Hindawi Publishing Corporation
Advances in Mathematical Physics
Volume 2015, Article ID 632745, 5 pages
http://dx.doi.org/10.1155/2015/632745

Research Article

Eigenvalues for a Neumann Boundary Problem Involving the $p(x)$-Laplacian

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Received 24 November 2014; Accepted 4 March 2015

Academic Editor: Remi Léandre

We study the existence of weak solutions to the following Neumann problem involving the $p(x)$-Laplacian operator: $-\Delta_{p(x)}u + e(x)|u|^{p(x)-2}u = \lambda a(x) f(u)$, in Ω, $\partial u/\partial \nu = 0$, on $\partial\Omega$. Under some appropriate conditions on the functions p, e, a, and f, we prove that there exists $\overline{\lambda} > 0$ such that any $\lambda \in (0, \overline{\lambda})$ is an eigenvalue of the above problem. Our analysis mainly relies on variational arguments based on Ekeland's variational principle.

1. Introduction

This paper is considered with the existence of weak solutions to the following Neumann problem:

$$\begin{gathered} -\Delta_{p(x)}u + e(x)|u|^{p(x)-2}u = \lambda a(x) f(u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with a smooth boundary, $p(x) \in C(\overline{\Omega})$ with $p(x) > 1$ on $\overline{\Omega}$, and $\lambda > 0$ is a parameter. $e(x) \in C(\overline{\Omega})$ is nonnegative, $f \in C(\mathbb{R})$, and $a(x) \in L^{r(x)}(\Omega)$ for some $r \in C(\overline{\Omega})$.

The operator $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called a $p(x)$-Laplacian. If $p(x) \equiv p$ is a constant, then operator is the well-known p-Laplacian, and (1) is the usual p-Laplacian equation. As a matter of fact, the $p(x)$-Laplacian has more complicated nonlinearities than the p-Laplacian. For example, it is not homogeneous. The study of differential equations with $p(x)$-growth conditions is an interesting and attractive topic and has been the object of considerable attention in recent years [1–6].

In [3], by applying a variational principle due to B. Ricceri and the theory of the variable exponent Sobolev spaces, the author considered the Neumann problem of $p(x)$-Laplacian:

$$\begin{gathered} -\Delta_{p(x)}u + \lambda(x)|u|^{p(x)-2}u = f(x,u) + g(x,u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2}$$

In [5], based on the three critical points theorem due to B. Ricceri, the author studied the problem

$$\begin{gathered} -\Delta_{p(x)}u + e(x)|u|^{p(x)-2}u \\ = \lambda a(x) f(x,u) + \mu g(x,u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3}$$

In [6], the author established the existence of at least three solutions of the problem

$$\begin{gathered} -\Delta_{p(x)}u + |u|^{p(x)-2}u = \lambda\alpha(x) f(u) + \beta(x) g(u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4}$$

However, in [3, 5, 6], the function $p(x)$ assumed that $p^- > N$. In this paper, by using Ekeland's variational principle, we show that when $1 < p(x) \le N$, there exists $\overline{\lambda} > 0$ such that any $\lambda \in (0, \overline{\lambda})$; problem (1) has at least one nontrivial weak solution. For more applications of Ekeland's variational principle to other problems, see, for example, [7–10]. Our result is partly motivated by these nice papers.

This paper is organized as follows. In Section 2, we recall some basic facts about the variable exponent Lebesgue and Sobolev spaces and also give some propositions that will be used later. In Section 3, we obtain the existence of weak solutions of problem (1).

2. Preliminaries

In order to deal with the $p(x)$-Laplacian problem, we need some theories on spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which will be used later. Let

$$C_+(\overline{\Omega}) = \{h \mid h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for } x \text{ on } \overline{\Omega}\}. \quad (5)$$

Through this paper, for any $h \in C(\overline{\Omega})$, we use the notations

$$h^+ := \sup_{x\in\overline{\Omega}} h(x), \qquad h^- := \inf_{x\in\overline{\Omega}} h(x); \quad (6)$$

denote

$$L^{p(x)}(\Omega) = \Big\{u \mid u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)}\, dx < \infty\Big\}. \quad (7)$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\left\{\lambda > 0 \mid \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}, \quad (8)$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space; we call it a generalized Lebesgue space.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\}, \quad (9)$$

and it can be equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega). \quad (10)$$

The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ are separable, reflexive Banach spaces [11].

