\| e^n \right\|^2 + \left\| e_x^{n+1} \right\|^2 + \left\| e_x^n \right\|^2 \right) + \tau \left\| r^n \right\|^2.
\end{aligned} \tag{43}
$$

Similar to the proof of Theorem 6, we have

$$
\begin{aligned}
\left\| e_x^{n+1} \right\|^2 &\le \frac{1}{2} \left(\left\| e^{n+1} \right\|^2 + \left\| e_{xx}^{n+1} \right\|^2 \right); \\
\left\| e_x^n \right\|^2 &\le \frac{1}{2} \left(\left\| e^n \right\|^2 + \left\| e_{xx}^n \right\|^2 \right).
\end{aligned} \tag{44}
$$

This yields

$$
\begin{aligned}
&\left\| e^{n+1} \right\|^2 + \left\| e_{xx}^{n+1} \right\|^2 - \left(\left\| e^n \right\|^2 + \left\| e_{xx}^n \right\|^2 \right) \\
&\quad \le C\tau \left(\left\| e^{n+1} \right\|^2 + \left\| e^n \right\|^2 + \left\| e_{xx}^{n+1} \right\|^2 + \left\| e_{xx}^n \right\|^2 \right) + \tau \left\| r^n \right\|^2.
\end{aligned} \tag{45}
$$

Let $B^n = \|e^n\|^2 + \|e_{xx}^n\|^2$. We claim that

$$
B^{n+1} - B^n \le C\tau \left(B^{n+1} + B^n \right) + \tau \left\| r^n \right\|^2, \tag{46}
$$

which yields

$$
(1 - C\tau) \left(\mathrm{B}^{n+1} - B^n \right) \le 2C\tau B^n + \tau \left\| r^n \right\|^2. \tag{47}
$$

If τ is sufficiently small which satisfies $1 - C\tau > 0$, then

$$
B^{n+1} - B^n \le C\tau B^n + C\tau \left\| r^n \right\|^2. \tag{48}
$$

Summing up (48) from 0 to $n-1$, we have

$$
B^n \le B^0 + C\tau \sum_{l=0}^{n-1} \left\| r^l \right\|^2 + C\tau \sum_{l=0}^{n-1} B^l. \tag{49}
$$

Since

$$
\tau \sum_{l=0}^{n-1} \left\| r^l \right\|^2 \le n\tau \max_{0 \le l \le n-1} \left\| r^l \right\|^2 \le T \cdot O\left(\tau^2 + h^2 \right)^2 \tag{50}
$$

and $B^0 = O(\tau^2 + h^2)^2$, we obtain

$$
B^n \le O\left(\tau^2 + h^2 \right)^2 + C\tau \sum_{l=0}^{n-1} B^l. \tag{51}
$$

By Lemma 8, we get

$$
B^n \le O\left(\tau^2 + h^2 \right)^2, \tag{52}
$$

which implies

$$
\left\| e^n \right\| \le O\left(\tau^2 + h^2 \right), \qquad \left\| e_{xx}^n \right\| \le O\left(\tau^2 + h^2 \right). \tag{53}
$$

From (44), we have

$$
\left\| e_x^n \right\| \le O\left(\tau^2 + h^2 \right). \tag{54}
$$

By Lemma 5 we get

$$
\left\| e^n \right\|_\infty \le O\left(\tau^2 + h^2 \right). \tag{55}
$$

□

Finally, we can similarly prove the results as follows.

Theorem 11. *The solution u^n of* (7)–(10) *is unique.*

4. Numerical Simulations

The difference scheme (7)–(10) is a nonlinear system about u_j^{n+1} that can be easily solved by the Newton iterative algorithm.

