Research Article

Construction of Analytic Solution for Time-Fractional Boussinesq Equation Using Iterative Method

Fei Xu, Yixian Gao, and Weipeng Zhang

School of Mathematics and Statistics, Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun, Jilin 130024, China

Correspondence should be addressed to Yixian Gao; gaoyx643@nenu.edu.cn

This paper is aimed at constructing analytical solution for both linear and nonlinear time-fractional Boussinesq equations by an iterative method. By the iterative process, we can obtain the analytic solution of the fourth-order time-fractional Boussinesq equation in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^n$, the sixth-order time-fractional Boussinesq equation, and the $2n$th-order time-fractional Boussinesq equation in $\mathbb{R}$. Through these examples, it shows that the method is simple and effective.

1. Introduction

Many phenomena in the physical, chemical, and biological sciences as well as in technologies are governed by differential equations. The idea of derivatives of noninteger order initially appeared in a letter from Leibniz to L'Hospital in 1695 about the notation $d^n y/dx^n$. L'Hospital posed a question to Leibniz: "what would the result be if $n = 1/2$?" Leibniz replied [1] "it follows that $d^{1/2} x$ will be equal to $x\sqrt{dx : x}$. This is an apparent paradox, from which, one day useful consequences will be drawn." In these words, fractional calculus was born. And so most authors on this topic will cite a particular date (September 30, 1695) as the birthday of so called "fractional calculus" [2]. However, at that time, there are few specific models based on this kind of derivative, so the study of fractional calculus attracts little attention.

Recently, fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in physics, biology, engineering, signal processing, systems identification, control theory, finance, and fractional dynamics, such as chaotic dynamics [3], mechanics of non-Hamiltonian systems [4], anomalous diffusion and transport theory [5], astrophysics [6], physical kinetics [7], plasma physics [8, 9], mechanics of fractional media [10], quantum mechanics [11], theory of long range interaction [12].

The Boussinesq equations arise in hydrodynamics to describe propagation of waves in nonlinear and dissipative media [13, 14]. They are suitable for problems in the percolation of water in porous subsurface strata and widely used in coastal and ocean engineering. Also, Boussinesq equations are the basis of several models used to describe unconfined groundwater flow and subsurface drainage problems. Recently, fractional differential equations have attracted many researchers' interest because of their ability to model particle transport in heterogeneous media and complex phenomena. The fractional Boussinesq equations are suitable for studying the water propagation through heterogeneous porous media. A fractional Boussinesq equation is obtained assuming power law changes of flux in a control volume and using a fractional Taylor series [15]. The fractional differential equations have been solved using several methods such as Laplace transformation method, Fourier transformation method, and operational method [16, 17]. In [18], El-Wakil and Abulwafa used the fractional variational principles and obtained solutions (which are described as periodic, soliton, and explosive waves) of the fractional Boussinesq equation. In [19], based on the finite volume and finite element methods, Zhuang et al. gave two novel numerical methods with a nonlocal operator (using nodal basis functions) for the space-fractional Boussinesq equation. In this paper, we will give a new iterative method to obtain the analytic solution of

the fourth-, sixth- and $2n$th-order fractional Boussinesq equation.

At first, we list some definitions that will be used in the theory of fractional differential equations [2].

Definition 1. The Gamma function is as follows:

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt, \quad (\operatorname{Re}(x) > 0). \tag{1}$$

Definition 2. The Beta function is as follows:

$$B(p,q) := \int_0^1 t^{p-1} (1-t)^{q-1} \, dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{2}$$

$$(\operatorname{Re}(p) > 0, \ \operatorname{Re}(q) > 0).$$

Definition 3. The Mittag-Leffler function is as follows:

$$E_\alpha(x) := \sum_{k=0}^\infty \frac{x^k}{\Gamma(\alpha k + 1)}, \quad (\alpha > 0). \tag{3}$$

Definition 4. The Riemann-Liouville fractional integral of order α ($\alpha > 0$) of a function $u(x,t)$ is denoted by $I_t^\alpha u(x,t)$ and defined as

$$I_t^\alpha u(x,t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(x,\tau) \, d\tau, \quad t > 0. \tag{4}$$

