Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2015, Article ID 387953, 8 pages
http://dx.doi.org/10.1155/2015/387953

Research Article

Estimates for Eigenvalues of the Elliptic Operator in Divergence Form on Riemannian Manifolds

Shenyang Tan,[1] Tiren Huang,[2] and Wenbin Zhang[1]

[1] *Taizhou Institute of Sci. & Tech., NUST., Taizhou 225300, China*
[2] *Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China*

Correspondence should be addressed to Shenyang Tan; ystsy@163.com

Received 5 December 2014; Accepted 19 January 2015

Academic Editor: Yao-Zhong Zhang

We investigate the Dirichlet weighted eigenvalue problem of the elliptic operator in divergence form on compact Riemannian manifolds $(M, g, e^{-\phi}dv)$. We establish a Yang-type inequality of this problem. We also get universal inequalities for eigenvalues of elliptic operators in divergence form on compact domains of complete submanifolds admitting special functions which include the Hadamard manifolds with Ricci curvature bounded below and any complete manifolds admitting eigenmaps to a sphere.

1. Introduction

Let $(M, \langle,\rangle)$ be an n-dimensional bounded compact Riemannian manifold, $\phi \in C^2(M)$, and $d\mu = e^{-\phi}dv$, where dv is the Riemannian volume measure on $(M, \langle,\rangle)$. Let Δ and ∇ be the Laplacian and the gradient operator on M, respectively. The witten Laplacian (or the drifting Laplacian) with respect to the weighted volume measure μ is given by

$$\Delta_\phi = \Delta - \langle \nabla\phi, \nabla(\cdot)\rangle. \quad (1)$$

In recent years, many mathematicians have paid their attention to the eigenvalue problem of the drifting Laplacian on Riemannian manifolds (see [1–3]). They have studied the following eigenvalue problem:

$$\Delta_\phi u = -\lambda u, \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0. \quad (2)$$

In particular in [4], Xia and Xu got a Payne-Plya-Weinberger-Yang-type inequality of the eigenvalues of this problem:

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \le \frac{1}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(4\lambda_i + 4\phi_0\lambda_i^{1/2} + n^2H_0^2 + \phi_0^2\right), \quad (3)$$

where $H_0 = \sup_\Omega |H|$, H is the mean curvature vector, and $\phi_0 = \max_{\overline{\Omega}}|\nabla\phi|$.

In this paper, we consider the following eigenvalue problem:

$$-\operatorname{div}(A\nabla u) + \langle A\nabla\phi, \nabla u\rangle + Vu = \lambda\rho u, \quad \text{in } M, \qquad u|_{\partial M} = 0, \quad (4)$$

where V is a nonnegative potential function, ρ is a positive function continuous on $\overline{M}$, and A is symmetric and positive definite matrices. Through integration by part, we can find

$$\int_M f\left(-\operatorname{div}(A\nabla g) + \langle A\nabla\phi, \nabla g\rangle + Vg\right)d\mu = \int_M g\left(-\operatorname{div}(A\nabla f) + \langle A\nabla\phi, \nabla f\rangle + Vf\right)d\mu,$$
$$\int_M f\left(-\operatorname{div}(A\nabla g) + \langle A\nabla\phi, \nabla g\rangle\right)d\mu = \int_M \langle A\nabla g, \nabla f\rangle d\mu, \quad (5)$$

where f and g are smooth functions on M with $f|_{\partial M} = g|_{\partial M} = 0$. As we know (see [5]), this problem has a real and discrete spectrum:

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_r \le \nearrow; \quad (6)$$

here each eigenvalue is repeated from its multiplicity.

In Section 2, we get a general inequality for the eigenvalue of the operator in divergence form $-\operatorname{div}(A\nabla(\cdot)) + \langle\nabla\phi,\nabla(\cdot)\rangle + V$ through the way of trial function. In Section 3, we obtain a Payne-Plya-Weinberger-Yang-type inequality through defining special trial function. In Section 4, we prove some universal inequalities for eigenvalues of the divergence operator on manifolds admitting special functions.

2. A General Inequality

Firstly, we give a useful inequality about the eigenvalues.

Theorem 1. *Let λ_i be the ith eigenvalue of problem (4) and let u_i be the orthonormal eigenfunction corresponding to λ_i; that is,*

$$-\operatorname{div}(A\nabla u_i) + \langle A\nabla\phi,\nabla u_i\rangle + Vu_i = \lambda_i\rho u_i, \quad \text{in } M;$$
$$\int_M \rho u_i u_j d\mu = \delta_{ij}, \tag{7}$$
$$u_i|_{\partial M} = 0.$$

Then, for any $h \in C^3(M)\cap C^2(\partial M)$ and any integer k, we have

$$\begin{aligned}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_M u_i^2\langle\nabla h,\nabla h\rangle d\mu &\le \epsilon\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_M u_i^2\langle A\nabla h,\nabla h\rangle d\mu \\ &\quad + \sum_{i=1}^{k}\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\int_M\frac{1}{\rho}\left(\langle\nabla u_i,\nabla h\rangle + \frac{1}{2}u_i\Delta h\right)^2 d\mu,\end{aligned} \tag{8}$$

where ϵ is any positive constant.

