Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2015, Article ID 301928, 9 pages
http://dx.doi.org/10.1155/2015/301928

Research Article

On the Deformation Retract of Eguchi-Hanson Space and Its Folding

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Received 20 December 2014; Revised 9 February 2015; Accepted 9 February 2015

Academic Editor: Ivan Avramidi

We introduce the deformation retract of the Eguchi-Hanson space using Lagrangian equations. The retraction of this space into itself and into geodesics has been presented. The deformation retract of the Eguchi-Hanson space into itself and after the isometric folding has been discussed. Theorems concerning these relations have been deduced.

1. Introduction

The real revolution in mathematical physics in the second half of twentieth century (and in pure mathematics itself) was algebraic topology and algebraic geometry [1]. In the nineteenth century, mathematical physics was essentially the classical theory of ordinary and partial differential equations. The variational calculus, as a basic tool for physicists in theoretical mechanics, was seen with great reservation by mathematicians until Hilbert set up its rigorous foundation by pushing forward functional analysis. This marked the transition into the first half of twentieth century, where, under the influence of quantum mechanics and relativity, mathematical physics turned mainly into functional analysis, complemented by the theory of Lie groups and by tensor analysis. All branches of theoretical physics still can expect the strongest impacts from use of the unprecedented wealth of results of algebraic topology and algebraic geometry of the second half of twentieth century [1].

Today, the concepts and methods of topology and geometry have become an indispensable part of theoretical physics. They have led to a deeper understanding of many crucial aspects in condensed matter physics, cosmology, gravity, and particle physics. Moreover, several intriguing connections between only apparently disconnected phenomena have been revealed based on these mathematical tools [2].

Topology enters general relativity through the fundamental assumption that spacetime exists and is organized as a manifold. This means that spacetime has a well-defined dimension, but it also carries with it the inherent possibility of modified patterns of global connectivity, such as distinguishing a sphere from a plane or a torus from a surface of higher genus. Such modifications can be present in the spatial topology without affecting the time direction, but they can also have a genuinely spacetime character in which case the spatial topology changes with time [3]. The topology change in classical general relativity has been discussed in [4]. See [5] for some applications of differential topology in general relativity.

In general relativity, boundaries that are S^1-bundles over some compact manifolds arise in gravitational thermodynamics [6]. The trivial bundle $\sum = S^1 \times S^2$ is a classic example. Manifolds with complete Ricci-flat metrics admitting such boundaries are known; they are the Euclideanised Schwarzschild metric and the flat metric with periodic identification. York [7] shows that there are in general two or no Schwarzschild solutions depending on whether the squashing (the ratio of the radius of the S^1-fibre to that of the S^2-base) is below or above a critical value. York's results in 4 dimensions extend readily to higher dimensions.

The simplest example of nontrivial bundles arises in quantum cosmology in which the boundary is a compact S^3,

that is, a nontrivial S^1 bundle over S^2. In the case of zero cosmological constant, regular 4 metrics admitting such an S^3 boundary are the Taub-Nut [8] and Taub-Bolt [9] metrics having zero- and two-dimensional (regular) fixed point sets of the $U(1)$ action, respectively.

The four-dimensional Riemannian manifolds for gravitational instantons can be asymptotically flat, asymptotically locally Euclidean, asymptotically locally flat, or compact without boundary [10]. Hawking's interpretation of the Taub-NUT solution [8] is an example of asymptotically locally flat space. The simplest nontrivial example of asymptotically locally Euclidean spaces is the metric of Eguchi-Hanson [11, 12]:

$$\begin{aligned} ds^2 &= \left(1-\frac{a^4}{R^4}\right)^{-1} dR^2 \\ &+ \frac{1}{4}R^2\left(1-\frac{a^4}{R^4}\right)(d\psi+\cos\theta d\phi)^2 \\ &+ \frac{1}{4}R^2\left(d\theta^2+\sin^2\theta d\phi^2\right), \end{aligned} \tag{1}$$

where a is a constant of integration. This Eguchi-Hanson metric is self-dual.