From now on, we denote X by the space $W^{1,p(x)}(\Omega)$. For any $u \in X$, define

$$\|u\|_e = \inf\left\{\lambda > 0 \mid \int_\Omega \left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} + e(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}; \quad (11)$$

then, it is easy to see that $\|u\|_e$ is a norm on X and equivalent to $\|u\|$. In the following, we will use $\|u\|_e$ instead of $\|u\|$ on X.

Proposition 1 (see [11, 12]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p^0(x)}(\Omega)$, where $1/p(x) + 1/p^0(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p^0(x)}(\Omega)$, one has the following Hölder-type inequality:*

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{(p^0)^-}\right)|u|_{p(x)}|v|_{p^0(x)} \le 2|u|_{p(x)}|v|_{p^0(x)}. \quad (12)$$

Moreover, if $h_i \in C_+(\overline{\Omega})$ with $1/h_1(x) + 1/h_2(x) + 1/h_3(x) = 1$, then, for $u \in L^{h_1(x)}(\Omega)$, $v \in L^{h_2(x)}(\Omega)$, $w \in L^{h_3(x)}(\Omega)$, one has

$$\left|\int_\Omega uvw\, dx\right| \le \left(\frac{1}{h_1^-} + \frac{1}{h_2^-} + \frac{1}{h_3^-}\right)|u|_{h_1(x)}|v|_{h_2(x)}|w|_{h_3(x)} \le 3|u|_{h_1(x)}|v|_{h_2(x)}|w|_{h_3(x)}. \quad (13)$$

Proposition 2 (see [11]). *Put $\rho(u) = \int_\Omega |u|^{p(x)}dx$, $\forall u \in L^{p(x)}(\Omega)$; then*

(i) $|u|_{p(x)} < 1\ (= 1; > 1) \Leftrightarrow \rho(u) < 1\ (= 1; > 1)$;

(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$; $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$;

(iii) $|u|_{p(x)} \to 0 \Leftrightarrow \rho(u) \to 0$; $|u|_{p(x)} \to \infty \Leftrightarrow \rho(u) \to \infty$.

From Proposition 2, the following inequalities hold:

$$\|u\|_e^{p^-} \le \int_\Omega |\nabla u(x)|^{p(x)} + e(x)|u(x)|^{p(x)}\, dx \le \|u\|_e^{p^+}, \quad \text{if } \|u\|_e \ge 1; \quad (14)$$

$$\|u\|_e^{p^+} \le \int_\Omega |\nabla u(x)|^{p(x)} + e(x)|u(x)|^{p(x)}\, dx \le \|u\|_e^{p^-}, \quad \text{if } \|u\|_e \le 1. \quad (15)$$

Proposition 3 (see [13]). *Let $q \in L^\infty(\Omega)$ be such that $1 \le p(x)q(x) \le \infty$ for a.e. $x \in \Omega$. Let $u \in L^{p(x)}(\Omega)$, $u \ne 0$. Then, one has the following:*

(i) *if $|u(x)|_{p(x)q(x)} \le 1$, then $|u|_{p(x)q(x)}^{q^+} \le \||u|^{q(x)}|_{p(x)} \le |u|_{p(x)q(x)}^{q^-}$;*

(ii) *if $|u(x)|_{p(x)q(x)} \ge 1$, then $|u|_{p(x)q(x)}^{q^-} \le \||u|^{q(x)}|_{p(x)} \le |u|_{p(x)q(x)}^{q^+}$.*

For any $x \in \overline{\Omega}$, let

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \ge N. \end{cases} \quad (16)$$

Proposition 4 (see [11, 14]). *Assume that $1 \leq q(x) \in C(\overline{\Omega})$ satisfy $q(x) < p^*(x)$ on $\overline{\Omega}$; then the imbedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.*