Definition 5. The Caputo partial fractional derivative of a function $u(x,t)$ is denoted by $D_t^\alpha u(x,t)$ and defined as

$$D_t^\alpha u(x,t)$$

$$:= \begin{cases} \dfrac{\partial^m}{\partial t^m} u(x,t), & \alpha = m, \ m \in \mathbb{N}; \\ I_t^{m-\alpha} \dfrac{\partial^m}{\partial t^m} u(x,t), & m-1 < \alpha < m, \ m \in \mathbb{N}. \end{cases} \tag{5}$$

Using Definitions 4 and 5, as in [20], we derive

$$I_t^\alpha D_t^\alpha u(x,t) = u(x,t) - \sum_{k=0}^{m-1} \left(\frac{\partial^k u(x,t)}{\partial t^k} \Big|_{t=0} \right) \frac{t^k}{k!}, \tag{6}$$

$$x \in \mathbb{R}^n, \ t > 0,$$

where $m-1 < \alpha \leqslant m, m \in \mathbb{N}$.

Lemma 6. *According to Definition 4, one has*

$$I_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}. \tag{7}$$

Proof. Consider

$$I_t^\alpha t^\beta = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^\beta d\tau$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t \left(1 - \left(\frac{\tau}{t}\right)\right)^{\alpha-1} \left(\frac{\tau}{t}\right)^{(\beta+1)-1} d\left(\frac{\tau}{t}\right) t^{\alpha+\beta}$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^1 x^{\alpha-1} (1-x)^{(\beta+1)-1} \, dx t^{\alpha+\beta} \tag{8}$$

$$= \frac{1}{\Gamma(\alpha)} B(\alpha, \beta+1) t^{\alpha+\beta}$$

$$= \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}.$$

$$\square$$

We will give an iterative method for the general functional differential equation [21]. Using this iterative process, we can construct the solution of the fractional differential equation.

Theorem 7. *Consider the functional equation*

$$u(x,t) = f(x,t) + L(u(x,t)) + N(u(x,t)), \tag{9}$$

where $f(x,t)$ is a known function, $(x,t) \in D = \{(x,t) : x \in \mathbb{R}^n, n \in \mathbb{N}, t \in (0, +\infty)\}$, and L and N are linear and nonlinear operators from a Banach space B to itself. When $u(x,t)$ is analytical about t, the solution of the functional equation (9) can be written into the series form:

$$u(x,t) = \sum_{n=0}^\infty u_n(x,t) = u_0 + u_1 + u_2 + \cdots, \tag{10}$$

where

$$u_0 = f,$$

$$u_1 = L(u_0) + N(u_0),$$

$$u_{m+1} = L(u_m) + \left(N\left(\sum_{n=0}^m u_n\right) - N\left(\sum_{n=0}^{m-1} u_n\right) \right), \tag{11}$$

$$(m = 1, 2 \cdots).$$

If the operators L and N are contractive, then the series $\sum_{n=0}^\infty u_n(x,t)$ converges absolutely and uniformly.

Proof. Since $u(x,t) = \sum_{n=0}^\infty u_n(x,t) = u_0 + u_1 + u_2 + \cdots$ and operator N is a nonlinear operator, N can be decomposed as follows:

$$N(u(x,t)) = N\left(\sum_{n=0}^\infty u_n(x,t)\right)$$

$$= N(u_0) \tag{12}$$

$$+ \sum_{n=1}^\infty \left(N\left(\sum_{j=0}^n u_j\right) - N\left(\sum_{j=0}^{n-1} u_j\right) \right).$$

For the linear operator L, it can also be written into the form

$$
\begin{aligned}
L\left(u\left(x,t\right)\right) &= L\left(\sum_{n=0}^{\infty} u_n\left(x,t\right)\right) \\
&= L\left(u_0\right) \\
&\quad + \sum_{n=1}^{\infty}\left(N\left(\sum_{j=0}^{n} u_j\right) - N\left(\sum_{j=0}^{n-1} u_j\right)\right).
\end{aligned}
\tag{13}
$$

Set the operator

$$
M\left(u\left(x,t\right)\right) := L\left(u\left(x,t\right)\right) + N\left(u\left(x,t\right)\right). \tag{14}
$$

According to (12), (13), and (14), we have

$$
\begin{aligned}
M\left(u\right) &= M\left(u_0\right) \\
&\quad + \sum_{n=1}^{\infty}\left(M\left(\sum_{j=0}^{n} u_j\right) - M\left(\sum_{j=0}^{n-1} u_j\right)\right).
\end{aligned}
\tag{15}
$$