In order to remove the apparent singularity, we take ψ to have period 2π. For large values of r, the metric tends towards Euclidean flat space. For surfaces of constant $r > a$, the topology is that of RP^3. The surface at $r = a$ is a 2-sphere. In other words, since ψ has a period of 2π, the level surfaces are S^3/Z_2, that is, RP^3, and hence the metric is asymptotically locally Euclidean and asymptotically looks like R^4/Z_2. The complete metric has the topology of $T^*(CP^1)$ [6]. The $(2n+n)$-dimensional metric has the succinct form

$$\begin{aligned} ds^2 &= \left(1-\frac{a^{2n+2}}{R^{2n+2}}\right)^{-1} dR^2 \\ &+ R^2\left(1-\frac{a^{2n+2}}{R^{2n+2}}\right)(d\psi+A)^2 + R^2 ds_M^2, \end{aligned} \tag{2}$$

where the bolt at $r = a$ is regular with ψ having a period of $2\pi/(n+1)$. The squashing

$$\frac{\beta^2}{\alpha^2} = 1-\left(\frac{1}{\rho}\right)^{2n+2} \tag{3}$$

with $\rho \equiv r/a \in [0,\infty)$ increases monotonically from zero and approaches unity as $\rho \to \infty$.

2. Deformation Retract

2.1. Deformation Retract: Definitions. The theory of deformation retract is very interesting topic in Euclidean and non-Euclidean spaces. It has been investigated from different points of view in many branches of topology and differential geometry. A retraction is a continuous mapping from the entire space into a subspace which preserves the position of all points in that subspace [13].

(i) Let M and N be two smooth manifolds of dimensions m and n, respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if and only if, for every piecewise geodesic path $\gamma : J \to M$, the induced path $fo\gamma : J \to N$ is a piecewise geodesic and of the same length as γ [14]. If f does not preserve the lengths, it is called topological folding. Many types of folding are discussed in [15, 16]. Some applications are discussed in [17–19].

(ii) A subset A of a topological space X is called a retract of X, if there exists a continuous map $r : X \to A$ such that [20]

  (a) X is open;
  (b) $r(a) = a, \forall a \in A$.

(iii) A subset A of a topological space X is said to be a deformation retract if there exists a retraction $r : X \to A$ and a homotopy $f : X \times I \to X$ such that [20]
  $f(x, 0) = x, \forall x \in X,$
  $f(x, 1) = r(x), \forall r \in X,$
  $f(a, t) = a, \forall a \in A, t \in [0, 1].$

The deformation retract is a particular case of homotopy equivalence, and two spaces are homotopy equivalent if and only if they are both deformation retracts of a single larger space.

Deformation retracts of Stein spaces have been studied in [21]. The deformation retract of the 4D Schwarzchild metric has been discussed in [22] where it was found that the retraction of the Schwarzchild space is spacetime geodesic. The deformation retract of Kerr spacetime and its folding has been discussed in [23]. In this paper we are going to discuss the retraction for the Eguchi-Hanson space.

Most of the studies on deformation retract and folding, if not all, are pure mathematical studies. The authors believe that these two concepts should be given more attention in modern mathematical physics. Topological studies of some famous metrics of mathematical physics could be a nice topological exploration start.

3. Eguchi-Hanson Metric

A four-dimensional flat metric can be written as

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2. \tag{4}$$

So the coordinates of the four-dimensional Eguchi-Hanson space (1) can be written as

$$\begin{aligned} x_1 = e\Bigg(& a^2 + R^2 - aR\tan^{-1}\left(\frac{R}{a}\right) \\ &+ \frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right) + \frac{a}{2}\ln(R-a)R \\ &- \frac{a^2}{2}\ln(R-a) - \frac{a}{2}\ln(R+a)R \end{aligned}$$

$$-\frac{a^2}{2}\ln(R+a)+C_1\Bigg)^{1/2},$$

$$x_2 = e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi + C_2},$$

$$x_3 = e\sqrt{\frac{1}{2}R^2\theta + C_3},$$

$$x_4 = e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta + \left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi + C_4}, \tag{5}$$

where C_1, C_2, C_3, and C_4 are constants of integration. Also $e = 1$ or -1.