We define $\Phi : X \to R$ by

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} + \frac{e(x)}{p(x)} |u(x)|^{p(x)} dx,$$
$$\Psi(u) = \int_\Omega a(x) F(u) dx, \qquad F(t) = \int_0^t f(s) ds, \tag{17}$$
$$I_\lambda(u) = \Phi(u) - \lambda \Psi(u).$$

Proposition 5 (see [15]). *(i) Φ is weakly lower semicontinuous, $\Phi \in C^1(X, \mathbb{R})$, and*

$$\left(\Phi'(u), v\right) = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v + e(x) |u|^{p(x)-2} uv\, dx, \quad \forall u, v \in X; \tag{18}$$

(ii) Φ' is a mapping of type (S_+); that is, if $u_n \rightharpoonup u$ and $\liminf_{n\to\infty}(\Phi'(u_n), u_n - u) \leq 0$, *then $u_n \to u$;*
(iii) $\Phi' : X \to X^*$ is a homeomorphism.

3. Main Result

We say that $u \in X$ is a weak solution of problem (1) if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v + e(x) |u|^{p(x)-2} uv\, dx = \lambda \int_\Omega a(x) f(u) v\, dx, \quad \forall v \in X. \tag{19}$$

It follows that we can seek for weak solutions of problem (1). We need the following assumptions.

(A) There exist $\gamma_1 \geq \gamma_2 > 0$, $0 < \beta < 1$, and $q_1, q_2, r \in C_+(\overline{\Omega})$ such that

$$1 < q_1(x) \leq q_2(x) < p(x) \leq N < r(x) \quad \text{on } \overline{\Omega}, \tag{20}$$

$$0 \leq F(t) \leq \gamma_1 |t|^{q_1(x)} \quad \text{for } t \in \mathbb{R}, \tag{21}$$

$$F(t) \geq \gamma_2 |t|^{q_2(x)} \quad \text{for } t \in [-\beta, \beta]. \tag{22}$$

(B) Consider $a(x) \in L^{r(x)}(\Omega)$ and there exists a subset $\Omega_1 \subset \Omega$ with $\operatorname{meas}(\Omega_1) > 0$ such that $a(x) > 0$ for $x \in \Omega_1$, where $\operatorname{meas}(\cdot)$ denotes the Lebesgue measure.

Remark 6. Regarding condition (A), we have

$$r^0(x) q_i(x) < p^*(x), \qquad s_i(x) = \frac{r(x) q_i(x)}{r(x) - q_i(x)} < p^*(x), \quad \text{for } i = 1, 2, \tag{23}$$

where $1/r(x) + 1/r^0(x) = 1$. Thus, by Proposition 4, the embeddings $X \hookrightarrow L^{r^0(x)q_i(x)}(\Omega)$ and $X \hookrightarrow L^{s_i(x)}\Omega$, $i = 1, 2$ are continuous and compact.

By a standard argument, we have that Ψ is weakly lower semicontinuous, $\Psi \in C^1(X, (R))$, and

$$\left(\Psi'(u), v\right) = \int_\Omega a(x) f(u) v\, dx, \quad \forall u, v \in X. \tag{24}$$

Remark 7. I_λ is weakly lower semicontinuous, $I_\lambda \in C^1(X, \mathbb{R})$, and

$$\left(I'_\lambda(u), v\right) = \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + e(x) |u|^{p(x)-2} uv\right) dx - \lambda \int_\Omega a(x) f(u) v\, dx, \quad \forall v \in X. \tag{25}$$

Thus, u is weak solution of problem (1) if and only if u is a critical point of I_λ.

