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Research Article

One-Dimensional Coulomb Multiparticle Systems

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We consider the system of particles with equal charges and nearest neighbour Coulomb interaction on the interval. We study local properties of this system, in particular the distribution of distances between neighbouring charges. For zero temperature case there is sufficiently complete picture and we give a short review. For Gibbs distribution the situation is more difficult and we present two related results.

1. Introduction

Many electric phenomena are not well understood and even might seem mysterious. More exactly, most are not still deduced from the microscale version of Maxwell equations on rigorous mathematical level. For example, even in the standard direct or alternative current, the electrons move along hundred kilometers of power lines, but the external accelerating force acts only on some meters of the wire. Here what one can read about this in "The Feynman Lectures on Physics" (see [1], volume 2, 16-2):

> "...The force pushes the electrons along the wire. But why does this move the galvanometer, which is so far from the force? Because when the electrons which feel the magnetic force try to move, they push - by electric repulsion - the electrons a little farther down the wire; they, in turn, repel the electrons a little farther on, and so on for a long distance. An amazing thing. It was so amazing to Gauss and Weber - who first built a galvanometer - that they tried to see how far the forces in the wire would go. They strung the wire all the way across the city...."

This was written by the famous physicist Richard Feynman. However, after that, this "amazing thing" was vastly ignored in the literature. For example, the Drude model (that can be found in any textbook on solid state physics, see, e.g., [2]), considers free (noninteracting) electrons and constant external accelerating force acting along all the wire, without mentioning where this force (or field) comes from.

Many more questions arise. For example, why the electrons in DC move slowly, but such stationary regime is being established almost immediately. In [3, 4] it was demonstrated rigorously that even on the classical (not quantum) level that the stationary and space homogeneous flow of charged particles may exist as a result of self-organization of strongly interacting (via Coulomb repulsion) system of electrons. This means that the field accelerating the electrons is created by the neighboring electrons via some multiscale self-organization.

In fact, Ohm's law is formulated on the macroscale (of order one); one-dimensional movement of N electrons is described on the microscale (of order N^{-1}), but the accelerating force is the corollary of the processes on the so-called submicroscale (of the order N^{-2}). To show this we used not only classical nonrelativistic physics, Newtonian dynamics and Coulomb's law, but also the simplest friction mechanism, ignoring where this friction mechanism comes from.

Besides other nonsolved dynamic problems like discharge, lightning, global current, bioelectricity, and so forth, also local and global properties of equilibrium configurations of charged particles in external electric fields are not at all studied (note that the mathematical part of equilibrium statistical physics has been developed mostly on the lattice). The equilibrium configurations can be either ground state (zero temperature) or Gibbs states. Ground states are easier to describe and we give short review of results in Section 2

of the paper. Study of local structure of Gibbs configurations is at the very beginning and we present in Section 3 two new results with complete proofs.

2. Part I: Ground State Configurations

Consider systems of N particles with equal charges, Coulomb interaction and external force F on an manifold. Even when there is no external force, the problem appears to be sufficiently difficult and was claimed important already long ago [5]. For example, J. J. Thomson (who discovered electron) suggested the problem of finding such configurations on the sphere, and the answer has been known for $N = 2, 3$, and 4 for more than 100 years, but for $N = 5$ the solution was obtained only quite recently [6].

More interesting is the case of large N, where the asymptotics $N \to \infty$ is of main interest. In one-dimensional case T. J. Stieltjes studied the problem with logarithmic interaction and found its connection with zeros of orthogonal polynomials on the corresponding interval (see [7, 8]). However, the problem of finding minimal energy configurations on two-dimensional sphere for any N and power interaction (sometimes it is called the seventh problem of Smale; it is also connected with the names of F. Risz and M. Fekete) was completely solved only for quadratic interaction (see [9–11] and review [12]). For more general compact manifolds see review [13].

In this section we review recent results concerning nonzero external force. Moreover, we consider not only global minima but even more interesting case of local energy minima. It appears that even in the simpler one-dimensional model with nearest neighbour interaction there is an interesting structure of fixed points (more exactly, fixed configurations), rich both in the number and in the charge distribution.

2.1. The Model. We consider the set of configurations of $N+1$ point particles:

$$-L \le x_N < \cdots < x_1 < x_0 \le 0 \tag{1}$$

with equal charges on the segment $[-L, 0]$. Here N is assumed to be sufficiently large. We assume repulsive Coulomb interaction of nearest neighbours and external force $\alpha_{\text{ext}} F_0(x)$; that is, the potential energy is given by

$$U = \sum_{i=1}^{N} V(x_{i-1} - x_i) - \sum_{i=0}^{N} \int_{-L}^{x_i} \alpha_{\text{ext}} F_0(x)\, dx, \qquad V(x) = \frac{\alpha_{\text{int}}}{|x|}, \tag{2}$$

where α_{ext}, α_{int} are positive constants. This defines the dynamics of the system of charges, if one defines exactly what occurs with particles 0 and N in the points 0 and $-L$ correspondingly. Namely, we assume completely inelastic boundary conditions. More exactly, when particle $x_0(t)$ at time t reaches point 0, having some velocity $v_0(t-0) \ge 0$, then its velocity $v_0(t)$ immediately becomes zero, and the particle itself stays at point 0 until the force acting on it (which varies accordingly to the motion of other particles) becomes negative. The same occurs for the particle $x_N(t)$ at point $-L$.

2.2. Problem of Many Local Minima. It is evident that if $F_0 \equiv 0$, then there is only one fixed point with

$$\delta_k = x_{k-1} - x_k = \frac{L}{N}, \quad k = 1, \ldots, N. \tag{3}$$

Thus it is the global energy minimum. More general result is the following.