Proof. We define a trial function

$$\varphi_i = hu_i - \sum_{j=1}^{k}a_{ij}u_j, \tag{9}$$

where $a_{ij} = \int_M \rho h u_i u_j d\mu = a_{ji}$, and then we have

$$\int_M \rho\varphi_i u_j d\mu = 0, \quad \varphi_i|_{\partial M} = 0, \quad \text{for } \forall i, j = 1,\ldots,k. \tag{10}$$

If we set $L = -\operatorname{div}(A\nabla(\cdot)) + \langle A\nabla\phi,\nabla(\cdot)\rangle + V$, then through direct calculation, we have

$$\begin{aligned}L\varphi_i &= \left(-\operatorname{div}(A\nabla(\cdot)) + \langle A\nabla\phi,\nabla(\cdot)\rangle + V\right)\varphi_i \\ &= hLu_i + u_iLh - Vhu_i - 2\langle A\nabla h,\nabla u_i\rangle - \sum_{j=1}^{k}a_{ij}\lambda_j u_j.\end{aligned} \tag{11}$$

Substituting (11) into the well known Rayleigh-Ritz inequality

$$\lambda_{k+1} \le \frac{\int_M \varphi_i L\varphi_i}{\int_M \rho\varphi_i^2 d\mu}, \tag{12}$$

we can get

$$\begin{aligned}(\lambda_{k+1}-\lambda_i)\int_M \rho\varphi_i^2 d\mu \le \int_M \varphi_i\left(u_iLh - Vhu_i + 2\langle A\nabla h,\nabla u_i\rangle\right)d\mu.\end{aligned} \tag{13}$$

We set

$$b_{ij} = \int_M\left(u_iLh - Vhu_i - 2\langle A\nabla h,\nabla u_i\rangle\right)u_j d\mu. \tag{14}$$

Through direct calculation, we have

$$b_{ij} = -b_{ji}, \qquad b_{ij} = (\lambda_i - \lambda_j)a_{ij}. \tag{15}$$

Combining with (13), we get

$$\begin{aligned}(\lambda_{k+1}-\lambda_i)\int_M \rho\varphi_i^2 d\mu &\le -\int_M hu_i\left(u_i\left(\operatorname{div}(A\nabla h) - \langle A\nabla\phi,\nabla h\rangle\right)\right. \\ &\qquad \left. - 2\langle\nabla u_i, A\nabla h\rangle\right)d\mu \\ &\quad + \sum_{j=1}^{k}(\lambda_i - \lambda_j)a_{ij}^2.\end{aligned} \tag{16}$$

Setting

$$c_{ij} = \int_M u_j\left(\langle\nabla u_i,\nabla h\rangle + \frac{1}{2}u_i\Delta h\right)d\mu, \tag{17}$$

then through direct calculation, we have

$$c_{ij} = -c_{ji}, \tag{18}$$

$$\begin{aligned}\int_M(-2)\varphi_i\left(\langle\nabla u_i,\nabla h\rangle + \frac{1}{2}u_i\Delta h\right)d\mu &= -2\int_M hu_i\left(\langle\nabla u_i,\nabla h\rangle + \frac{1}{2}u_i\Delta h\right)d\mu + 2\sum_{j=1}^{k}a_{ij}c_{ij} \\ &= \int_M u_i^2\langle\nabla h,\nabla h\rangle d\mu + 2\sum_{j=1}^{k}a_{ij}c_{ij}.\end{aligned} \tag{19}$$