Note from Remark 6 that the embedding $X \hookrightarrow L^{r^0(x)q_1(x)}(\Omega)$ is continuous; then, there exists a positive constant $d > 1$ such that

$$|u|_{r^0(x)q_1(x)} \leq d \|u\|_e, \quad \text{for any } u \in X. \tag{26}$$

Lemma 8. *For any $\alpha \in (0, 1/d)$, there exist $\overline{\lambda} > 0$ and $\gamma > 0$ such that $I_\lambda(u) \geq \gamma$ for any $\lambda \in (0, \overline{\lambda})$ and $u \in X$ with $\|u\|_e = \alpha$.*

Proof. Let $\alpha \in (0, 1/d)$ be fixed. Then, $\alpha < 1$, and, from (26), we know that

$$|u|_{r^0(x)q_1(x)} < 1, \quad \text{for any } u \in X, \quad \|u\|_e = \alpha. \tag{27}$$

From Propositions 1 and 3, (15), (21), and (26), it follows that

$$\begin{aligned}
I_\lambda(u) &\geq \frac{1}{p^+} \int_\Omega \left(|\nabla u|^{p(x)} + e(x) |u|^{p(x)}\right) dx - \lambda \int_\Omega |a(x)| F(u) dx \\
&\geq \frac{1}{p^+} \|u\|_e^{p^+} - \lambda\gamma_1 \int_\Omega |a(x)| |u|^{q_1(x)} dx \\
&\geq \frac{1}{p^+} \|u\|_e^{p^+} - 2\lambda\gamma_1 |a(x)|_{r(x)} \left||u|^{q_1(x)}\right|_{r^0(x)} \\
&\geq \frac{1}{p^+} \|u\|_e^{p^+} - 2\lambda\gamma_1 |a|_{r(x)} |u|^{q_1^-}_{r^0(x)q_1(x)} \\
&\geq \frac{1}{p^+} \|u\|_e^{p^+} - 2\lambda\gamma_1 d^{q_1^-} |a|_{r(x)} \|u\|_e^{q_1^-} \\
&= \frac{1}{p^+} \alpha^{p^+} - 2\lambda\gamma_1 d^{q_1^-} |a|_{r(x)} \alpha^{q_1^-}.
\end{aligned} \tag{28}$$

Hence, if we let

$$\overline{\lambda} = \frac{2\alpha^{p^+ - q_1^-}}{5p^+ \gamma_1 d^{q_1^-} |a|_{r(x)}}, \tag{29}$$

then, for any $\lambda \in (0, \overline{\lambda})$ and $u \in X$ with $\|u\|_e = \alpha$, there exists $\gamma = \alpha^{p^+}/5p^+ > 0$ such that $I_\lambda(u) \geq \gamma$. □

Lemma 9. *There exists $\phi \in X$ such that $\phi \geq 0$, $\phi \neq 0$, and $I_\lambda(t\phi) < 0$ for $t > 0$ small enough.*

Proof. From condition (B) and (20), there exists a subset $\Omega_1 \subset \Omega$, and $q_2(x) < p(x)$ on Ω_1. If we let $\widehat{q}_2 = \min_{x\in\overline{\Omega}_1} q_2(x)$ and $\widehat{p} = \min_{x\in\overline{\Omega}_1} p(x)$, then there exists $\delta > 0$ such that $\widehat{q}_2 + \delta < \widehat{p}$. Moreover, since $q_2(x) \in C(\Omega_1)$, there exists an open set $\Omega_2 \subset \Omega_1$, $\text{meas}(\Omega_2) > 0$, and $|q_2(x) - \widehat{q}_2| < \delta$ for $x \in \Omega_2$. Thus, $q_2(x) < \widehat{q}_2 + \delta < \widehat{p}$ in Ω_2.

Let $\phi \in C_0^\infty(\Omega)$ be nontrivial such that $\text{supp}(\phi) \subset \Omega_2 \subset \Omega_1$, $\phi \geq 0$ and $\phi \neq 0$ in Ω_2. From (22), then for $0 < t < \min\{1, \beta/(\max_{x\in\Omega_2}\phi(x))\}$, we have