4. Using Euler-Lagrange Equation

In general relativity, the geodesic equation is equivalent to the Euler-Lagrange equations

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0, \quad i = 1,2,3,4 \tag{6}$$

associated with the Lagrangian

$$L\left(x^\mu, \dot{x}^\mu\right) = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu. \tag{7}$$

To find a geodesic which is a subset of the 6D Schwarzchild space, the Lagrangian could be written as

$$\begin{aligned} L = {} & \left(1-\frac{a^4}{R^4}\right)^{-1}\dot{R}^2 + \frac{1}{4}R^2\left(1-\frac{a^4}{R^4}\right)\dot{\psi}^2 \\ & + \frac{1}{4}R^2\dot{\theta}^2 + \frac{1}{4}R^2\left[\sin^2\theta + \left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\dot{\phi}^2 \\ & + \frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\cos\theta\dot{\psi}\dot{\phi}. \end{aligned} \tag{8}$$

since there is no explicit dependence on ϕ, or ψ, $\partial L/\partial\dot{\phi}$ and $\partial L/\partial\dot{\psi}$ could be considered as constants of motion; thus

$$\begin{aligned} & \frac{1}{2}R^2\left[\sin^2\theta + \left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\dot{\phi} \\ & \quad + \frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\cos\theta\dot{\psi} = h, \\ & \frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\left[\dot{\psi} + \cos\theta\dot{\phi}\right] = k. \end{aligned} \tag{9}$$

The R-component gives

$$\begin{aligned} & \frac{d}{d\lambda}2\dot{R}\left(1-\frac{a^4}{R^4}\right)^{-1} \\ & - \Bigg[\frac{-4a^4}{\left(1-a^4/R^4\right)^2 R^5}\dot{R}^2 \\ & + \left(\frac{1}{2}R\left(1-\frac{a^4}{R^4}\right) + \frac{a^4}{R^3}\right)\dot{\psi}^2 + \frac{1}{2}R\dot{\theta}^2 \\ & + \left(\frac{1}{2}R\left(\sin^2\theta + \left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right) + \frac{a^4}{R^3}\cos^2\theta\right)\dot{\phi}^2 \\ & + \left(R\left(1-\frac{a^4}{R^4}\right)\cos^2\theta + \frac{2a^4}{R^3}\cos^2\theta\right)\dot{\psi}\dot{\phi}\Bigg] = 0. \end{aligned} \tag{10}$$

The θ-component gives

$$\frac{d}{d\lambda}\left(R^2\dot{\theta}\right) - \frac{1}{2}\dot{\phi}^2\sin 2\theta\left(\frac{a^4}{R^2}\right) + R^2\sin\theta\left(1-\frac{a^4}{R^4}\right)\dot{\psi}\dot{\phi} = 0. \tag{11}$$

For $\theta = \pi/2$, (9), (10), and (11) reduce to

$$\begin{aligned} & \frac{1}{2}R^2\dot{\phi} = h, \\ & \frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\dot{\psi} = k, \\ & \frac{d}{d\lambda}2\dot{R}\left(1-\frac{a^4}{R^4}\right)^{-1} \\ & - \Bigg[\frac{-4a^4}{\left(1-a^4/R^4\right)^2 R^5}\dot{R}^2 \\ & + \left(\frac{1}{2}R\left(1-\frac{a^4}{R^4}\right) + \frac{a^4}{R^3}\right)\dot{\psi}^2 + \frac{1}{2}R\dot{\phi}^2\Bigg] = 0, \\ & -\frac{1}{2}\dot{\phi}^2\left(\frac{a^4}{R^2}\right) + R^2\left(1-\frac{a^4}{R^4}\right)\dot{\psi}\dot{\phi} = 0. \end{aligned} \tag{12}$$

From $(1/2)R^2(1-a^4/R^4)\dot{\psi} = k$, if $k = 0$ then $\dot{\psi} = 0$ or $\psi = A$; if $A = 0$ we get the following coordinates:

$$x_1 = e\Bigg(a^2 + R^2 - aR\tan^{-1}\left(\frac{R}{a}\right) + \frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right) + \frac{a}{2}\ln\left(R-a\right)R$$

$$
\begin{aligned}
&- \frac{a^2}{2}\ln(R-a) - \frac{a}{2}\ln(R+a)R \\
&\left. - \frac{a^2}{2}\ln(R+a) + C_1\right)^{1/2}, \\
x_2 &= e\sqrt{C_2}, \\
x_3 &= e\sqrt{\frac{1}{2}R^2\theta + C_3}, \\
x_4 &= e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta + \left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi + C_4}.
\end{aligned}
\tag{13}
$$