Theorem 1. *Assume that $F_0(x)$ is continuous, nonnegative, and monotonic. Then for any N, L, and α_{ren} the fixed point exists and is unique.*

However, the monotonicity assumption in this theorem is very essential. An example of strong nonuniqueness (where the number of fixed points is of the order N) is very simple—for a function $F_0(x)$ with the only maximum inside the interval. Namely, on the interval $[-1, 1]$, put for $b > a > 0$

$$\begin{aligned} F_0(x) &= a - 2ax, \quad x \ge 0, \\ F_0(x) &= a + 2bx, \quad x \le 0. \end{aligned} \tag{4}$$

Then there exists $C_{\text{cr}} > 0$ such that for all sufficiently large N and $\alpha_{\text{ren}} = cN$, $c > C_{\text{cr}}$, one can show using similar techniques that for any odd $N_1 < N$ there exists fixed point such that

$$\begin{aligned} -1 = x_N < \cdots < x_{N_1} < 0 < x_{N_1-1} < \cdots < x_{(N_1+1)/2} \\ = \frac{1}{2} < \cdots < x_0 < 1. \end{aligned} \tag{5}$$

Moreover, any such point will give local minimum of the energy.

One-dimensional case shows what can be expected in multidimensional case, which is more complicated but has great interest in connection to the static charge distribution in the atmosphere or in the living organism.

2.3. Phase Transitions. To discover phase transitions one should consider asymptotics $N \to \infty$, with the parameters $L, F_0(x)$ being fixed. Then the fixed points will depend only on the "renormalized force" $F = (\alpha_{\text{ext}}/\alpha_{\text{int}})F_o$, and we assume that the renormalized constant $\alpha_{\text{ren}} = \alpha_{\text{ext}}/\alpha_{\text{int}}$ can tend to infinity together with N, namely, as $\alpha_{\text{ren}} = cN^{\gamma}$, where $c, \gamma > 0$.

The necessity to consider cases when α_{ren} depends on N issues from concrete examples, where $\alpha_{\text{ren}} \gg N$. For example, the linear density of electrons in some conductors, see [2], is of the order $N \approx 10^9\ \text{m}^{-1}$, $\alpha_{\text{int}} = e^2/\epsilon_0 \approx 10^{-28}$, and $\alpha_{\text{ext}} = 220$ (volt/meter) where $e = 220 \times 10^{-19}$ (in SI system). Thus α_{ren} has the order 10^{11}. This is close to the critical point of our model, which, as it will be shown, is asymptotically $c_{\text{cr}}N$ that is close to 4×10^9 in our case.

Below this section we assume for simplicity that $F_0 > 0$ is constant. We formulate now the assertions proven in [14–16].

Critical Force. For any N, L there exists $F_{\mathrm{cr}} = F_{\mathrm{cr}}(N, L)$ such that for the fixed point the following holds: $x_N > -L$ for $F > F_{\mathrm{cr}}$ and $x_N = -L$ for $F \leq F_{\mathrm{cr}}$. If $F = cN^{\gamma}$, $\gamma > 1$, then for any $c > 0$ we have $x_N \to 0$. At the same time $F_{\mathrm{cr}} \sim_{N\to\infty} c_{\mathrm{cr}} N$, where

$$c_{\mathrm{cr}} = \frac{4}{L^2}. \tag{6}$$

Multiscale Phase. The case when α_{ren} does not depend on N was discussed in detail in [14, 15]; there are no phase transitions, but it is discovered that the structure of the fixed configuration differs from (3) only on the submicroscale of the order N^{-2}. More exactly, consider more general case when $V(x) = |x|^{-b}$, $b > 0$. Then the following holds: if F does not depend on N, then for any $k = 1, \ldots, N$

$$(x_{k-1} - x_k) - \frac{L}{N} \sim \frac{FL^{1+b}}{1+b} N^{-b} \left(k - \frac{N}{2}\right). \tag{7}$$

Uniform Density. We define the density $\rho(x)$ so that for any subintervals $I \subset [-L, 0]$ there exist the limits

$$\rho(I) = \int_I \rho(x)\, dx = \lim_{N\to\infty} \frac{\#\{i : x_i \in I\}}{N}. \tag{8}$$

Then if $F = o(N)$, then the density exists and is strictly uniform, that is, for all $k = 1, \ldots, N$ as $N \to \infty$,

$$\max_k \left|(x_{k-1} - x_k) - \frac{L}{N}\right| = o\left(\frac{1}{N}\right). \tag{9}$$

Nonuniform Density. If $F = cN$ and $0 < c \leq c_{\mathrm{cr}}$, then $x_N = -L$ and the density of particles exists and is nowhere zero but is not uniform (not constant in x).

Weak Contraction. If $F = cN$ and $c > c_{\mathrm{cr}}$, then, as $N \to \infty$,

$$-L < x_N \longrightarrow -\frac{2}{\sqrt{c}} \tag{10}$$

and the density on the interval $(-2/\sqrt{c}, 0)$ is not uniform.

Strong Contraction. If $F = cN^{\gamma}$, $\gamma > 1$, then the density $\rho(x) \to \delta(x)$ in the sense of distributions.

Both contraction cases are related to the discharge possibility; as after disappearance of the external force, discharge can be produced, the strength of which depends on the initial concentration of charged particles.

3. Part II: Gibbs Distribution

We consider the set $\Omega = \Omega_N = \{\omega = (x_0, \ldots, x_N)\}$, $N \geq 2$, of configurations of $N+1$ points particles on the segment $[0, L]$ such that

$$0 = x_0 < \cdots < x_N = L. \tag{11}$$

Introducing new variables $u_k = x_k - x_{k-1}$, $k = 1, \ldots, N$, one sees that Ω_N is an open simplex:

$$u_1 + \cdots + u_N = L, \quad u_k > 0, \tag{12}$$

which is denoted by $S(N, L)$.