$$\begin{aligned}
I_\lambda(t\phi) &= \int_\Omega \frac{1}{p(x)} \left(|\nabla(t\phi)|^{p(x)} + e(x)|t\phi|^{p(x)} \right) dx \\
&\quad - \lambda \int_\Omega a(x) F(t\phi)\, dx \\
&= \int_{\Omega_2} \frac{t^{p(x)}}{p(x)} \left(|\nabla(\phi)|^{p(x)} + e(x)|\phi|^{p(x)} \right) dx \\
&\quad - \lambda \int_{\Omega_2} a(x) F(t\phi)\, dx \\
&\leq \frac{t^{\widehat{p}}}{p^-} \int_{\Omega_2} \left(|\nabla(\phi)|^{p(x)} + e(x)|\phi|^{p(x)} \right) dx \\
&\quad - \lambda\gamma_2 \int_{\Omega_2} a(x)|t\phi|^{q_2(x)}\, dx \\
&\leq \frac{t^{\widehat{p}}}{p^-} \int_{\Omega_2} \left(|\nabla(\phi)|^{p(x)} + e(x)|\phi|^{p(x)} \right) dx \\
&\quad - \lambda\gamma_2 t^{\widehat{q}_2+\delta} \int_{\Omega_2} a(x)|\phi|^{q_2(x)}\, dx.
\end{aligned} \tag{30}$$

Since $\int_{\Omega_2}(|\nabla(\phi)|^{p(x)} + e(x)|\phi|^{p(x)})dx > 0$, in fact, if this is not true, $\int_{\Omega_2}(|\nabla(\phi)|^{p(x)} + e(x)|\phi|^{p(x)})dx = 0$; by Proposition 2, we have $\|\phi\|_e = 0$ and so $\phi \equiv 0$ in Ω, which is a contradiction. Hence, $I_\lambda(t\phi) < 0$ for

$$\begin{aligned}
0 < t &< \min\Bigg\{1, \frac{\beta}{\max_{x\in\Omega_2}\phi(x)}, \\
&\quad \left(\frac{\lambda\gamma_2 p^- \int_{\Omega_2} a(x)|\phi|^{q_2(x)}}{\int_{\Omega_2}\left(|\nabla(\phi)|^{p(x)} + e(x)|\phi|^{p(x)}\right)dx} \right)^{1/(\widehat{p}-\widehat{q}_2-\delta)} \Bigg\}.
\end{aligned} \tag{31}$$

□

In this paper, we study problem (1) by using Ekeland's variation principle, which is recalled below.

Lemma 10 (see [16]). *Let M be a complete metric space and let $J : M \to (\infty, \infty]$ be a lower semicontinuous functional, bounded from below, and not identically equal to ∞. Let $\epsilon > 0$ be given and $z \in M$ be such that*

$$J(z) \leq \inf_M J + \epsilon. \tag{32}$$

Then, there exists $v \in M$ such that

$$J(v) \leq J(z) \leq \inf_M J + \epsilon, \quad d(z, v) \leq 1, \tag{33}$$

and for any $u \neq v$ in M,

$$J(v) < J(u) + \epsilon d(v, u), \tag{34}$$

where $d(\cdot,\cdot)$ denotes the distance between two elements in M.

We now state our main theorem.

Theorem 11. *Assume that conditions (A) and (B) hold. Then, there exists $\overline{\lambda} > 0$ such that any $\lambda \in (0, \overline{\lambda})$; problem (1) has at least one nontrivial weak solution.*

Proof. Let $\overline{\lambda}$ be defined by (29), and $\lambda \in (0, \overline{(\lambda)})$. By Lemma 8, we have

$$\inf_{\partial B_\alpha(0)} I_\lambda > 0, \tag{35}$$

where $B_\alpha(0)$ is the ball in X and $\partial B_\alpha(0)$ is the boundary of $B_\alpha(0)$. For any $u \in B_\alpha(0)$, by an argument similar to those used in Lemma 8, we can obtain that

$$I_\lambda(u) \geq \frac{1}{p^+}\|u\|_e^{p^+} - 2\lambda\gamma_1 d^{q_1^-} |a|_{r(x)} \|u\|_e^{q_1^-}. \tag{36}$$

Note from Lemma 9 that there exists $\phi \in X$ such that $I_\lambda(t\phi) < 0$ for $t > 0$ small enough. Then, from (35) and (36),

$$-\infty < \underline{c}_\lambda := \inf_{\overline{\partial B_\alpha}(0)} I_\lambda < 0. \tag{37}$$