Since $x_1^2 + x_2^2 + x_3^2 - x_o^2 > 0$ which is the great circle S_1 in the Eguchi-Hanson space S, this geodesic is a retraction in Eguchi-Hanson space; $ds^2 > 0$. From $(1/2)R^2\dot{\phi} = h$, if $h = 0$ then $\dot{\phi} = 0$ or $\phi = B$; if $B = 0$ we get the following coordinates:

$$
\begin{aligned}
x_1 = e\Bigg(&a^2 + R^2 - aR\tan^{-1}\left(\frac{R}{a}\right) \\
&+ \frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right) + \frac{a}{2}\ln(R-a)R \\
&- \frac{a^2}{2}\ln(R-a) - \frac{a}{2}\ln(R+a)R \\
&- \frac{a^2}{2}\ln(R+a) + C_1\Bigg)^{1/2},
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
x_2 &= e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi + C_2}, \\
x_3 &= e\sqrt{\frac{1}{2}R^2\theta + C_3}, \\
x_4 &= e\sqrt{C_4}.
\end{aligned}
$$

Since $x_1^2 + x_2^2 + x_3^2 - x_o^2 > 0$ which is the great circle S_1 in the Eguchi-Hanson space S, this geodesic is a retraction in Eguchi-Hanson space; $ds^2 > 0$. Then the following theorem has been proved.

Theorem 1. *The retraction of the Eguchi-Hanson spacetime is the great circle in spacetime geodesic in Eguchi-Hanson space. The deformation retract of the Eguchi-Hanson space EH is defined by*

$$
\phi : EH \times I \longrightarrow EH, \tag{15}
$$

where EH is the Eguchi-Hanson space and I is the closed interval $[0,1]$. *The retraction of the Eguchi-Hanson space EH is given by*

$$
R : EH \longrightarrow S_1, S_2. \tag{16}
$$

The deformation retract of the Eguchi-Hanson space EH into a geodesic $S_1 \subset EH$ *is given by*

$$
\begin{aligned}
\phi(m,c) = \cos\frac{\pi c}{2}
\times \Bigg\{ & e\Bigg(a^2 + R^2 - aR\tan^{-1}\left(\frac{R}{a}\right) \\
&+ \frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right) + \frac{a}{2}\ln(R-a)R \\
&- \frac{a^2}{2}\ln(R-a) - \frac{a}{2}\ln(R+a)R \\
&- \frac{a^2}{2}\ln(R+a) + C_1\Bigg)^{1/2}, \\
& e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi + C_2}, \\
& e\sqrt{\frac{1}{2}R^2\theta + C_3}, \\
& e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta + \left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi + C_4}\Bigg\} \\
+ \sin\frac{\pi c}{2}
\times \Bigg\{ & e\Bigg(a^2 + R^2 - aR\tan^{-1}\left(\frac{R}{a}\right) \\
&+ \frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right) + \frac{a}{2}\ln(R-a)R \\
&- \frac{a^2}{2}\ln(R-a) - \frac{a}{2}\ln(R+a)R \\
&- \frac{a^2}{2}\ln(R+a) + C_1\Bigg)^{1/2}, \\
& e\sqrt{C_2}, e\sqrt{\frac{1}{2}R^2\theta + C_3}, \\
& e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta + \left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi + C_4}\Bigg\}.
\end{aligned}
\tag{17}
$$

The deformation retract of the Eguchi-Hanson space EH into a geodesic $S_2 \subset EH$ *is given by*

$$
\begin{aligned}
\phi(m,c) = \cos\frac{\pi c}{2}
\cdot \Bigg\{ & e\Bigg(a^2 + R^2 - aR\tan^{-1}\left(\frac{R}{a}\right)
\end{aligned}
$$

$$+\frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right)+\frac{a}{2}\ln(R-a)R$$
$$-\frac{a^2}{2}\ln(R-a)-\frac{a}{2}\ln(R+a)R$$
$$-\frac{a^2}{2}\ln(R+a)+C_1\Bigg)^{1/2},$$
$$e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi+C_2},$$
$$e\sqrt{\frac{1}{2}R^2\theta+C_3},$$
$$e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta+\left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi+C_4}\Bigg\}$$
$$+\sin\frac{\pi c}{2}$$
$$\times\Bigg\{e\Bigg(a^2+R^2-aR\tan^{-1}\left(\frac{R}{a}\right)$$
$$+\frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right)+\frac{a}{2}\ln(R-a)R$$
$$-\frac{a^2}{2}\ln(R-a)-\frac{a}{2}\ln(R+a)R$$
$$-\frac{a^2}{2}\ln(R+a)+C_1\Bigg)^{1/2},$$
$$e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi+C_2},e\sqrt{\frac{1}{2}R^2\theta+C_3}\Bigg\}. \tag{18}$$