We will consider the probability density:

$$P(\omega) = Z_N^{-1} \exp(-\beta U(\omega)) \tag{13}$$

on $S(N, L)$ with respect to the Lebesgue measure ν on $S(N, L)$, where

$$\begin{aligned} Z_N &= \int_{S(N,L)} \exp(-\beta U(\omega))\, d\nu \\ &= \int_{0<x_1<\cdots<x_{N-1}<L} \exp(-\beta U(\omega))\, dx_1 \cdots dx_{N-1} \\ &= \int_{S(N,L)} \prod_{k=1}^{N} \exp(-\beta V(u_k))\, du_1 \cdots du_N, \\ U(\omega) &= V(x_1 - x_0) + \cdots + V(x_N - x_{N-1}) \\ &= \sum_{k=1}^{N} V(u_k) \end{aligned} \tag{14}$$

is the function on the set of configurations called the energy and V is the function on the segment $[0, \infty]$, called potential. The most interesting case for us is the Coulomb repulsive potential:

$$V(u) = \frac{1}{u}, \quad u > 0. \tag{15}$$

Equivalently one could say that we consider the sum

$$S_N = \xi_1 + \cdots + \xi_N \tag{16}$$

of N independent identically distributed positive random variables ξ_k, each having density $g(u)$, $u > 0$, further assumed to be smooth, for simplicity. Then the conditional density of the vector $\{\xi_1, \ldots, \xi_N\}$, under the condition $S_N = L$,

$$P(\omega) = \frac{g(u_1) \cdots g(u_N)}{\int_{S(N,L)} g(u_1) \cdots g(u_N)\, du_1 \cdots du_N} \tag{17}$$

that coincides with (13), if we put

$$\begin{aligned} g(u) &= Z_1^{-1} \exp(-\beta V(u)), \\ Z_1 &= \int_0^L \exp(-\beta V(u))\, du. \end{aligned} \tag{18}$$

It is clear that conditional distributions $P(\xi_k < x \mid S_N = L)$ are the same for all k. In particular

$$\langle \xi_k \mid S_N = L \rangle = \frac{L}{N}. \tag{19}$$

Note that in the limiting case $\beta = \infty$ the distribution is concentrated in the unique fixed point $u_k = L/N$.

Below we put $L = 1$. Let

$$f^{\star n}(x) = f \star f \star f \star \cdots \star f \tag{20}$$

be the n times convolution of $f(x)$. Then the conditional variance is

$$\begin{aligned} d_N &= D(\xi_1 \mid S_N = 1) \\ &= \int_0^1 \left(x - \frac{1}{N}\right)^2 \frac{g(x)\, g^{\star(N-1)}(1-x)}{g^{\star N}(1)}\, dx. \end{aligned} \tag{21}$$

We want to note here that there exist many papers, related to the famous Kac mean field model, where conditional independence (chaos) of ξ_k is proved under various conditions (see, e.g., [17] and the references therein). We follow here another goal, that is, revealing possible multiscale local structure in the Gibbs situation, which could resemble zero temperature case structure, discussed in Section 2.

3.1. Results. We consider the densities having the following asymptotic behaviour as $x \to 0$:

$$g(x) \sim c_0 x^{\alpha-1} e^{-\beta/x}, \quad \alpha \in R,\ \beta \geq 0. \tag{22}$$

Theorem 2. *Under this condition and if $\alpha > 0$ and $\beta = 0$, as $N \to \infty$,*

$$d_N \sim c_1 N^{-2} \tag{23}$$

for some constant $c_1 = c_1(\alpha) > 0$ depending only on α.

Theorem 3. *If $\beta > 0$, then for any $\alpha \in R$, as $N \to \infty$,*

$$d_N \sim c_1 N^{-3} \tag{24}$$

for some constant $c_1 = c_1(\beta) > 0$ depending only on β.

It is of interest to know the behaviour of the covariance for densities with hyperexponential decrease at zero.

3.2. Proofs

3.2.1. General Power Asymptotics. We will prove here Theorem 2. Instead of one density $g(x)$ it is useful to consider the family of densities (such trick has been used in some large deviations problems, see [18, 19])

$$h_\lambda(x) = e^{-\lambda x} g(x)\, z^{-1}(\lambda), \tag{25}$$

where $\lambda \geq 0$ and

$$z(\lambda) = \int_0^1 e^{-\lambda x} g(x)\, dx. \tag{26}$$

Let $\xi_{\lambda,k}$ be random variables with density $h_\lambda(x)$. Put $m_\lambda = E\xi_{\lambda,k}$, $\sigma_\lambda^2 = D\xi_{\lambda,k}$ and denote the conditional densities of $\xi_{\lambda,k}$ under the condition that $S_N = 1$:

$$\begin{aligned} f_\lambda(x) &= \frac{h_\lambda(x)\, h_\lambda^{\star(N-1)}(1-x)}{h_\lambda^{\star N}(1)} \\ &= \frac{g(x)\, g^{\star(n-1)}(1-x)}{g^{\star n}(1)}, \quad x \in [0,1]. \end{aligned} \tag{27}$$

It is easy to check that $f_\lambda(x)$ in fact does not depend on λ.