Let

$$0 < \epsilon < \inf_{\partial B_\alpha(0)} I_\lambda - \underline{c}_\lambda. \tag{38}$$

Applying Lemma 10, we see that there exists $u_\epsilon \in \overline{B_\alpha(0)}$ such that

$$\begin{gathered}
\underline{c}_\lambda \leq I_\lambda(u_\epsilon) \leq \underline{c}_\lambda + \epsilon, \\
I_\lambda(u_\epsilon) < I_\lambda(u) + \epsilon\|u - u_\epsilon\|_e \quad \text{for } u \neq u_\epsilon.
\end{gathered} \tag{39}$$

Since $I_\lambda(u_\epsilon) \leq \underline{c}_\lambda + \epsilon < \inf_{\partial B_\alpha(0)} I_\lambda$, we have $u_\epsilon \in B_\alpha(0)$.

Now, define a functional $J_\lambda : \overline{B_\alpha}(0) \to \mathbb{R}$ by

$$J_\lambda(u) = I_\lambda(u) + \epsilon\|u - u_\epsilon\|_e. \tag{40}$$

By (40), u_ϵ is a minimum point of J_λ, and for $t > 0$ small enough and all $v \in B_\alpha(0)$, we have

$$\frac{J_\lambda(u_\epsilon + tv) - J_\lambda(u_\epsilon)}{t} \geq 0; \tag{41}$$

then

$$\frac{I_\lambda(u_\epsilon + tv) - I_\lambda(u_\epsilon)}{t} + \epsilon \|v\|_e \geq 0; \tag{42}$$

when $t \to 0$, we have

$$\left(I'_\lambda(u_\epsilon), v\right) + \epsilon \|v\|_e \geq 0 \quad \forall v \in B_\alpha(0). \tag{43}$$

Together with (39), there exists a sequence $\{u_n\} \subset B_\alpha(0)$ such that

$$I_\lambda(u_n) \longrightarrow \underline{c}_\lambda, \qquad I'_\lambda(u_n) \longrightarrow 0. \tag{44}$$

By the reflexivity of X, there exists $u_0 \in X$ such that $u_n \rightharpoonup u_0$. Note that

$$\begin{aligned}\left|I'_\lambda(u_n), u_n - u_0\right| &\leq \left|I'_\lambda(u_n), u_n\right| + \left|I'_\lambda(u_n), u_0\right| \\ &\leq \left\|I'_\lambda(u_n)\right\| \|u_n\| + \left\|I'_\lambda(u_n)\right\| \|u_0\|.\end{aligned} \tag{45}$$

Then, from (44), we have

$$\lim_{n\to\infty} \left(I'_\lambda(u_n), u_n - u_0\right) = 0. \tag{46}$$

From (21) and Propositions 1 and 3, we have

$$\begin{aligned}&\left|\Psi'(u_n), u_n - u_0\right| \\ &\leq 3q_1^+ \gamma_1 |a|_{r(x)} \max\left\{|u_n|_{q_1(x)}^{q_1^+ - 1}, |u_n|_{q_1(x)}^{q_1^- - 1}\right\} |u_n - u_0|_{s_1(x)},\end{aligned} \tag{47}$$

where $s_1(x)$ is defined in Remark 6; by the continuous and compact embedding of $X \hookrightarrow L^{q_1(x)}(\Omega)$ and $X \hookrightarrow L^{s_1(x)}(\Omega)$, we can get

$$\lim_{n\to\infty} \left(\Psi'(u_n), u_n - u_0\right) = 0. \tag{48}$$

Now, we conclude that

$$\lim_{n\to\infty} \left(\Phi'(u_n), u_n - u_0\right) = 0. \tag{49}$$

Thus, by Proposition 5, we have $u_n \to u_0$. Hence,

$$I_\lambda(u_0) = \underline{c}_\lambda < 0, \qquad I'_\lambda(u_0) = 0. \tag{50}$$

Therefore, u_0 is a nontrivial weak solution of problem (1). □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors are thankful to the referees for their helpful suggestions. Project is supported by the Applied Basic Research Projects of Yunnan Province (no. 2013FD031) and the National Natural Science Foundation of China (no. 11461083).

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