According to (11), (14), and (15), we can get

$$
\begin{aligned}
u_0 &= f, \\
u_1 &= M\left(u_0\right), \\
u_{m+1} &= M\left(\sum_{n=0}^{m} u_n\right) - M\left(\sum_{n=0}^{m-1} u_n\right), \quad (m = 1, 2 \cdots).
\end{aligned}
\tag{16}
$$

Since operators L and N are contractive, M is also contractive; that is, there exists a constant $0 < K < 1$, such that

$$
\left\| M\left(v_i\right) - M\left(v_j\right) \right\| \le K \left\| v_i - v_j \right\|, \quad \forall v_i, v_j \in B, \tag{17}
$$

where $\|\cdot\|$ denotes the norm on Banach space B. Finally, let us give the estimation of u_{m+1}:

$$
\begin{aligned}
\left\| u_{m+1} \right\| &= \left\| M\left(\sum_{n=0}^{m} u_n\right) - M\left(\sum_{n=0}^{m-1} u_n\right) \right\| \le K \left\| u_m \right\| \\
&\le K^{m+1} \left\| u_0 \right\|.
\end{aligned}
\tag{18}
$$

Since $0 < K < 1$, the series $\sum_{m=1}^{\infty} K^{m+1} \|u_0\|$ converges absolutely as well as uniformly.

According to Weierstrass' criterion, we show that the series $\sum_{i=0}^{\infty} u_i$ converges absolutely as well as uniformly. $\qquad \square$

In fact, using (11) along with Theorem 7, a new iterative method for constructing analytical solutions of the functional equation (12) is given:

$$
u\left(x,t\right) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots, \tag{19}
$$

where

$$
\begin{aligned}
u_0 &= f, \\
u_1 &= L\left(u_0\right) + N\left(u_0\right), \\
u_2 &= L\left(u_1\right) + \left(N\left(u_0 + u_1\right) - N\left(u_0\right)\right), \\
u_3 &= L\left(u_2\right) + \left(N\left(u_0 + u_1 + u_2\right) - N\left(u_0 + u_1\right)\right), \\
&\vdots, \\
u_{m+1} &= L\left(u_m\right) + \left(N\left(u_0 + u_1 + u_2 + \cdots + u_m\right)\right. \\
&\quad \left. - N\left(u_0 + u_1 + \cdots + u_{m-1}\right)\right), \\
&\vdots.
\end{aligned}
\tag{20}
$$

According to Theorem 7, we know that the iterative process is reasonable. In the following, we apply the iterative method to the linear and nonlinear time-fractional Boussinesq equation.

2. Application to the Time-Fractional Boussinesq Equations

In this section, using the iterative method, we will construct the solution of the time-fractional Boussinesq equation.

2.1. Fourth-Order Time-Fractional Boussinesq Equation in $\mathbb{R}$. Consider the general fourth-order time-fractional Boussinesq equation with one-dimensional space variable

$$
\begin{aligned}
D_t^{(\alpha)} u\left(x,t\right) &= \beta D_x^{(4)} u\left(x,t\right) + \gamma D_x^{(2)} u\left(x,t\right) \\
&\quad + \theta D_x^{(2)} u^2\left(x,t\right) - 4\theta u^2\left(x,t\right),
\end{aligned}
\tag{21}
$$

with the initial conditions

$$
\begin{aligned}
u\left(x,0\right) &= e^x, \\
u_t\left(x,0\right) &= 0, \\
x &\in \mathbb{R}, \quad t > 0,
\end{aligned}
\tag{22}
$$

where $1 < \alpha \le 2$ describing the order of the fractional time derivative, $u(x,t)$ is a field function, and β, γ, and θ are constant coefficients.