Lemma 4. *(1) $m_\lambda \sim \alpha\lambda^{-1}$ and $\sigma_\lambda^2 \sim \alpha\lambda^{-2}$ as $\lambda \to \infty$. (2) there exists a unique λ_N, such that $m_{\lambda_N} = 1/N$. Then $\lambda_N \sim \alpha N$, $\sigma_{\lambda_n}^2 \sim \alpha^{-1} N^{-2}$ as $N \to \infty$.*

Proof. (1) By abelian theorem, see [20, page 445, Theorem 3], and the condition $g(x) \sim c_0 x^{\alpha-1}$ as $x \to 0$ we have as $\lambda \to \infty$

$$\begin{aligned} z(\lambda) &\sim \frac{\Gamma(\alpha)\, c_0}{\lambda^N}, \\ -z'(\lambda) &\sim \frac{\Gamma(\alpha+1)\, c_0}{\lambda^{N+1}}, \\ z''(\lambda) &\sim \frac{\Gamma(\alpha+2)\, c_0}{\lambda^{N+2}}. \end{aligned} \tag{28}$$

It follows that

$$\begin{aligned} m_\lambda &= -\frac{z'(\lambda)}{z(\lambda)} \sim \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\lambda} = \frac{\alpha}{\lambda}, \\ \sigma_\lambda^2 &= \frac{c''(\lambda)}{c(\lambda)} - \left(\frac{c'(\lambda)}{c(\lambda)}\right)^2 \\ &\sim \left(\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} - \left(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}\right)^2\right)\lambda^{-2} = \frac{\alpha^2}{\lambda^2} \end{aligned} \tag{29}$$

as $\lambda \to \infty$.

(2) The function m_λ is monotonically decreasing in λ. Thus for any N there exists λ_N such that

$$m_{\lambda_N} = \frac{1}{N}. \tag{30}$$

From (1) it follows that $\lambda_N \sim \alpha N$ and $\sigma_{\lambda_N}^2 \sim \alpha^{-1} N^{-2}$ as $N \to \infty$. □

Let $\phi_\lambda(t)$ be the characteristic function of ξ_λ.

Lemma 5. *The family of densities (25) has the following properties:*

(1) *The normalized moment $a_\lambda = E|\xi_\lambda - m_\lambda|^4/\sigma_\lambda^4$ is bounded uniformly in $\lambda > 0$.*

(2) *For any $\delta > 0$ there exists $\lambda_0 > 0$ such that $\sup_{\lambda>\lambda_0} \sup_{t>\delta} \phi_\lambda(t/\sigma_\lambda) < 1$.*

(3) *For some $q \geq 1$*

$$\int_{-\infty}^{\infty} |\phi_\lambda(t)|^q\, dt = O(\lambda^{q\alpha}). \tag{31}$$

Proof. (1) It is similar to the proof of (1) in Lemma 4.

(2) Put

$$f(t,\lambda) = \phi_\lambda\left(t\sigma_\lambda^{-1}\right) = z^{-1}(\lambda) \int_0^1 e^{it\sigma_\lambda^{-1}x} e^{-\lambda x} g(x)\, dx. \tag{32}$$

Let us show that for some $\delta > 0$

$$|f(t,\lambda)| \leq \left|\frac{1}{1 - it\alpha^{-1/2}}\right|^\alpha + O\left(e^{-\delta\sqrt{\lambda}}\right) \tag{33}$$

as $\lambda \to \infty$. For this we can write $f(t,\lambda)$ as

$$f(t,\lambda) = J_1(\lambda) + J_2(\lambda), \tag{34}$$

where

$$\begin{aligned} J_1(\lambda) &= z^{-1}(\lambda)\int_0^{\lambda^{-1/2}} e^{it\sigma_\lambda^{-1}x}e^{-\lambda x}g(x)\,dx, \\ J_2(\lambda) &= z^{-1}(\lambda)\int_{\lambda^{-1/2}}^{1} e^{it\sigma_\lambda^{-1}x}e^{-\lambda x}g(x)\,dx. \end{aligned} \tag{35}$$

Taking into account that

$$\begin{aligned} z(\lambda) &\sim c_0\Gamma(\alpha)\lambda^{-\alpha}, \\ \sigma_\lambda^2 &\sim \alpha\lambda^{-2}, \\ &\lambda \longrightarrow \infty \end{aligned} \tag{36}$$

and condition $g(x) \sim c_0 x^{\alpha-1}$ as $x \to 0$ we have that as $\lambda \to \infty$

$$|J_1(\lambda)| \sim (\Gamma(\alpha))^{-1}\lambda^\alpha\left|\int_0^{\lambda^{-1/2}} e^{it\alpha^{-1/2}\lambda x}e^{-\lambda x}x^{\alpha-1}\,dx\right|. \tag{37}$$

Putting $y = \lambda x$ in the last integral we get

$$\begin{aligned} &(\Gamma(\alpha))^{-1}\lambda^\alpha\int_0^{\lambda^{-1/2}} e^{it\alpha^{-1/2}\lambda x}e^{-\lambda x}x^{\alpha-1}\,dx \\ &\quad = (\Gamma(\alpha))^{-1}\int_0^{\lambda^{1/2}} e^{it\alpha^{-1/2}y}e^{-y}y^{\alpha-1}\,dy. \end{aligned} \tag{38}$$

As

$$\begin{aligned} &(\Gamma(\alpha))^{-1}\int_0^{\lambda^{1/2}} e^{it\alpha^{-1/2}y}e^{-y}y^{\alpha-1}\,dy \\ &\quad = (\Gamma(\alpha))^{-1}\int_0^{\infty} e^{it\alpha^{-1/2}y}e^{-y}y^{\alpha-1}\,dy \\ &\qquad - (\Gamma(\alpha))^{-1}\int_{\lambda^{1/2}}^{\infty} e^{it\alpha^{-1/2}y}e^{-y}y^{\alpha-1}\,dy, \\ &(\Gamma(\alpha))^{-1}\int_0^{\infty} e^{it\alpha^{-1/2}y}e^{-y}y^{\alpha-1}\,dy = \left(\frac{1}{1-it\alpha^{-1/2}}\right)^\alpha, \\ &(\Gamma(\alpha))^{-1}\left|\int_{\lambda^{1/2}}^{\infty} e^{it\alpha^{-1/2}y}e^{-y}y^{\alpha-1}\,dy\right| = O\left(e^{-\delta\sqrt{\lambda}}\right) \end{aligned} \tag{39}$$

for some $\delta > 0$, then

$$|J_1(\lambda)| \le \left|\frac{1}{1-it\alpha^{-1/2}}\right|^\alpha + O\left(e^{-\delta\sqrt{\lambda}}\right). \tag{40}$$