Multiplying (19) by $(\lambda_{k+1}-\lambda_i)^2$, we get

$$\begin{aligned}
&(\lambda_{k+1}-\lambda_i)^2\left(\int_M u_i^2\langle\nabla h,\nabla h\rangle\,d\mu+2\sum_{j=1}^k a_{ij}c_{ij}\right)\\
&=(\lambda_{k+1}-\lambda_i)^2\\
&\quad\cdot\int_M(-2)\sqrt{\rho}\phi_i\left(\frac{1}{\sqrt{\rho}}\left(\langle\nabla u_i,\nabla h\rangle+\frac{1}{2}u_i\Delta h\right)\right.\\
&\quad\left.-\sum_{j=1}^k c_{ij}\sqrt{\rho}u_j\right)d\mu\\
&\le\epsilon(\lambda_{k+1}-\lambda_i)^3\int_M\rho\phi_i^2\,d\mu\\
&\quad+\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\int_M\left(\frac{1}{\sqrt{\rho}}\left(\langle\nabla u_i,\nabla h\rangle+\frac{1}{2}u_i\Delta h\right)\right.\\
&\quad\left.-\sum_{j=1}^k c_{ij}\sqrt{\rho}u_j\right)^2d\mu\\
&=\epsilon(\lambda_{k+1}-\lambda_i)^3\int_M\rho\phi_i^2\,d\mu\\
&\quad+\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\left(\int_M\frac{1}{\rho}\left(\langle\nabla u_i,\nabla h\rangle+\frac{1}{2}u_i\Delta h\right)^2d\mu\right.\\
&\quad\left.-\sum_{j=1}^k c_{ij}^2\right)\\
&\le\epsilon(\lambda_{k+1}-\lambda_i)^2\left(-\int_M hu_i\left(u_i\left(\operatorname{div}(A\nabla h)\right.\right.\right.\\
&\quad\left.-\langle A\nabla\phi,\nabla h\rangle\right)\\
&\quad\left.+2\langle\nabla u_i,A\nabla h\rangle\,d\mu\right)\\
&\quad\left.+\sum_{j=1}^k(\lambda_i-\lambda_j)a_{ij}^2\right)\\
&\quad+\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\\
&\quad\cdot\left(\int_M\frac{1}{\rho}\left(\langle\nabla u_i,\nabla h\rangle+\frac{1}{2}u_i\Delta h\right)^2d\mu-\sum_{j=1}^k c_{ij}^2\right),
\end{aligned}\tag{20}$$

where ϵ is any positive constant. Summing over i from 1 to k, we have

$$\begin{aligned}
&\sum_{j=1}^k(\lambda_{k+1}-\lambda_i)^2\int_M u_i^2\langle\nabla h,\nabla h\rangle\,d\mu\\
&\quad-2\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)(\lambda_i-\lambda_j)a_{ij}c_{ij}\\
&\le\epsilon\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\left(-\int_M hu_i\left(u_i\left(\operatorname{div}(A\nabla h)\right.\right.\right.\\
&\quad\left.-\langle A\nabla\phi,\nabla h\rangle\right)\\
&\quad\left.\left.+2\langle\nabla u_i,A\nabla h\rangle\right)d\mu\right)\\
&\quad+\sum_{i=1}^k\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\left(\int_M\frac{1}{\rho}\left(\langle\nabla u_i,\nabla h\rangle+\frac{1}{2}u_i\Delta h\right)^2d\mu\right.\\
&\quad\left.-\sum_{i,j=1}^k(\lambda_{k+1}-\lambda_i)\epsilon(\lambda_i-\lambda_j)^2a_{ij}^2\right)\\
&\quad-\sum_{i,j=1}^k\frac{(\lambda_{k+1}-\lambda_i)}{\epsilon}c_{ij}^2.
\end{aligned}\tag{21}$$

Because of $a_{ij}=a_{ji}$, $c_{ij}=-c_{ji}$, we infer

$$\begin{aligned}
&\sum_{j=1}^k(\lambda_{k+1}-\lambda_i)^2\int_M u_i^2\langle\nabla h,\nabla h\rangle\,d\mu\\
&\le\epsilon\sum_{i=1}^k(\lambda_{k+1}-\lambda_i)^2\\
&\quad\cdot\left(-\int_M hu_i\left(u_i\left(\operatorname{div}(A\nabla h)\right.\right.\right.\\
&\quad\left.\left.\left.-\langle A\nabla\phi,\nabla h\rangle\right)+2\langle\nabla u_i,A\nabla h\rangle\right)d\mu\right)\\
&\quad+\sum_{i=1}^k\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\int_M\frac{1}{\rho}\left(\langle\nabla u_i,\nabla h\rangle+\frac{1}{2}u_i\Delta h\right)^2d\mu.
\end{aligned}\tag{22}$$

Considering the property of the measure on this weighted manifold that $d\mu=e^{-\phi}dv$, we can refer to the fact that

$$\begin{aligned}
&\int_M hu_i^2\operatorname{div}(A\nabla h)\,d\mu\\
&=\int_M hu_i^2\operatorname{div}(A\nabla h)\,e^{-\phi}dv\\
&=-\int_M\langle A\nabla h,\nabla h\rangle u_i^2e^{-\phi}dv\\
&\quad-\int_M\langle A\nabla h,\nabla u_i\rangle 2hu_ie^{-\phi}dv+\int_M\langle A\nabla h,\nabla\phi\rangle hu_i^2e^{-\phi}dv\\
&=-\int_M\langle A\nabla h,\nabla h\rangle u_i^2d\mu-\int_M\langle A\nabla h,\nabla u_i\rangle 2hu_i\,d\mu\\
&\quad+\int_M\langle A\nabla h,\nabla\phi\rangle hu_i^2d\mu.
\end{aligned}\tag{23}$$

Substituting (23) into (22), we can finish the proof of Theorem 1. □

3. The Main Theorem and the Proof

In this section, we give some estimates about the eigenvalues of the operator in divergence form.