Now we are going to discuss the folding f of the Eguchi-Hanson space EH. Let

$$f: EH \longrightarrow EH, \tag{19}$$

where

$$f(x_1,x_2,x_3,x_4)=(|x_1|,x_2,x_3,x_4). \tag{20}$$

An isometric folding of the Eguchi-Hanson space EH into itself may be defined by

$$f:\Bigg\{e\Bigg(a^2+R^2-aR\tan^{-1}\left(\frac{R}{a}\right)$$
$$+\frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right)+\frac{a}{2}\ln(R-a)R$$
$$-\frac{a^2}{2}\ln(R-a)-\frac{a}{2}\ln(R+a)R$$
$$-\frac{a^2}{2}\ln(R+a)+C_1\Bigg)^{1/2},$$
$$e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi+C_2},$$
$$e\sqrt{\frac{1}{2}R^2\theta+C_3},$$
$$e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta+\left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi+C_4}\Bigg\}$$
$$\longrightarrow\Bigg\{\Bigg|\Bigg(a^2+R^2-aR\tan^{-1}\left(\frac{R}{a}\right)$$
$$+\frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right)+\frac{a}{2}\ln(R-a)R$$
$$-\frac{a^2}{2}\ln(R-a)-\frac{a}{2}\ln(R+a)R$$
$$-\frac{a^2}{2}\ln(R+a)+C_1\Bigg)^{1/2}\Bigg|,$$
$$e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi+C_2},$$
$$e\sqrt{\frac{1}{2}R^2\theta+C_3},$$
$$e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta+\left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi+C_4}\Bigg\}. \tag{21}$$

The deformation retract of the folded Eguchi-Hanson space EH into the folded geodesic $f(s^1)$ is

$$\phi f:\Bigg\{\Bigg|\Bigg(a^2+R^2-aR\tan^{-1}\left(\frac{R}{a}\right)$$
$$+\frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right)+\frac{a}{2}\ln(R-a)R$$
$$-\frac{a^2}{2}\ln(R-a)-\frac{a}{2}\ln(R+a)R$$
$$-\frac{a^2}{2}\ln(R+a)+C_1\Bigg)^{1/2}\Bigg|,$$
$$e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi+C_2},$$
$$e\sqrt{\frac{1}{2}R^2\theta+C_3},$$
$$e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta+\left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi+C_4}\Bigg\}\times I$$

$$
\begin{aligned}
\longrightarrow \Bigg\{ \Bigg| \Bigg(& a^2 + R^2 - aR\tan^{-1}\left(\frac{R}{a}\right) \\
& + \frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right) + \frac{a}{2}\ln(R-a)R \\
& - \frac{a^2}{2}\ln(R-a) - \frac{a}{2}\ln(R+a)R \\
& - \frac{a^2}{2}\ln(R+a) + C_1 \Bigg)^{1/2} \Bigg|, \\
& e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi + C_2}, \\
& e\sqrt{\frac{1}{2}R^2\theta + C_3}, \\
& e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta + \left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi + C_4} \Bigg\}
\end{aligned} \tag{22}
$$

with

$$
\begin{aligned}
\phi f\,(m,c) & \\
= \cos\frac{\pi c}{2} & \\
\cdot \Bigg\{ \Bigg(& a^2 + R^2 - aR\tan^{-1}\left(\frac{R}{a}\right) \\
& + \frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right) + \frac{a}{2}\ln(R-a)R \\
& - \frac{a^2}{2}\ln(R-a) - \frac{a}{2}\ln(R+a)R \\
& - \frac{a^2}{2}\ln(R+a) + C_1 \Bigg)^{1/2}, \\
& e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi + C_2}, \\
& e\sqrt{\frac{1}{2}R^2\theta + C_3}, \\
& e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta + \left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi + C_4} \Bigg\} \\
+ \sin\frac{\pi c}{2} & \\
\cdot \Bigg\{ \Bigg| \Bigg(& a^2 + R^2 - aR\tan^{-1}\left(\frac{R}{a}\right) \\
& + \frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right) + \frac{a}{2}\ln(R-a)R \\
& - \frac{a^2}{2}\ln(R-a) - \frac{a}{2}\ln(R+a)R \\
& - \frac{a^2}{2}\ln(R+a) + C_1 \Bigg)^{1/2} \Bigg|, \\
& e\sqrt{C_2}, e\sqrt{\frac{1}{2}R^2\theta + C_3}, \\
& e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta + \left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi + C_4} \Bigg\}.
\end{aligned} \tag{23}
$$