Applying I_t^α to both sides of (21), we derive

$$
\begin{aligned}
I_t^\alpha D_t^\alpha u\left(x,t\right) &= \beta I_t^\alpha D_x^{(4)} u\left(x,t\right) + \gamma I_t^\alpha D_x^{(2)} u\left(x,t\right) \\
&\quad + \theta I_t^\alpha D_x^{(2)} u^2\left(x,t\right) - 4\theta I_t^\alpha u^2\left(x,t\right).
\end{aligned}
\tag{23}
$$

Denoting

$$
\begin{aligned}
L\left(u\left(x,t\right)\right) &= \beta I_t^\alpha D_x^{(4)} u\left(x,t\right) + \gamma I_t^\alpha D_x^{(2)} u\left(x,t\right), \\
N\left(u\left(x,t\right)\right) &= \theta I_t^\alpha D_x^{(2)} u^2\left(x,t\right) - 4\theta I_t^\alpha u^2\left(x,t\right)
\end{aligned}
\tag{24}
$$

and using (9), we can get

$$
I_t^\alpha D_t^\alpha u\left(x,t\right) = L\left(u\left(x,t\right)\right) + N\left(u\left(x,t\right)\right). \tag{25}
$$

According to the iterative method (19)-(20), we obtain

$$u_0 = e^x,$$

$$u_1 = e^x \frac{(\beta + \gamma) t^\alpha}{\Gamma(\alpha + 1)},$$

$$u_2 = e^x \frac{((\beta + \gamma) t^\alpha)^2}{\Gamma(2\alpha + 1)}, \dots .$$ (26)

Generally,

$$u_m = e^x \frac{((\beta + \gamma) t^\alpha)^m}{\Gamma(m\alpha + 1)}.$$ (27)

So the solution of (21) is

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} e^x \frac{((\beta + \gamma) t^\alpha)^n}{\Gamma(n\alpha + 1)}$$

$$= e^x E_\alpha((\beta + \gamma) t^\alpha).$$ (28)

Remark 8. Particularly, if $\alpha = 2$, then the fourth-order time-fractional Boussinesq equation is the regular Boussinesq equation. The solution is $u(x,t) = e^x E_2((\beta + \gamma)t^2)$, which is consistent with the solutions in [22].

Remark 9. If $\beta = 0$, $\gamma \neq 0$, then the fourth-order time-fractional Boussinesq equation becomes fractional wave equation, and the solution is $u(x,t) = e^x E_\alpha(\gamma t^\alpha)$.

Remark 10. If $\theta = 0$, then the fourth-order time-fractional Boussinesq equation becomes fractional linear beam equation, and the solution is $u(x,t) = e^x E_\alpha((\beta + \gamma)t^\alpha)$.

2.2. Fourth-Order Time-Fractional Boussinesq Equation in $\mathbb{R}^2$. Consider the fourth-order time-fractional Boussinesq equation with two-dimensional space variables

$$D_t^{(\alpha)} u(\mathbf{x}, t) = \beta_1 D_{x_1}^{(4)} u(\mathbf{x}, t) + \beta_2 D_{x_2}^{(4)} u(\mathbf{x}, t)$$

$$+ \gamma_1 D_{x_1}^{(2)} u(\mathbf{x}, t) + \gamma_2 D_{x_2}^{(2)} u(\mathbf{x}, t)$$

$$+ \theta_1 D_{x_1}^{(2)} u^2(\mathbf{x}, t) + \theta_2 D_{x_2}^{(2)} u^2(\mathbf{x}, t)$$

$$- 4\theta_1 u^2(\mathbf{x}, t) - 4\theta_2 u^2(\mathbf{x}, t),$$

$$\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \ t > 0,$$ (29)

with the initial conditions

$$u(x_1, x_2, 0) = e^{x_1 + x_2},$$

$$u_t(x_1, x_2, 0) = 0,$$ (30)

where $1 < \alpha \leq 2$ and the coefficients β_i, γ_i, and $\theta_i \in \mathbb{R}$, $(i = 1, 2)$.

Applying I_t^α to both sides of (29), we derive

$$I_t^\alpha D_t^\alpha u(\mathbf{x}, t) = \beta_1 I_t^\alpha D_{x_1}^{(4)} u(\mathbf{x}, t) + \beta_2 I_t^\alpha D_{x_2}^{(4)} u(\mathbf{x}, t)$$

$$+ \gamma_1 I_t^\alpha D_{x_1}^{(2)} u(\mathbf{x}, t) + \gamma_2 I_t^\alpha D_{x_2}^{(2)} u(\mathbf{x}, t)$$

$$+ \theta_1 I_t^\alpha D_{x_1}^{(2)} u^2(\mathbf{x}, t) + \theta_2 I_t^\alpha D_{x_2}^{(2)} u^2(\mathbf{x}, t)$$

$$- 4\theta_1 I_t^\alpha u^2(\mathbf{x}, t) - 4\theta_2 I_t^\alpha u^2(\mathbf{x}, t).$$ (31)