Lemma 2. *Let M be an n-dimensional complete Riemannian manifold and let Ω be a bounded domain with smooth boundary and let ϕ be a smooth function on Ω in M; A is a symmetry and positive definite matrix; suppose λ_i be the ith eigenvalue of the problem:*

$$-\operatorname{div}(A\nabla u_i) + \langle A\nabla\phi, \nabla u_i\rangle + Vu_i = \lambda\rho u_i, \quad \text{in } \Omega;$$
$$\int_\Omega \rho u_i u_j d\mu = \delta_{ij}; \tag{24}$$
$$u_i|_{\partial\Omega} = 0.$$

If M is isometrically immersed in R^m with mean curvature vector H, then

$$\begin{aligned} n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2 d\mu &\le \epsilon\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2 \operatorname{tr}(A)\,d\mu \\ &+ \sum_{i=1}^{k}\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\int_\Omega \frac{1}{\rho}\left(|\nabla u_i|^2 + \frac{1}{4}u_i^2 n^2 H^2\right)d\mu. \end{aligned} \tag{25}$$

Proof. Let x_α, $\alpha = 1, 2, \ldots, m$ be the standard coordinate functions of R^m. Taking $h = x_\alpha$ in (8), summing over α from 1 to m, we have

$$\begin{aligned} \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\sum_{\alpha=1}^{m}\int_\Omega u_i^2\langle\nabla x_\alpha, \nabla x_\alpha\rangle d\mu &\le \epsilon\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2\sum_{\alpha=1}^{m}\langle A\nabla x_\alpha, \nabla x_\alpha\rangle d\mu \\ &+ \sum_{i=1}^{k}\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\int_\Omega\frac{1}{\rho}\sum_{\alpha=1}^{m}\left(\langle\nabla u_i, \nabla x_\alpha\rangle + \frac{1}{2}u_i\Delta x_\alpha\right)^2 d\mu \\ &= \epsilon\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2\sum_{\alpha=1}^{m}\langle A\nabla x_\alpha, \nabla x_\alpha\rangle d\mu \\ &+ \sum_{i=1}^{k}\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\int_\Omega\frac{1}{\rho}\sum_{\alpha=1}^{m}(\langle\nabla u_i, \nabla x_\alpha\rangle)^2 + \frac{1}{4}u_i^2(\Delta x_\alpha)^2 \\ &+ \langle u_i\nabla u_i, \Delta x_\alpha\nabla x_\alpha\rangle\,d\mu. \end{aligned} \tag{26}$$

Since M is isometrically immersed in R^m, we have

$$\sum_{\alpha=1}^{m}|\nabla x_\alpha|^2 = n, \tag{27}$$

and then,

$$\int_\Omega u_i^2\sum_{\alpha=1}^{m}|\nabla x_\alpha|^2 d\mu = n\int_\Omega u_i^2 d\mu. \tag{28}$$

Also, we have

$$\Delta(x_1, x_2, \ldots, x_m) \equiv (\Delta x_1, \Delta x_2, \ldots, \Delta x_m) = nH. \tag{29}$$

Let $e_1, \ldots, e_n$ be orthonormal tangent vector fields locally defined on M; we have

$$\begin{aligned} \sum_{\alpha=1}^{n+1}\langle A\nabla x_\alpha, \nabla x_\alpha\rangle &= \sum_{\alpha=1}^{n+1}\left\langle A\left(\sum_{i=1}^{n}\langle\nabla x_\alpha, e_i\rangle e_i\right), \sum_{j=1}^{n}\langle\nabla x_\alpha, e_j\rangle e_j\right\rangle \\ &= \sum_{i,j=1}^{n}\sum_{\alpha=1}^{n+1}(e_i x_\alpha)(e_j x_\alpha)\langle Ae_i, e_j\rangle \\ &= \sum_{i,j=1}^{n}\langle e_i, e_j\rangle\langle Ae_i, e_j\rangle = \sum_{i}^{n}\langle Ae_i, e_i\rangle \\ &= \operatorname{tr}(A), \end{aligned} \tag{30}$$

and then,

$$\begin{gathered} \sum_{\alpha=1}^{m}\int_\Omega u_i^2\langle A\nabla x_\alpha, \nabla x_\alpha\rangle d\mu \le \int_\Omega u_i^2\operatorname{tr}(A)\,d\mu, \\ \sum_{\alpha=1}^{m}\Delta x_\alpha\langle\nabla x_\alpha, \nabla u_i\rangle = \sum_{\alpha=1}^{m}\Delta x_\alpha\nabla u_i(x_\alpha) = \langle nH, \nabla u_i\rangle = 0. \end{gathered} \tag{31}$$