The deformation retract of the folded Eguchi-Hanson space EH into the folded geodesic $f(s^2)$ is

$$
\begin{aligned}
\phi f\,(m,c) & \\
= \cos\frac{\pi c}{2} & \\
\cdot \Bigg\{ \Bigg| \Bigg(& a^2 + R^2 - aR\tan^{-1}\left(\frac{R}{a}\right) \\
& + \frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right) + \frac{a}{2}\ln(R-a)R \\
& - \frac{a^2}{2}\ln(R-a) - \frac{a}{2}\ln(R+a)R \\
& - \frac{a^2}{2}\ln(R+a) + C_1 \Bigg)^{1/2} \Bigg|, \\
& e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi + C_2}, \\
& e\sqrt{\frac{1}{2}R^2\theta + C_3}, \\
& e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta + \left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi + C_4} \Bigg\} \\
+ \sin\frac{\pi c}{2} & \\
\cdot \Bigg\{ \Bigg| \Bigg(& a^2 + R^2 - aR\tan^{-1}\left(\frac{R}{a}\right) \\
& + \frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right) + \frac{a}{2}\ln(R-a)R \\
& - \frac{a^2}{2}\ln(R-a) - \frac{a}{2}\ln(R+a)R \\
& - \frac{a^2}{2}\ln(R+a) + C_1 \Bigg)^{1/2} \Bigg|,
\end{aligned}
$$

$$e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi+C_2},$$
$$e\sqrt{\frac{1}{2}R^2\theta+C_3}, e\sqrt{C_4}\Bigg\}. \tag{24}$$

Then we reach the following theorem.

Theorem 2. *The folding of the Eguchi-Hanson space (26) and any folding homeomorphic to that folding have the same deformation retract of the Eguchi-Hanson space onto a geodesic.*

Now let the folding be defined by

$$f^{*}: EH \longrightarrow EH, \tag{25}$$

where

$$f^{o}(x_1, x_2, x_3, x_4) = (x_1, |x_2|, x_3, x_4). \tag{26}$$

The isometric folded Eguchi-Hanson space $f^{*}(EH)$ is

$$\Bigg\{e\Bigg(a^2+R^2-aR\tan^{-1}\left(\frac{R}{a}\right) + \frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right)+\frac{a}{2}\ln(R-a)R - \frac{a^2}{2}\ln(R-a)-\frac{a}{2}\ln(R+a)R - \frac{a^2}{2}\ln(R+a)+C_1\Bigg)^{1/2}, \tag{27}$$
$$\left|\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi+C_2}\right|,$$
$$e\sqrt{\frac{1}{2}R^2\theta+C_3},$$
$$e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta+\left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi+C_4}\Bigg\}.$$

The deformation retract of the folded Eguchi-Hanson space $f^{*}(EH)$ into the folded geodesic $f^{*}(s^1)$ is given by

$$\phi f^{*}: \Bigg\{\Bigg(a^2+R^2-aR\tan^{-1}\left(\frac{R}{a}\right) + \frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right)+\frac{a}{2}\ln(R-a)R - \frac{a^2}{2}\ln(R-a)-\frac{a}{2}\ln(R+a)R - \frac{a^2}{2}\ln(R+a)+C_1\Bigg)^{1/2},$$
$$\left|e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi+C_2}\right|,$$
$$e\sqrt{\frac{1}{2}R^2\theta+C_3},$$
$$e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta+\left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi+C_4}\Bigg\}\times I$$
$$\longrightarrow \Bigg\{\Bigg(a^2+R^2-aR\tan^{-1}\left(\frac{R}{a}\right) + \frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right)+\frac{a}{2}\ln(R-a)R - \frac{a^2}{2}\ln(R-a)-\frac{a}{2}\ln(R+a)R - \frac{a^2}{2}\ln(R+a)+C_1\Bigg)^{1/2},$$
$$\left|e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi+C_2}\right|,$$
$$e\sqrt{\frac{1}{2}R^2\theta+C_3},$$
$$e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta+\left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi+C_4}\Bigg\} \tag{28}$$