For $J_2(\lambda)$ we get the estimate

$$\begin{aligned} |J_2(\lambda)| &= c^{-1}(\lambda)\left|\int_{\lambda^{-1/2}}^{1} e^{it\sigma_\lambda^{-1}x}e^{-\lambda x}g(x)\,dx\right| \\ &\le C\lambda^\alpha e^{-\sqrt{\lambda}} = O\left(e^{-\delta\sqrt{\lambda}}\right). \end{aligned} \tag{41}$$

From these estimates (33) follows.

(3) Note that always

$$\int_0^1 |h_\lambda(x)|^p\,dx < \infty \tag{42}$$

for some $p > 1$. Without loss of generality one can assume that $1 < p \le 2$. By Hausdorff-Young inequality

$$\left(\int_{-\infty}^{\infty}|\phi_\lambda(t)|^q\,dt\right)^{1/q} \le \left(\frac{1}{2\pi}\int_0^1 |h_\lambda(x)|^p\,dx\right)^{1/p}, \tag{43}$$

where $1 < p \le 2$ and $1/p + 1/q = 1$. As

$$\begin{aligned} &\left(\frac{1}{2\pi}\int_0^1 |h_\lambda(x)|^p\,dt\right)^{1/p} \\ &\quad = z^{-1}(\lambda)\left(\frac{1}{2\pi}\int_0^1 \left|e^{-\lambda x}g(x)\right|^p dt\right)^{1/p} \\ &\quad \le z^{-1}(\lambda)\left(\frac{1}{2\pi}\int_0^1 |g(x)|^p\,dt\right)^{1/p} \end{aligned} \tag{44}$$

and $z(\lambda) \sim C\lambda^{-\alpha}$, then

$$\int_{-\infty}^{\infty}|\phi_\lambda(t)|^q\,dt = O\left(\lambda^{q\alpha}\right). \tag{45}$$

□

Lemma 6. *Assume conditions (1)–(3) of Lemma 5 and that λ_N are defined by the condition $m_{\lambda_N} = 1/N$. Then*

$$h_{\lambda_N}^{*N}(x) = \frac{1}{\sigma_{\lambda_N}\sqrt{2\pi N}}\left(\exp\left(-\frac{(x-1)^2}{2\sigma_{\lambda_N}^2 N}\right) + o(1)\right), \tag{46}$$

where $o(1)$ tends to 0 uniformly in $x \ge 0$.

Proof. We change a bit the standard proof of local limit theorem. Let $p_N(z)$ be the density of the standard deviation

$$\frac{S_{N,\lambda_N} - Nm_{\lambda_N}}{\sqrt{N}\sigma_{\lambda_N}}, \tag{47}$$

where $S_{N,\lambda} = \xi_{1,\lambda} + \cdots + \xi_{N,\lambda}$ is the sum independent random variables having density (25). Let $q(z)$ be the standard Gaussian density. The inverse Fourier transform gives

$$\begin{aligned} &\sup_{z\in R}|p_N(z) - q(z)| \\ &\quad \le \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\psi_{\lambda_N}^N\left(\frac{t}{\sqrt{N}\sigma_{\lambda_N}}\right) - \frac{1}{\sqrt{2\pi}}e^{-t^2/2}\right|dt, \end{aligned} \tag{48}$$

where $\psi_\lambda(t) = \phi_\lambda(t)e^{-itm_\lambda}$. Denote

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int_{|t|\le\sqrt{N}a_{\lambda_N}^{-1}}\left|\psi_{\lambda_N}^N\left(\frac{t}{\sqrt{N}\sigma_{\lambda_N}}\right)-\frac{1}{\sqrt{2\pi}}e^{-t^2/2}\right|dt,\\ I_2 &= \frac{1}{2\pi}\int_{|t|>\sqrt{N}a_{\lambda_N}^{-1}}\left|\psi_{\lambda_N}\left(\frac{t}{\sqrt{N}\sigma_{\lambda_N}}\right)\right|^N dt,\\ I_3 &= \frac{1}{(2\pi)^{3/2}}\int_{|t|>\sqrt{N}a_{\lambda_N}^{-1}}e^{-t^2/2}\,dt, \end{aligned} \tag{49}$$

where $a_\lambda = E|\xi_\lambda - m_\lambda|^4/\sigma_\lambda^4$ is bounded uniformly in λ (Lemma 5, part (1)). Then

$$\sup_{z\in R}\left|p_N(z)-q(z)\right| \le I_1 + I_2 + I_3. \tag{50}$$

For I_1 we have the estimate (see [21, page 109, lemma 1])

$$I_1 \le \frac{a_{\lambda_N}}{2\pi\sqrt{N}}\int_{|t|\le\sqrt{N}a_{\lambda_N}^{-1}}|t|^3 e^{-t^3/3}dt \le \frac{Ca_{\lambda_N}}{\sqrt{N}}. \tag{51}$$

For I_2 we have by parts (2) and (3) of Lemma 5,

$$\begin{aligned} I_2 &= \frac{1}{2\pi}\int_{|t|>\sqrt{N}a_{\lambda_N}^{-1}}\left|\phi_{\lambda_N}\left(\frac{t}{\sqrt{N}\sigma_{\lambda_N}}\right)\right|^N dt\\ &\le N^{-1/2}\gamma^{N-q}\int_{-\infty}^{\infty}\left|\phi_{\lambda_N}(t)\right|^q dt \le CN^{-1/2}\gamma^{N-q}N^{q\alpha} \end{aligned} \tag{52}$$

for N being sufficiently large, where $\gamma < 1$ and $q > 1$.