Let $x_L = -60, x_R = 90$, and $T = 40$. According to [3, 4], when $p = 3$, the soliton solution is as follows:

$$
\begin{aligned}
u(x,t) &= \frac{1}{4} \sqrt{-15 + 3\sqrt{41}} \operatorname{sech}^2 \frac{1}{4} \\
&\quad \times \sqrt{\frac{-5 + \sqrt{41}}{2}} \left[x - \frac{1}{10} \left(5 + \sqrt{41} \right) t \right],
\end{aligned} \tag{56}
$$

and the initial condition is

$$
u_0(x) = \frac{1}{4} \sqrt{-15 + 3\sqrt{41}} \operatorname{sech}^2 \frac{1}{4} \sqrt{\frac{-5 + \sqrt{41}}{2}} x. \tag{57}
$$

When $p = 5$, the soliton solution is as follows:

$$
\begin{aligned}
u(x,t) &= \sqrt[4]{\frac{4}{15} \left(-5 + \sqrt{34} \right)} \operatorname{sech} \frac{1}{3} \\
&\quad \times \sqrt{-5 + \sqrt{34}} \left[x - \frac{1}{10} \left(5 + \sqrt{34} \right) t \right],
\end{aligned} \tag{58}
$$

Table 1: The error at various time step.

	$\tau = h = 0.25$		$\tau = h = 0.125$		$\tau = h = 0.0625$	
	$p = 3$	$p = 5$	$p = 3$	$p = 5$	$p = 3$	$p = 5$
$t = 10$	$2.53343e-3$	$3.43739e-3$	$6.35973e-4$	$8.64177e-4$	$1.59182e-4$	$2.16316e-4$
$t = 20$	$4.40914e-3$	$6.31248e-3$	$1.10927e-3$	$1.58976e-3$	$2.77703e-4$	$3.98160e-4$
$t = 30$	$6.03109e-3$	$9.13274e-3$	$1.51828e-3$	$2.30402e-3$	$3.80216e-4$	$5.77288e-4$
$t = 40$	$7.53941e-3$	$1.20204e-2$	$1.89987e-3$	$3.03743e-3$	$4.75848e-4$	$7.61418e-4$

Table 2: The error comparison when $t = 40$.

	$\tau = h = 0.25$		$\tau = h = 0.125$		$\tau = h = 0.0625$	
	$p = 3$	$p = 5$	$p = 3$	$p = 5$	$p = 3$	$p = 5$
Scheme I	$7.53941e-3$	$1.20204e-2$	$1.89987e-3$	$3.03743e-3$	$4.75848e-4$	$7.61418e-4$
Scheme II	$1.34986e-2$	$1.79985e-2$	$3.42489e-3$	$4.56804e-3$	$8.59570e-4$	$1.14689e-3$

Table 3: The verification of the second convergence.

	$\Vert e^{n}(h,\tau)\Vert / \Vert e^{2n}(h/2,\tau/2)\Vert$					
		$p = 3$			$p = 5$	
	$\tau = h = 0.1$	$\tau = h = 0.05$	$\tau = h = 0.025$	$\tau = h = 0.1$	$\tau = h = 0.05$	$\tau = h = 0.025$
$t = 10$	—	3.98355	3.99527	—	3.97764	3.99497
$t = 20$	—	3.97483	3.99443	—	3.97073	3.99275
$t = 30$	—	3.97233	3.99319	—	3.96382	3.99112
$t = 40$	—	3.96837	3.99261	—	3.95742	3.98917

Table 4: Numerical simulations on conservation invariant E^n.

	$\tau = h = 0.25$		$\tau = h = 0.125$		$\tau = h = 0.0625$	
	$p = 3$	$p = 5$	$p = 3$	$p = 5$	$p = 3$	$p = 5$
$t = 0$	1.68252899330	3.11067490241	1.68254308255	3.11070293879	1.68254661109	3.11070996431
$t = 10$	1.68252899329	3.11067490241	1.68254308255	3.11070293879	1.68254661108	3.11070996430
$t = 20$	1.68252899328	3.11067490240	1.68254308255	3.11070293879	1.68254661095	3.11070996426
$t = 30$	1.68252899327	3.11067490240	1.68254308255	3.11070293879	1.68254661102	3.11070996417
$t = 40$	1.68252899325	3.11067490240	1.68254308254	3.11070293879	1.68254661095	3.11070996435