Substituting (28), (29), and (31) into (26), we can finish the proof of Lemma 2. □

Theorem 3. *Under the same assumption of Lemma 2, let $\tau = (\sup_\Omega\rho)^{-1}$, $\sigma = (\inf_\Omega\rho)^{-1}$, $V_0 = \min_\Omega V$, $|\nabla\phi| \le C_0$, $|H| \le H_0$, $\xi_1 I \le A \le \xi_2 I$, and then one has*

$$\begin{aligned} \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 &\le \frac{4\xi_2\sigma^2}{n\tau^2} \cdot \sum_{i=1}^{n}(\lambda_{k+1}-\lambda_i) \\ &\cdot \left\{\frac{1}{2\xi_1}\left[\xi_2 C_0\sigma + \left(\xi_2^2 C_0^2\sigma - 4\xi_1 V_0\tau + 4\xi_1\lambda_i\right)^{1/2}\right] + \frac{1}{4}\sigma n^2 H_0^2\right\}. \end{aligned} \tag{32}$$

Proof. Obviously, we have

$$\sigma = \sigma\int_\Omega \rho u_i^2 d\mu \ge \int_\Omega u_i^2 d\mu \ge \tau\int_\Omega \rho u_i^2 d\mu = \tau. \tag{33}$$

Multiplying the equation

$$-\operatorname{div}(A\nabla u_i) + \langle A\nabla\phi, \nabla u_i\rangle + Vu_i = \lambda_i \rho u_i \quad (34)$$

by u_i and integrating on Ω, we have

$$\begin{aligned}\lambda_i &= \int_\Omega \langle A\nabla u_i, \nabla u_i\rangle\, d\mu + \int_\Omega u_i \langle A\nabla\phi, \nabla u_i\rangle\, d\mu + \int_\Omega Vu_i^2 d\mu \\ &\geq \xi_1 \int_\Omega |\nabla u_i|^2\, d\mu + \int_\Omega u_i \langle A\nabla\phi, \nabla u_i\rangle\, d\mu + \int_\Omega Vu_i^2 d\mu.\end{aligned} \quad (35)$$

Considering

$$\begin{aligned}&\left|\int_\Omega u_i \langle A\nabla\phi, \nabla u_i\rangle\, d\mu\right| \\ &\quad\leq \xi_2 \int_\Omega |u_i|\,|\nabla\phi|\,|\nabla u_i|\, d\mu \\ &\quad\leq \xi_2 C_0 \left(\int_\Omega u_i^2 d\mu\right)^{1/2} \left(\int_\Omega |\nabla u_i|^2\, d\mu\right)^{1/2} \\ &\quad\leq \xi_2 C_0 \sigma \left(\int_\Omega |\nabla u_i|^2\, d\mu\right)^{1/2},\end{aligned} \quad (36)$$

then we can obtain

$$\begin{aligned}\lambda_i &\geq \xi_1 \int_\Omega |\nabla u_i|^2\, d\mu + \int_\Omega u_i \langle A\nabla\phi, \nabla u_i\rangle\, d\mu + \int_\Omega Vu_i^2 d\mu \\ &\geq \xi_1 \int_\Omega |\nabla u_i|^2\, d\mu - \xi_2 C_0 \sigma \left(\int_\Omega |\nabla u_i|^2\, d\mu\right)^{1/2} + V_0\tau.\end{aligned} \quad (37)$$

Solving this inequality, we have

$$\begin{aligned}&\int_\Omega |\nabla u_i|^2\, d\mu \\ &\quad\leq \frac{1}{2\xi_1}\left\{\xi_2 C_0 \sigma + \left(\xi_2^2 C_0^2 \sigma^2 - 4\xi_1 V_0 \tau + 4\xi_1 \lambda_i\right)^{1/2}\right\}.\end{aligned} \quad (38)$$