The estimate for I_3 is trivial. Thus,

$$p_N(x) = q(x) + O\left(N^{-1/2}\right), \tag{53}$$

where $O(N^{-1/2})$ does not depend on $x \in R$. Lemma follows as

$$h_{\lambda_N}^{\star N}(x) = \frac{1}{\sqrt{N}\sigma_{\lambda_N}}p_N\left(\frac{x-1}{\sqrt{N}\sigma_{\lambda_N}}\right). \tag{54}$$

The lemma is proved. □

Lemma 7. *Assume conditions (1)–(3) of Lemma 5. Then the conditional variance*

$$d_N = D_N + o\left(\sigma_{\lambda_N}^2\right), \tag{55}$$

where

$$D_N = \int_0^1\left(x-\frac{1}{N}\right)^2 h_{\lambda_N}(x)\exp\left(-\frac{(x-1/N)^2}{2\sigma_{\lambda_N}^2(N-1)}\right)dx. \tag{56}$$

Proof. By Lemma 6 we have

$$\begin{aligned} h_{\lambda_N}^{\star(N-1)}(1-x) &= \frac{1}{\sigma_{\lambda_N}\sqrt{2\pi(N-1)}}\left(\exp\left(-\frac{(x-1/N)^2}{2\sigma_{\lambda_N}^2(N-1)}\right)\right.\\ &\quad\left.+o(1)\right), \end{aligned} \tag{57}$$

$$h_{\lambda_N}^{\star N}(1) = \frac{1}{\sigma_{\lambda_N}\sqrt{2\pi N}}(1+o(1)), \tag{58}$$

where $m_{\lambda_N} = 1/N$. The division gives

$$\begin{aligned} \frac{h_{\lambda_N}^{\star(N-1)}(1-x)}{h_{\lambda_N}^{\star N}(1)} &= \sqrt{\frac{N}{N-1}}\exp\left(-\frac{(x-1/N)^2}{2\sigma_{\lambda_N}^2(N-1)}\right)\\ &\quad+o(1), \end{aligned} \tag{59}$$

where the term $o(1)$ tends to 0 uniformly in $x \in [0,1]$. By (27) the conditional variance is

$$d_N = \int_0^1\left(x-\frac{1}{N}\right)^2\frac{h_{\lambda_N}(x)\,h_{\lambda_N}^{\star(N-1)}(1-x)}{h_{\lambda_N}^{\star N}(1)}dx. \tag{60}$$

Substituting (59) into this expression we get

$$\begin{aligned} d_N &= \int_0^1\left(x-\frac{1}{N}\right)^2 h_{\lambda_N}(x)\exp\left(-\frac{(x-1/N)^2}{2\sigma_{\lambda_N}^2(N-1)}\right)dx\\ &\quad+o\left(\sigma_{\lambda_N}^2\right). \end{aligned} \tag{61}$$

The lemma is proved. □

By Lemma 4 $\sigma_{\lambda_N}^2 \sim \alpha^{-1}N^{-2}$ as $N \to \infty$. Thus

$$d_N = D_N + o\left(N^{-2}\right), \tag{62}$$

where

$$D_N = \int_0^1\left(x-\frac{1}{N}\right)^2 h_{\lambda_N}(x)\exp\left(-\frac{(x-1/N)^2}{2\sigma_{\lambda_N}^2(N-1)}\right)dx. \tag{63}$$

Lemma 8. *As $N \to \infty$,*

$$D_N \sim c_1N^{-2}. \tag{64}$$

Proof. By definition (25) and Lemma 4 as $N \to \infty$,

$$\begin{aligned} D_N &\sim \frac{(\alpha N)^\alpha}{c_0\Gamma(\alpha)}\int_0^1\left(x-\frac{1}{N}\right)^2 e^{-\alpha Nx}g(x)\\ &\quad\cdot\exp\left(-\frac{\alpha N(x-1/N)^2}{2}\right)dx. \end{aligned} \tag{65}$$

Because of $g(x) \sim c_0 x^{\alpha-1}$ as $x \to 0$ we have

$$D_N \sim \frac{(\alpha N)^\alpha e^{-\alpha}}{\Gamma(\alpha)} \int_0^{N^{-1}} \left(x - \frac{1}{N}\right)^2 x^{\alpha-1}\, dx. \tag{66}$$

Changing variable $y = Nx$ we have

$$\begin{aligned} &\frac{(\alpha N)^\alpha e^{-\alpha}}{\Gamma(\alpha)} \int_0^{N^{-1}} \left(x - \frac{1}{N}\right)^2 x^{\alpha-1}\, dx \\ &\quad = \frac{\alpha^\alpha e^{-\alpha}}{\Gamma(\alpha) N^2} \int_0^1 (1-y)^2 y^{\alpha-1}\, dy \end{aligned} \tag{67}$$

which gives the lemma and the theorem. □

3.2.2. Coulomb Case. Again we introduce the exponential family of densities

$$h_\lambda(x) = e^{-\lambda x} g(x) Z^{-1}(\lambda), \quad \lambda \geq 0, \tag{68}$$

where the function $g(x)$ satisfies condition (22) with $\beta > 0$ and

$$Z(\lambda) = \int_0^1 e^{-\lambda x} g(x)\, dx. \tag{69}$$

We will use modified Bessel function of the second kind defined as follows:

$$K_\alpha(z) = \frac{1}{2} \int_0^\infty x^{\alpha-1} e^{-(z/2)(x+1/x)} dx, \tag{70}$$

where $\alpha \in R$ and $z > 0$. For $K_\alpha(z)$ we know the asymptotic expansion as $z \to \infty$:

$$K_\alpha(z) \sim \sqrt{\frac{\pi}{2}} \frac{e^{-z}}{\sqrt{z}} \left(1 + \frac{4\alpha^2 - 1}{8z} + \cdots\right). \tag{71}$$

Let again ξ_λ be a random variable with the density $h_\lambda(x)$ and put $m_\lambda = E\xi_\lambda$ and $\sigma_\lambda^2 = D\xi_\lambda$.