Now, let us denote that

$$L(u(\mathbf{x}, t)) := \beta_1 I_t^\alpha D_{x_1}^{(4)} u(\mathbf{x}, t) + \beta_2 I_t^\alpha D_{x_2}^{(4)} u(\mathbf{x}, t)$$

$$+ \gamma_1 I_t^\alpha D_{x_1}^{(2)} u(\mathbf{x}, t)$$

$$+ \gamma_2 I_t^\alpha D_{x_2}^{(2)} u(\mathbf{x}, t),$$ (32)

$$N(u(\mathbf{x}, t)) := \theta_1 I_t^\alpha D_{x_1}^{(2)} u^2(\mathbf{x}, t) + \theta_2 I_t^\alpha D_{x_2}^{(2)} u^2(\mathbf{x}, t)$$

$$- 4\theta_1 I_t^\alpha u^2(\mathbf{x}, t) - 4\theta_2 I_t^\alpha u^2(\mathbf{x}, t).$$

So we can get

$$I_t^\alpha D_t^\alpha u(\mathbf{x}, t) = L(u(\mathbf{x}, t)) + N(u(\mathbf{x}, t)).$$ (33)

According to the iterative method (19)-(20), we can get

$$u_0 = e^{x_1 + x_2},$$

$$u_1 = e^{x_1 + x_2} \frac{(\beta_1 + \beta_2 + \gamma_1 + \gamma_2) t^\alpha}{\Gamma(\alpha + 1)},$$

$$u_2 = e^{x_1 + x_2} \frac{((\beta_1 + \beta_2 + \gamma_1 + \gamma_2) t^\alpha)^2}{\Gamma(2\alpha + 1)}, \dots .$$ (34)

Generally,

$$u_m = e^{x_1 + x_2} \frac{((\beta_1 + \beta_2 + \gamma_1 + \gamma_2) t^\alpha)^m}{\Gamma(m\alpha + 1)}.$$ (35)

So the solution of (29) is

$$u(\mathbf{x}, t) = \sum_{n=0}^{\infty} u_n(\mathbf{x}, t)$$

$$= \sum_{n=0}^{\infty} e^{x_1 + x_2} \frac{((\beta_1 + \beta_2 + \gamma_1 + \gamma_2) t^\alpha)^n}{\Gamma(n\alpha + 1)}$$

$$= e^{x_1 + x_2} E_\alpha((\beta_1 + \beta_2 + \gamma_1 + \gamma_2) t^\alpha).$$ (36)

2.3. Fourth-Order Time-Fractional Boussinesq Equation in $\mathbb{R}^n$. Consider the fourth-order time-fractional Boussinesq equation with n-dimensional space variables

$$D_t^{(\alpha)} u(\mathbf{x}, t) = \sum_{i=0}^{n} \beta_i D_{x_i}^{(4)} u(\mathbf{x}, t) + \sum_{i=0}^{n} \gamma_i D_{x_i}^{(2)} u(\mathbf{x}, t)$$

$$+ \sum_{i=0}^{n} \theta_i D_{x_i}^{(2)} u^2(\mathbf{x}, t) - \sum_{i=0}^{n} 4\theta_i u^2(\mathbf{x}, t),$$ (37)

with the initial conditions

$$u(\mathbf{x}, 0) = \exp\left(\sum_{i=0}^{n} x_i\right),$$

$$u_t(\mathbf{x}, 0) = 0,$$

(38)

where $1 < \alpha \le 2$, $\beta_i, \gamma_i, \theta_i \in \mathbb{R}$, $(i = 1, 2, \ldots, n)$, and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, t > 0$.