Substituting (33) and (38) into (25) and taking

$$\epsilon = \left(\frac{A}{B}\right)^{1/2}, \quad (39)$$

where

$$\begin{aligned}A &= \sum_{i=1}^{n} (\lambda_{k+1} - \lambda_i)\, \sigma \\ &\quad\cdot \left\{\frac{1}{2\xi_1}\left[\xi_2 C_0 \sigma \right.\right. \\ &\qquad\left. + \left(\xi_2^2 C_0^2 \sigma - 4\xi_1 V_0 \tau + 4\xi_1 \lambda_i\right)^{1/2}\right] \\ &\quad\left. + \frac{1}{4}\sigma n^2 H_0^2\right\}, \\ B &= \sum_{i=1}^{n} (\lambda_{k+1} - \lambda_i)^2\, n \xi_2 \sigma,\end{aligned} \quad (40)$$

we can finish the proof of Theorem 3. □

Remark 4. If we set ϕ = constant, $A = I$, $\rho = 1$ and $V = 0$, the divergence operator becomes the usual laplace operator on Riemannian manifolds and we can find our result is sharper than the result in [4, 6].

Remark 5. For some of the recent developments about universal inequalities for eigenvalues on Riemannian manifolds, we refer to [6–12] and the references therein.

4. Eigenvalues on Manifolds Admitting Special Functions

In this section, we get some universal inequalities for eigenvalues of the divergence operator on manifolds admitting special functions.

Theorem 6. *Let M be an n-dimensional complete Riemannian manifold and let Ω be a bounded domain with smooth boundary and let ϕ be a smooth function on Ω in M; A is a symmetry and positive definite matrix; let $\tau = (\sup_\Omega \rho)^{-1}$, $\sigma = (\inf_\Omega \rho)^{-1}$, $V_0 = \min_\Omega V$, $|\nabla\phi| \leq C_0$, $|H| \leq H_0$, $\xi_1 I \leq A \leq \xi_2 I$; suppose λ_i be the ith eigenvalue of the problem:*

$$\begin{aligned}-\operatorname{div}(A\nabla u_i) + \langle A\nabla\phi, \nabla u_i\rangle + Vu_i &= \lambda\rho u_i, \quad \text{in } \Omega; \\ \int_\Omega \rho u_i u_j d\mu &= \delta_{ij}; \\ u_i|_{\partial\Omega} &= 0;\end{aligned} \quad (41)$$

if there exists a function $\theta : \Omega \to R$ and a constant A_0 such that

$$|\nabla\theta| = 1, \quad |\Delta\theta| \leq A_0, \quad \text{on } \Omega, \quad (42)$$

then

$$\begin{aligned}&\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \\ &\quad\leq \frac{8\xi_2 \sigma^2}{n\tau^2} \sum_{i=1}^{n} (\lambda_{k+1} - \lambda_i) \\ &\qquad\cdot \left\{\frac{1}{2\xi_1}\left[\xi_2 C_0 \sigma \right.\right. \\ &\qquad\qquad\left. + \left(\xi_2^2 C_0^2 \sigma - 4\xi_1 V_0 \tau + 4\xi_1 \lambda_i\right)^{1/2}\right] \\ &\qquad\left. + \frac{1}{4}\sigma A_0^2\right\}.\end{aligned} \quad (43)$$

Proof. Taking $h = \theta$ in (8) and considering (38), (42), and $\langle \nabla\theta, A\nabla\theta\rangle \le \xi_2 |\nabla\theta|^2$, we have

$$\begin{aligned}
&\tau\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\
&\quad\le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2 d\mu \\
&\quad\le \epsilon\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2\langle\nabla\theta, A\nabla\theta\rangle d\mu \\
&\qquad+\sum_{i=1}^{k}\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\int_\Omega\frac{1}{\rho}\left(\langle\nabla u_i,\nabla\theta\rangle+\frac{1}{2}u_i\Delta\theta\right)^2 d\mu \\
&\quad\le \epsilon\sigma\xi_2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\
&\qquad+2\sigma\sum_{i=1}^{k}\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\int_\Omega|\nabla u_i|^2+\frac{A_0^2u_i^2}{4}d\mu \\
&\quad\le \epsilon\sigma\xi_2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\
&\qquad+2\sigma\sum_{i=1}^{k}\frac{\lambda_{k+1}-\lambda_i}{\epsilon} \\
&\qquad\quad\cdot\left(\frac{1}{2\xi_1}\left(\xi_2C_0\sigma\right.\right. \\
&\qquad\quad\left.+\left(\xi_2^2C_0^2\sigma^2-4\xi_1V_0\tau+4\xi_1\lambda_i\right)^{1/2}\right) \\
&\qquad\quad\left.+\sigma\frac{A_0^2}{4}\right).
\end{aligned} \tag{44}$$