Lemma 9. *(1) As $\lambda \to \infty$,*

$$Z(\lambda) \sim c_0 \sqrt{\frac{\pi}{2}} \left(\frac{\beta}{\lambda}\right)^{\alpha/2} \frac{e^{-2\sqrt{\lambda\beta}}}{(\lambda\beta)^{1/4}}. \tag{72}$$

(2) As $\lambda \to \infty$,

$$\begin{aligned} m_\lambda &\sim \sqrt{\frac{\beta}{\lambda}}, \\ \sigma_\lambda^2 &\sim \frac{1}{2\beta^{1/2}\lambda^{3/2}}. \end{aligned} \tag{73}$$

(3) There exists a unique λ_N such that $m_{\lambda_N} = 1/N$ such that $\lambda_N \sim \beta N^2$ and $\sigma_{\lambda_N}^2 \sim 2^{-1}\beta^{-2}N^{-3}$ as $N \to \infty$.

Proof. (1) We can write

$$Z(\lambda) = I_1(\lambda) + I_2(\lambda), \tag{74}$$

where

$$\begin{aligned} I_1(\lambda) &= \int_0^{\lambda^{-\epsilon}} e^{-\lambda x} g(x)\, dx, \\ I_2(\lambda) &= \int_{\lambda^{-\epsilon}}^1 e^{-\lambda x} g(x)\, dx \end{aligned} \tag{75}$$

and $\epsilon > 0$ is small enough. By (22)

$$I_1(\lambda) \sim c_0 I_1'(\lambda) = c_0 \int_0^{\lambda^{-\epsilon}} e^{-\lambda x - \beta/x} x^{\alpha-1}\, dx \tag{76}$$

as $\lambda \to \infty$. One can write

$$I_1'(\lambda) = Z_1(\lambda) - Z_1'(\lambda), \tag{77}$$

where

$$\begin{aligned} Z_1(\lambda) &= \int_0^\infty e^{-\lambda x - \beta/x} x^{\alpha-1}\, dx, \\ Z_1'(\lambda) &= \int_{\lambda^{-\epsilon}}^\infty e^{-\lambda x - \beta/x} x^{\alpha-1}\, dx. \end{aligned} \tag{78}$$

Find asymptotics of $Z_1(\lambda)$ as $\lambda \to \infty$. Changing variable $y = \beta^{-1}x$ gives

$$Z_1(\lambda) = \beta^\alpha \int_0^\infty e^{-\lambda\beta y - 1/y} y^{\alpha-1} dy \tag{79}$$

and changing variable $t = (2/z)x$ in (70) gives

$$K_\alpha(z) = 2^{-\alpha-1} z^\alpha \int_0^\infty t^{\alpha-1} e^{-(z^2/4)t - 1/t} dt. \tag{80}$$

Put $z = 2\sqrt{\lambda\beta}$. Then

$$Z_1(\lambda) = \frac{2\beta^{\alpha/2} K_\alpha\left(2\sqrt{\lambda\beta}\right)}{\lambda^{\alpha/2}} \tag{81}$$

and using (71) we get

$$Z_1(\lambda) \sim \sqrt{\frac{\pi}{2}} \left(\frac{\beta}{\lambda}\right)^{\alpha/2} \frac{e^{-2\sqrt{\lambda\beta}}}{(\lambda\beta)^{1/4}}, \quad \lambda \longrightarrow \infty. \tag{82}$$

Taking into account $I_2(\lambda) = O(e^{-\lambda^{1-\epsilon}})$ and $Z_1'(\lambda) = O(e^{-\lambda^{1-\epsilon}})$ for some small enough $\epsilon > 0$ we come to (72).

(2) Mathematical expectation is

$$m_\lambda = Z^{-1}(\lambda) \int_0^1 x e^{-\lambda x} g(x)\, dx. \tag{83}$$

By part (1) of this lemma we have

$$\int_0^1 x e^{-\lambda x} g(x)\, dx \sim \sqrt{\frac{\pi}{2}} \left(\frac{\beta}{\lambda}\right)^{(\alpha+1)/2} \frac{e^{-2\sqrt{\lambda\beta}}}{(\lambda\beta)^{1/4}}, \quad \lambda \longrightarrow \infty. \tag{84}$$

So

$$m_\lambda \sim \sqrt{\frac{\beta}{\lambda}}, \quad \lambda \longrightarrow \infty. \tag{85}$$

Covariance is equal to

$$\sigma_\lambda^2 = Z^{-1}(\lambda)\int_0^1 x^2 e^{-\lambda x} g(x)\,dx - Z^{-2}(\lambda)\left(\int_0^1 x e^{-\lambda x} g(x)\,dx\right)^2. \tag{86}$$

It follows from part (1) that, as $\lambda \to \infty$,

$$\sigma_\lambda^2 \sim \frac{\beta}{\lambda}\left(\frac{K_{\alpha+2}\left(2\sqrt{\lambda\beta}\right)}{K_\alpha\left(2\sqrt{\lambda\beta}\right)} - \left(\frac{K_{\alpha+1}\left(2\sqrt{\lambda\beta}\right)}{K_\alpha\left(2\sqrt{\lambda\beta}\right)}\right)^2\right). \tag{87}$$

Using asymptotic expansion (71) we get

$$\sigma_\lambda^2 \sim \frac{1}{2\beta^{1/2}\lambda^{3/2}} \tag{88}$$

as $\lambda \to \infty$.