Applying I_t^α to both sides of (37), we have

$$I_t^\alpha D_t^\alpha u(\mathbf{x}, t) = \sum_{i=0}^{n} \beta_i I_t^\alpha D_{x_i}^{(4)} u(\mathbf{x}, t) + \sum_{i=0}^{n} \gamma_i I_t^\alpha D_{x_i}^{(2)} u(\mathbf{x}, t)$$

$$+ \sum_{i=0}^{n} \theta_i I_t^\alpha D_{x_i}^{(2)} u^2(\mathbf{x}, t)$$

(39)

$$- 4 \sum_{i=0}^{n} \theta_i I_t^\alpha u^2(\mathbf{x}, t).$$

Denoting

$$L(u(\mathbf{x}, t)) = \sum_{i=0}^{n} \beta_i I_t^\alpha D_{x_i}^{(4)} u(\mathbf{x}, t) + \sum_{i=0}^{n} \gamma_i I_t^\alpha D_{x_i}^{(2)} u(x, t),$$

$$N(u(\mathbf{x}, t)) = \sum_{i=0}^{n} \theta_i I_t^\alpha D_{x_i}^{(2)} u^2(\mathbf{x}, t) - 4 \sum_{i=0}^{n} \theta_i I_t^\alpha u^2(\mathbf{x}, t),$$

(40)

we can get

$$I_t^\alpha D_t^\alpha u(\mathbf{x}, t) = L(u(\mathbf{x}, t)) + N(u(\mathbf{x}, t)).$$

(41)

According to the iterative method (19)-(20), we obtain

$$u_0 = \exp\left(\sum_{i=0}^{n} x_i\right),$$

$$u_1 = \exp\left(\sum_{i=0}^{n} x_i\right) \frac{\left(\sum_{i=0}^{n} \beta_i + \sum_{i=0}^{n} \gamma_i\right) t^\alpha}{\Gamma(\alpha + 1)},$$

(42)

$$u_2 = \exp\left(\sum_{i=0}^{n} x_i\right) \frac{\left(\left(\sum_{i=0}^{n} \beta_i + \sum_{i=0}^{n} \gamma_i\right) t^\alpha\right)^2}{\Gamma(2\alpha + 1)}, \ldots.$$

Generally,

$$u_m = \exp\left(\sum_{i=0}^{n} x_i\right) \frac{\left(\left(\sum_{i=0}^{n} \beta_i + \sum_{i=0}^{n} \gamma_i\right) t^\alpha\right)^m}{\Gamma(m\alpha + 1)}.$$

(43)

So the solution of (37) is

$$u(\mathbf{x}, t) = \sum_{n=0}^{\infty} u_n(\mathbf{x}, t)$$

$$= \sum_{n=0}^{\infty} \exp\left(\sum_{i=0}^{n} x_i\right) \frac{\left(\left(\sum_{i=0}^{n} \beta_i + \sum_{i=0}^{n} \gamma_i\right) t^\alpha\right)^m}{\Gamma(m\alpha + 1)}$$

(44)

$$= \exp\left(\sum_{i=0}^{n} x_i\right) E_\alpha\left(\left(\sum_{i=0}^{n} \beta_i + \sum_{i=0}^{n} \gamma_i\right) t^\alpha\right).$$

2.4. Sixth-Order Time-Fractional Boussinesq Equation in $\mathbb{R}$*.* Let us consider the sixth-order time-fractional Boussinesq equation

$$D_t^{(\alpha)} u(x, t) = D_x^{(6)} u(x, t) + D_x^{(4)} u(x, t) + D_x^{(2)} u(x, t)$$

$$+ \theta D_x^{(2)} u^2(x, t) - 4\theta u^2(x, t),$$

(45)

where $x \in \mathbb{R}, t > 0, 1 < \alpha \le 2$, and the parameter $\theta \in \mathbb{R}$, with the initial conditions

$$u(x, 0) = e^x,$$

$$u_t(x, 0) = 0.$$

(46)

Applying I_t^α to both sides of (45), we have

$$I_t^\alpha D_t^\alpha u(x, t) = I_t^\alpha D_x^{(6)} u(x, t) + I_t^\alpha D_x^{(4)} u(x, t)$$

$$+ I_t^\alpha D_x^{(2)} u(x, t) + \theta I_t^\alpha D_x^{(2)} u^2(x, t)$$

$$- I_t^\alpha 4\theta u^2(x, t).$$

(47)

Denote that

$$L(u(x, t)) = I_t^\alpha D_x^{(6)} u(x, t) + I_t^\alpha D_x^{(4)} u(x, t)$$

$$+ \theta I_t^\alpha D_x^{(2)} u(x, t),$$

(48)

$$N(u(x, t)) = \theta I_t^\alpha D_x^{(2)} u^2(x, t) - I_t^\alpha 4\theta u^2(x, t).$$