Taking

$$\begin{aligned}
\epsilon = &\left\{\left(2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\right.\right. \\
&\cdot\left(\frac{1}{2\xi_1}\left(\xi_2C_0\sigma\right.\right. \\
&\left.\left.\left.+\left(\xi_2^2C_0^2\sigma^2-4\xi_1V_0\tau+4\xi_1\lambda_i\right)^{1/2}\right)+\sigma\frac{A_0^2}{4}\right)\right) \\
&\left.\cdot\left(\xi_2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\right)^{-1}\right\}^{1/2},
\end{aligned} \tag{45}$$

we can complete the proof of Theorem 6. □

Remark 7. Let M be an n-dimensional connected complete Riemannian manifold; suppose its Ricci curvature satisfies $\mathrm{Ric}_m \ge -(n-1)c^2$, $c \ge 0$. If there exists a smooth function θ satisfying $|\nabla\theta| = 1$, then $|\Delta\theta| \le (n-1)c^2$. So the Bussemann functions on Cartan-Hadamard manifolds with Ricci curvature bounded below satisfy the condition in Theorem 6.

Theorem 8. *Let M be an n-dimensional complete Riemannian manifold and let Ω be a bounded domain with smooth boundary; let ϕ be a smooth function on Ω in M; A is a symmetry and positive definite matrix; let $\tau = (\sup_\Omega\rho)^{-1}$, $\sigma = (\inf_\Omega\rho)^{-1}$, $V_0 = \min_\Omega V$, $|\nabla\phi| \le C_0$, $|H| \le H_0$, $\xi_1 I \le A \le \xi_2 I$; suppose λ_i be the ith eigenvalue of the problem:*

$$\begin{gathered}
-\operatorname{div}(A\nabla u_i)+\langle A\nabla\phi,\nabla u_i\rangle+Vu_i=\lambda\rho u_i, \quad \text{in } \Omega; \\
\int_\Omega\rho u_iu_jd\mu=\delta_{ij}; \\
u_i|_{\partial\Omega}=0;
\end{gathered} \tag{46}$$

if Ω admits an eigenmap $f = (f_1, f_2, \dots, f_{m+1}) : \Omega \to S^m$ corresponding to an eigenvalue η, that is

$$\Delta f_\alpha=-\eta f_\alpha, \quad \alpha=1,\dots,m+1, \qquad \sum_{\alpha=1}^{m+1}f_\alpha^2=1, \tag{47}$$

then

$$\begin{aligned}
&\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\
&\quad\le\frac{2\xi_2\sigma^2}{\tau^2} \\
&\qquad\cdot\sum_{i=1}^{n}(\lambda_{k+1}-\lambda_i) \\
&\qquad\quad\cdot\left\{\frac{1}{\xi_1}\left[\xi_2C_0\sigma\right.\right. \\
&\qquad\qquad\left.\left.+\left(\xi_2^2C_0^2\sigma-4\xi_1V_0\tau+4\xi_1\lambda_i\right)^{1/2}\right]+\frac{\sigma\eta}{2}\right\}.
\end{aligned} \tag{48}$$

Proof. Because of (47), we obtain

$$\sum_{\alpha=1}^{m+1}|\nabla f_\alpha|^2=\eta. \tag{49}$$

Taking $h = f_\alpha$ in (8) and summing over α from 1 to $m+1$, we get

$$\begin{aligned}
&\eta\tau\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\
&\le \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\sum_{\alpha=1}^{m+1}\int_\Omega u_i^2\left|\nabla f_\alpha\right|^2 d\mu \\
&\le \epsilon\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\int_\Omega u_i^2\sum_{\alpha=1}^{m+1}\langle\nabla f_\alpha, Af_\alpha\rangle d\mu \\
&\quad + \sum_{i=1}^{k}\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\int_\Omega \frac{1}{\rho}\sum_{\alpha=1}^{m+1}\left(\langle\nabla u_i,\nabla f_\alpha\rangle - \frac{1}{2}\eta f_\alpha u_i\right)^2 d\mu \\
&\le \epsilon\eta\sigma\xi_2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\
&\quad + 2\sigma\sum_{i=1}^{k}\frac{\lambda_{k+1}-\lambda_i}{\epsilon}\int_\Omega \eta\left|\nabla u_i\right|^2 + \frac{\eta^2u_i^2}{4}d\mu \\
&\le \epsilon\eta\sigma\xi_2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \\
&\quad + \sigma\sum_{i=1}^{k}\frac{\lambda_{k+1}-\lambda_i}{\epsilon} \\
&\qquad \cdot\left(\frac{\eta}{\xi_1}\left(\xi_2C_0\sigma\right.\right. \\
&\qquad\qquad \left.+\left(\xi_2^2C_0^2\sigma^2 - 4\xi_1V_0\tau + 4\xi_1\lambda_i\right)^{1/2}\right) \\
&\qquad\qquad \left.+\sigma\frac{\eta^2}{2}\right).
\end{aligned} \tag{50}$$