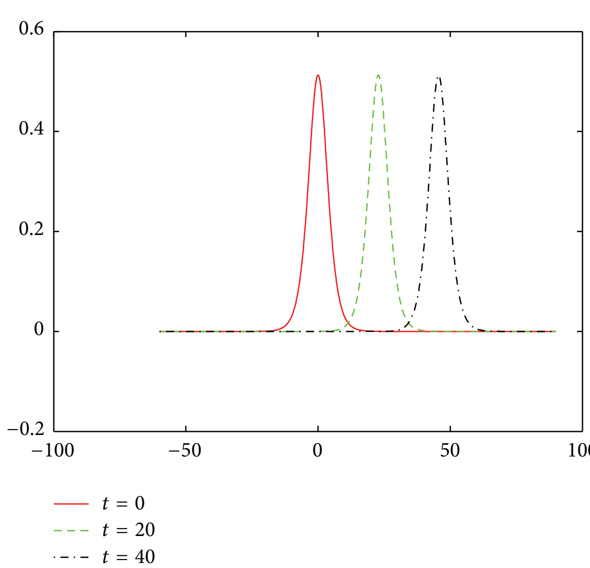

Figure 1: When $p = 3$ and $\tau = h = 0.125$, the wave graph of $u(x,t)$ at various times.

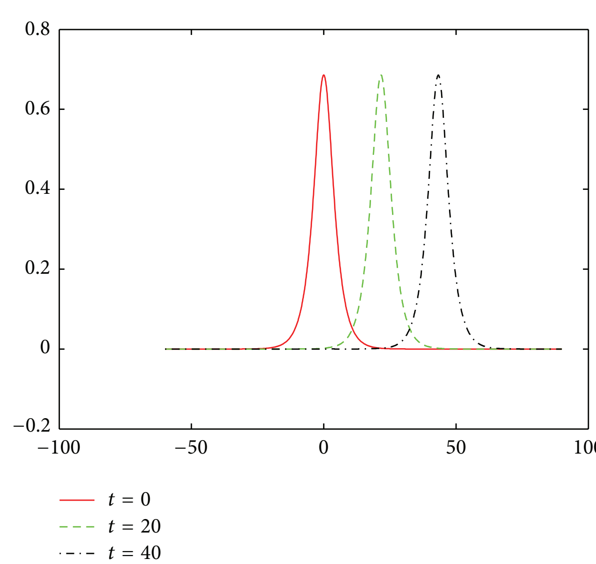

Figure 2: When $p = 5$ and $\tau = h = 0.125$, the wave graph of $u(x,t)$ at various times.

and the initial condition is

$$u_0(x) = \sqrt[4]{\frac{4}{15}\left(-5+\sqrt{34}\right)}\operatorname{sech}\frac{1}{3}\sqrt{-5+\sqrt{34}}x. \tag{59}$$

In Table 1 we give the error at various time step. We denote the C-N scheme in this paper as scheme I and the difference scheme in [7] as scheme II. In Table 2 we give the error comparison between scheme I and scheme II. It is easy to see that the calculation results of scheme I are slightly better than scheme II. Using the method in [20, 21], we verified the second convergence of the difference scheme in Table 3. Numerical simulations on the conservation invariant E^n are given in Table 4.

The wave graph comparison of $u(x,t)$ at various times is given in Figures 1 and 2 when $p = 3$ and $p = 5$.

5. Conclusions

In this paper, we propose a conservative Crank-Nicolson finite difference scheme for the initial-boundary value problem of the generalized Rosenau-KdV equation. The two-level finite difference scheme is of second order convergence and unconditionally stable, which can start by itself. From Table 2 we conclude that the C-N scheme is more efficient than scheme II in [7]. From Table 3 we conclude that the C-N scheme is of second order convergence obviously. Numerical simulations on the conservation invariant E^n are given in Table 4. Figures 1 and 2 show that the height of the wave graph at different time is almost identical. Table 4 and Figures 1 and 2 imply that the finite difference scheme is conservative and efficient.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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