with

$$\phi f^{*}(m,c) = \cos\frac{\pi c}{2} \cdot \Bigg\{e\Bigg(a^2+R^2-aR\tan^{-1}\left(\frac{R}{a}\right) + \frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right)+\frac{a}{2}\ln(R-a)R - \frac{a^2}{2}\ln(R-a)-\frac{a}{2}\ln(R+a)R - \frac{a^2}{2}\ln(R+a)+C_1\Bigg)^{1/2},$$
$$\left|e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi+C_2}\right|,$$
$$e\sqrt{\frac{1}{2}R^2\theta+C_3},$$

$$e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta+\left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi+C_4}\Bigg\}$$
$$+\sin\frac{\pi c}{2}$$
$$\cdot\Bigg\{e\Bigg(a^2+R^2-aR\tan^{-1}\left(\frac{R}{a}\right)$$
$$+\frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right)+\frac{a}{2}\ln(R-a)R$$
$$-\frac{a^2}{2}\ln(R-a)-\frac{a}{2}\ln(R+a)R$$
$$-\frac{a^2}{2}\ln(R+a)+C_1\Bigg)^{1/2},$$
$$\left|\sqrt{C_2}\right|, e\sqrt{\frac{1}{2}R^2\theta+C_3},$$
$$e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta+\left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi+C_4}\Bigg\}. \tag{29}$$

The deformation retract of the folded Eguchi-Hanson space $f^*(EH)$ into the folded geodesic $f^*(s^2)$ is given by

$$\phi f^*(m,c)$$
$$=\cos\frac{\pi c}{2}$$
$$\cdot\Bigg\{e\Bigg(a^2+R^2-aR\tan^{-1}\left(\frac{R}{a}\right)$$
$$+\frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right)+\frac{a}{2}\ln(R-a)R$$
$$-\frac{a^2}{2}\ln(R-a)-\frac{a}{2}\ln(R+a)R$$
$$-\frac{a^2}{2}\ln(R+a)+C_1\Bigg)^{1/2},$$
$$\left|\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi+C_2}\right|,$$
$$e\sqrt{\frac{1}{2}R^2\theta+C_3},$$
$$e\sqrt{\frac{1}{2}R^2\left[\sin^2\theta+\left(1-\frac{a^4}{R^4}\right)\cos^2\theta\right]\phi+C_4}\Bigg\}$$
$$+\sin\frac{\pi c}{2}\Bigg\{e\Bigg(a^2+R^2-aR\tan^{-1}\left(\frac{R}{a}\right)$$
$$+\frac{a^2}{2}\ln\left(1+\frac{a^2}{2}\right)+\frac{a}{2}\ln(R-a)R$$
$$-\frac{a^2}{2}\ln(R-a)-\frac{a}{2}\ln(R+a)R$$
$$-\frac{a^2}{2}\ln(R+a)+C_1\Bigg)^{1/2},$$
$$\left|e\sqrt{\frac{1}{2}R^2\left(1-\frac{a^4}{R^4}\right)\psi+C_2}\right|,$$
$$e\sqrt{\frac{1}{2}R^2\theta+C_3}, e\sqrt{C_4}\Bigg\}. \tag{30}$$

Then the following theorem has been proved.

Theorem 3. *The deformation retract of the isometric folding of Eguchi-Hanson space and any folding homeomorphic to this type of folding is different from the deformation retract of Eguchi-Hanson space under condition (26).*

5. Conclusion

The deformation retract of the Eguchi-Hanson space has been investigated by making use of Lagrangian equations. The retraction of this space into itself and into geodesics has been presented. The deformation retraction of the Eguchi-Hanson space is a geodesic which is found to be a great circle. The folding of the Eguchi-Hanson space has been discussed and it was found that this folding and any folding homeomorphic to that folding have the same deformation retract of the Eguchi-Hanson space onto a geodesic. Also, the deformation retract of the isometric folding of Eguchi-Hanson space and any folding homeomorphic to this type of folding is found to be different from the deformation retract of Eguchi-Hanson space under condition (26).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are deeply indebted to the team work at the deanship of the scientific research Taibah University for their valuable help and critical guidance and for facilitating many administrative procedures. This research work was financially supported by Grant no. 1435/6164 from the deanship of the scientific research at Taibah University, Al-Madinah Al-Munawwarah, Saudi Arabia.