Taking

$$\begin{aligned}
\epsilon = &\left\{\left(\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\right.\right. \\
&\quad\cdot\left(\frac{1}{\xi_1}\left(\xi_2C_0\sigma\right.\right. \\
&\qquad\qquad \left.\left.\left.+\left(\xi_2^2C_0^2\sigma^2 - 4\xi_1V_0\tau + 4\xi_1\lambda_i\right)^{1/2}\right)+\sigma\frac{\eta}{2}\right)\right) \\
&\quad\left.\cdot\left(\xi_2\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\right)^{-1}\right\}^{1/2},
\end{aligned} \tag{51}$$

then the proof of Theorem 8 is finished. □

Remark 9. Any compact homogeneous Riemannian manifold admits eigenmaps to some unit sphere for the first positive eigenvalues of the Laplacian which satisfy the condition in Theorem 8 [13].

5. Physical Interpretation

In quantum mechanics, eigenvalue is the dynamics of macro possible values. The wave function is superposition of a number of eigenstates. Different eigenstate is corresponding to the specific eigenvalue (of course there may be degenerate case; namely, the same eigenvalue corresponds to different intrinsic state). The experimental measurement of the mechanical quantity must be one of eigenvalues, and wave function in the measurement is the eigenstate of the corresponding eigenvalue. The gap between different eigenvalues means the difference between the energy levels. That is why many researchers pay much attention to this problem. In this paper, we find a relatively accurate upper bound between any two different eigenvalues.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to express their gratitude to the referee for his valuable comments and suggestions. The paper is supported by the National Natural Science Foundation of China (Grant no. 11401531) and the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (no. 14KJD110004).

References

[1] L. Ma and B. Y. Liu, "Convex eigenfunction of a drifting Laplacian operator and the fundamental gap," *Pacific Journal of Mathematics*, vol. 240, no. 2, pp. 343–361, 2009.

[2] L. Ma and B. Liu, "Convexity of the first eigenfunction of the drifting Laplacian operator and its applications," *New York Journal of Mathematics*, vol. 14, pp. 393–401, 2008.

[3] L. Ma and S.-H. Du, "Extension of Reilly formula with applications to eigenvalue estimates for drifting Laplacians," *Comptes Rendus Mathematique*, vol. 348, no. 21-22, pp. 1203–1206, 2010.

[4] C. Xia and H. Xu, "Inequalities for eigenvalues of the drifting Laplacian on Riemannian manifolds," *Annals of Global Analysis and Geometry*, vol. 45, no. 3, pp. 155–166, 2014.

[5] K. O. Friedrichs, *Spectral Theory of Operators in Hilbert Space*, vol. 9 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1980.

[6] Q. L. Wang and C. Y. Xia, "Universal bounds for eigenvalues of Schrödinger operator on Riemannian manifolds," *Annales Academiae Scientiarum Fennicae Mathematica*, vol. 33, pp. 319–336, 2008.

[7] M. S. Ashbaugh, "The universal eigenvalue bounds of Payne-Pólya-Weinberger, Hile-PROtter, and H. C. Yang," *Proceedings*

of the Indian Academy of Sciences—Mathematical Sciences, vol. 112, no. 1, pp. 3–30, 2002.

[8] D. Chen and Q.-M. Cheng, "Extrinsic estimates for eigenvalues of the Laplace operator," *Journal of the Mathematical Society of Japan*, vol. 60, no. 2, pp. 325–339, 2008.

[9] Q.-M. Cheng and H. C. Yang, "Estimates on eigenvalues of Laplacian," *Mathematische Annalen*, vol. 331, no. 2, pp. 445–460, 2005.

[10] Q.-M. Cheng and H. Yang, "Inequalities for eigenvalues of Laplacian on domains and compact complex hypersurfaces in complex projective spaces," *Journal of the Mathematical Society of Japan*, vol. 58, no. 2, pp. 545–561, 2006.

[11] M. P. do Carmo, Q. Wang, and C. Xia, "Inequalities for eigenvalues of elliptic operators in divergence form on Riemannian manifolds," *Annali di Matematica Pura ed Applicata*, vol. 189, no. 4, pp. 643–660, 2010.

[12] E. M. Harrell II, "Commutators, eigenvalue gaps, and mean curvature in the theory of Schrödinger operators," *Communications in Partial Differential Equations*, vol. 32, no. 1–3, pp. 401–413, 2007.

[13] P. Li, "Eigenvalue estimates on homogeneous manifolds," *Commentarii Mathematici Helvetici*, vol. 55, no. 3, pp. 347–363, 1980.