(3) It follows from continuity of m_λ as function of λ and (85) that there exists λ_N such that $m_{\lambda_N} = 1/N$ and then $\lambda_N \sim \beta N^2$ as $N \to \infty$. By (88)

$$\sigma_{\lambda_N}^2 \sim \frac{1}{2\beta^2 N^3}, \quad N \longrightarrow \infty. \tag{89}$$

Lemma is proved. □

Lemma 10. *The exponential family (68) has the following properties:*

(1) *The normalized moment* $a_\lambda = E|\xi_\lambda - m_\lambda|^4/\sigma_\lambda^4$ *is bounded uniformly in* $\lambda > 0$.

(2) *For any* $\delta > 0$ *there exists* $\lambda_0 > 0$ *such that* $\sup_{\lambda>\lambda_0}\sup_{t>\delta}\phi_\lambda(t/\sigma_\lambda) < 1$.

(3) *For some* $q \geq 1$, *as* $\lambda \to \infty$,

$$\int_{-\infty}^{\infty} |\phi_\lambda(t)|^q\,dt = O\left(\lambda^{q\alpha}\right). \tag{90}$$

Proof. It is similar to Lemma 5.

Using Lemmas 6, 7, and 9 we find

$$d_N = D_N + o\left(N^{-3}\right), \tag{91}$$

where

$$D_N = \int_0^1 \left(x - \frac{1}{N}\right)^2 h_{\lambda_N}(x)\exp\left(-\frac{(x-1/N)^2}{2\sigma_{\lambda_N}^2(N-1)}\right)dx. \tag{92}$$

By Lemma 9 $\lambda_N \sim \beta N^2$ and $\sigma_{\lambda_N}^2 \sim 2^{-1}\beta^{-2}N^{-3}$ as $N \to \infty$, then

$$D_N \sim Z^{-1}(\lambda_N)\int_0^1 \left(x - \frac{1}{N}\right)^2 e^{-\beta N^2 x} g(x)\,x^{\alpha-1} \cdot \exp\left(-\beta^2 N^2\left(x - \frac{1}{N}\right)^2\right)dx. \tag{93}$$

We split the integral in (93) into two integrals:

$$D_N^{(1)} = \int_0^{N^{-1/2}} \left(x - \frac{1}{N}\right)^2 e^{-\beta N^2 x} g(x)\,x^{\alpha-1} \cdot \exp\left(-\beta^2 N^2\left(x - \frac{1}{N}\right)^2\right)dx,$$
$$D_N^{(2)} = \int_{N^{-1/2}}^1 \left(x - \frac{1}{N}\right)^2 e^{-\beta N^2 x} g(x)\,x^{\alpha-1} \cdot \exp\left(-\beta^2 N^2\left(x - \frac{1}{N}\right)^2\right)dx. \tag{94}$$

By condition (22)

$$Z^{-1}(\lambda_N)D_N^{(1)} \sim Z^{-1}(\lambda_N)c_0\int_0^{N^{-1/2}} \left(x - \frac{1}{N}\right)^2 \cdot e^{-\beta(1/x+N^2x)}x^{\alpha-1}\exp\left(-\beta^2 N^2\left(x - \frac{1}{N}\right)^2\right)dx. \tag{95}$$

To find the asymptotics of $D_N^{(1)}$ we use Laplace's method. Consider the function $s(x) = \beta(1/x + N^2 x)$. Its derivative

$$s'(x) = \beta\left(-\frac{1}{x^2} + N^2\right) \tag{96}$$

equals 0 at the point N^{-1}. The second derivative

$$s''(x) = \frac{2\beta}{x^3}. \tag{97}$$

The function $s(x)$ can be expanded at the neighborhood of N^{-1} using Taylor's formula:

$$s(x) = 2\beta N + \beta N^3\left(x - \frac{1}{N}\right)^2 + O\left(\left(x - \frac{1}{N}\right)^3\right). \tag{98}$$

By (72) we have

$$Z^{-1}(\lambda_N) \sim c_0^{-1}\sqrt{\frac{2\beta}{\pi}}e^{2\beta N}N^{\alpha+1/2} \tag{99}$$

and so we get, as $N \to \infty$,

$$Z^{-1}(\lambda_N)D_N^{(1)} \sim c_0^{-1}\sqrt{\frac{2\beta}{\pi}}e^{2\beta N}N^{\alpha+1/2}c_0\int_0^1 \left(x - \frac{1}{N}\right)^2 \cdot N^{-\alpha+1}e^{-2\beta N - \beta N^3(x-1/N)^2}dx. \tag{100}$$

After cancellations

$$\begin{aligned} Z^{-1}(\lambda_N) D_N^{(1)} &\sim \sqrt{\frac{2\beta}{\pi}} N^{3/2} \int_0^1 \left(x - \frac{1}{N}\right)^2 e^{-N^3(x-1/N)^2} dx \\ &\sim \sqrt{\frac{\beta}{2}} N^{-3} \end{aligned} \tag{101}$$

as $N \to \infty$. For the second integral $D_N^{(2)}$ we have, as $N \to \infty$,

$$Z^{-1}(\lambda_N) D_N^{(2)} = O\left(e^{-\beta\sqrt{N}}\right). \tag{102}$$

So

$$D_N \sim \sqrt{\frac{\beta}{2}} N^{-3}, \quad N \longrightarrow \infty. \tag{103}$$

Theorem is proved. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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