So we can get

$$I_t^\alpha D_t^\alpha u(x, t) = L(u(x, t)) + N(u(x, t)).$$

(49)

According to the iterative method (19)-(20), we have

$$u_0 = e^x,$$

$$u_1 = e^x \frac{3t^\alpha}{\Gamma(\alpha + 1)},$$

(50)

$$u_2 = e^x \frac{(3t^\alpha)^2}{\Gamma(2\alpha + 1)}, \ldots.$$

Generally,

$$u_m = e^x \frac{(3t^\alpha)^m}{\Gamma(m\alpha + 1)}.$$

(51)

So, the solution of (45) is

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = \sum_{n=0}^{\infty} e^x \frac{(3t^\alpha)^n}{\Gamma(n\alpha + 1)}$$

$$= e^x E_\alpha(3t^\alpha).$$

(52)

Remark 11. If $\alpha = 2$, the sixth-order time-fractional Boussinesq equation is the regular Boussinesq equation, and the solution is $u(x, t) = e^x E_2(3t^2)$.

2.5. 2nth-Order Time-Fractional Boussinesq Equation in $\mathbb{R}$. Consider the 2nth-order time-fractional Boussinesq equation

$$
\begin{aligned}
D_t^{(\alpha)} u(x,t) = {} & D_t^{(2n)} u(x,t) + D_t^{(2n-2)} u(x,t) + \cdots \\
& + D_t^{(2)} u(x,t) + \theta D_x^{(2)} u^2(x,t) \\
& - 4\theta u^2(x,t),
\end{aligned}
\tag{53}
$$

where $1 < \alpha \le 2$, $t > 0$, and $x \in \mathbb{R}$, with the initial conditions

$$
\begin{aligned}
u(x,0) &= e^x, \\
u_t(x,0) &= 0.
\end{aligned}
\tag{54}
$$

And we apply I_t^α to both sides of (53):

$$
\begin{aligned}
I_t^\alpha D_t^\alpha u(x,t) = {} & I_t^\alpha D_x^{(2n)} u(x,t) + \cdots + I_t^\alpha D_x^{(2)} u(x,t) \\
& + \theta I_t^\alpha D_x^{(2)} u^2(x,t) - I_t^\alpha 4\theta u^2(x,t).
\end{aligned}
\tag{55}
$$

Denoting

$$
\begin{aligned}
L(u(x,t)) = {} & I_t^\alpha D_x^{(2n)} u(x,t) + I_t^\alpha D_x^{(2n-2)} u(x,t) + \cdots \\
& + I_t^\alpha D_x^{(2)} u(x,t),
\end{aligned}
\tag{56}
$$

$$
N(u(x,t)) = \theta I_t^\alpha D_x^{(2)} u^2(x,t) - I_t^\alpha 4\theta u^2(x,t),
$$

we can get

$$
I_t^\alpha D_t^\alpha u(x,t) = L(u(x,t)) + N(u(x,t)).
\tag{57}
$$

According to iterative method (19)-(20), we derive

$$
\begin{aligned}
u_0 &= e^x, \\
u_1 &= e^x \frac{n t^\alpha}{\Gamma(\alpha+1)}, \\
u_2 &= e^x \frac{(n t^\alpha)^2}{\Gamma(2\alpha+1)}, \dots .
\end{aligned}
\tag{58}
$$

Generally,

$$
u_m = e^x \frac{(n t^\alpha)^m}{\Gamma(m\alpha+1)}.
\tag{59}
$$

So, the solution of (53) is

$$
u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) = \sum_{i=0}^{\infty} e^x \frac{(n t^\alpha)^i}{\Gamma(i\alpha+1)} = e^x E_\alpha(n t^\alpha).
\tag{60}
$$

Remark 12. If $\alpha = 2$, the 2nth-order time-fractional Boussinesq equation is the regular Boussinesq equation, and the solution is $u(x,t) = e^x E_2(n t^2)$.

3. Conclusion

For a given order time-fractional Boussinesq equation with the initial value $u(x,0) = e^x$, $u_t(x,0) = 0$, such as the fourth-order time-fractional Boussinesq equation, the sixth-order time-fractional Boussinesq equation, and the 2nth-order time-fractional Boussinesq equation, by constructing iterative processes, we can get the analytical solution successfully. This method is effective and direct, which can be applied to other kinds of nonlinear evolution equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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