References

[1] H. Eschrig, *Topology and Geometry for Physics*, vol. 822 of *Lecture Notes in Physics*, Springer, Heidelberg, Germany, 1st edition, 2011.

[2] E. Bick and F. D. Steffen, Eds., *Topology and Geometry in Physics*, vol. 659 of *Lecture Notes in Physics*, Springer, Berlin, Germany, 2005.

[3] R. D. Sorkin, *Consequences of Spacetime Topology*, Syracuse University, Syracuse, NY, USA, 1989.

[4] A. Borde, "Topology change in classical general relativity," http://arxiv.org/abs/gr-qc/9406053.

[5] A. Chamblin, "Some applications of differential topology in general relativity," *Journal of Geometry and Physics*, vol. 13, no. 4, pp. 357–377, 1994.

[6] M. M. Akbar and G. W. Gibbons, "Ricci-flat metrics with *U(1)* action and Dirichlet boundary-value problem in Riemannian quantum gravity and isoperimetric inequalities," *Classical and Quantum Gravity*, vol. 20, no. 9, pp. 1787–1822, 2003.

[7] J. W. York, "Black-hole thermodynamics and the euclidean einstein action," *Physical Review D: Particles and Fields*, vol. 33, no. 8, pp. 2092–2099, 1986.

[8] S. W. Hawking, "Gravitational instantons," *Physics Letters. A*, vol. 60, no. 2, pp. 81–83, 1977.

[9] D. N. Page, "Taub-nut instanton with an horizon," *Physics Letters B*, vol. 78, no. 2-3, pp. 249–251, 1978.

[10] S. H. Hendi, R. B. Mann, N. Riazi, and B. Eslam Panah, "Eguchi-Hanson-like space-times in $F(R)$ gravity," *Physical Review D*, vol. 86, Article ID 104034, 2012.

[11] T. Eguchi and A. J. Hanson, "Asymptotically flat self-dual solutions to euclidean gravity," *Physics Letters B*, vol. 74, no. 3, pp. 249–251, 1978.

[12] T. Eguchi and A. J. Hanson, "Self-dual solutions to Euclidean gravity," *Annals of Physics*, vol. 120, no. 1, pp. 82–106, 1979.

[13] K. Borsuk, "Sur les rtractes," *Fundamenta Mathematicae*, vol. 17, p. 220, 1931.

[14] S. A. Robertson, "Isometric folding of Riemannian manifolds," *Proceedings of the Royal Society of Edinburgh, Section A: Mathematics*, vol. 79, no. 3-4, pp. 275–284, 1977.

[15] H. Rafat and F. Salama, "Fractal dimension and exponential folding of chaotic manifold," *Applied Mathematical Sciences*, vol. 6, no. 89–92, pp. 4431–4441, 2012.

[16] H. Rafat, "On tiling for some types of manifolds and their folding," *Applied Mathematical Sciences*, vol. 3, no. 26, pp. 1275–1284, 2009.

[17] J. Nešetřil and P. O. de Mendez, "Folding," *Journal of Combinatorial Theory Series B*, vol. 96, no. 5, pp. 730–739, 2006.

[18] P. Di Francesco, "Folding and coloring problems in mathematics and physics," *American Mathematical Society, Bulletin: New Series*, vol. 37, no. 3, pp. 251–307, 2000.

[19] E. Fieux and J. Lacaze, "Foldings in graphs and relations with simplicial complexes and posets," *Discrete Mathematics*, vol. 312, no. 17, pp. 2639–2651, 2012.

[20] W. S. Massey, *Algebraic Topology: An Introduction*, Harcourt, New York, NY, USA, 1967.

[21] H. A. Hamm and N. Mihalache, "Deformation retracts of Stein spaces," *Mathematische Annalen*, vol. 308, no. 2, pp. 333–345, 1997.

[22] H. Rafat, "A spacetime geodesics of the schwarzschild space and its deformation retract," *International Journal of Mathematical Combinatorics*, vol. 3, pp. 23–31, 2009.

[23] H. Rafat, "On the deformation retract of Kerr spacetime and its folding," *Advances in Mathematical Physics*, Article ID 673768, 8 pages, 2014.