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Review Article

Uniqueness and Nonuniqueness in Inverse Problems for Elliptic Partial Differential Equations and Related Medical Imaging

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Unique determination issues about inverse problems for elliptic partial differential equations in divergence form are summarized and discussed. The inverse problems include medical imaging problems including electrical impedance tomography (EIT), diffuse optical tomography (DOT), and inverse scattering problem (ISP) which is an elliptic inverse problem closely related with DOT and EIT. If the coefficient inside the divergence is isotropic, many uniqueness results are known. However, it is known that inverse problem with anisotropic coefficients has many possible coefficients giving the same measured data for the inverse problem. For anisotropic coefficient with anomaly with or without jumps from known or unknown background, nonuniqueness of the inverse problems is discussed and the relation to cloaking or illusion of the anomaly is explained. The uniqueness and nonuniqueness issues are discussed firstly for EIT and secondly for ISP in similar arguments. Arguing the relation between source-to-detector map and Dirichlet-to-Neumann map in DOT and the uniqueness and nonuniqueness of DOT are also explained.

1. Introduction

Let us consider the following second order elliptic partial differential equations with Diriclet boundary value in a Lipschitz domain Ω compactly embedded in $\mathbb{R}^n$:

$$-\nabla\cdot(\sigma\nabla u)+k^2\tau u=q \quad \text{in } \Omega, \tag{1a}$$

$$u=f \quad \text{on } \partial\Omega, \tag{1b}$$

where k is a nonnegative number, q, f are real functions satisfying $q\in H^{-1}(\Omega)$, $f\in H^{1/2}(\partial\Omega)$, and σ is a real matrix and τ is a nonnegative function such that

$$L\le\frac{y^t\sigma y}{y^t y}\le U, \quad 0\le\tau\le U \tag{2}$$

for all $y\in\mathbb{R}^n\setminus\{\mathbb{O}\}$ and some positive constants L and U. It is known that (1a) and (1b) have a unique solution $u\in H^1(\Omega)$ [1]. Therefore, we can define the Dirichlet-to-Neumann map $\Lambda_{\sigma,\tau}: H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)$ as $\Lambda_{\sigma,\tau}(f)=\nu\cdot(\sigma\nabla u)|_{\partial\Omega}$ using the boundary trace operator. We will explain EIT, ISP, and DOT using (1a) and (1b) and Dirichlet-to-Neumann map or corresponding measurements map.

When $k=q=0$, EIT is formulated as to find σ such that $\Lambda_\sigma(:=\Lambda_{\sigma,0})=\Lambda$ for given Dirichlet-to-Neumann map Λ. In finite measurements case, EIT is to find the conductivity σ satisfying $\Lambda_\sigma(f_i)=\Lambda(f_i)$, $i=1,\dots,N$, for given finite Dirichlet and Neumann boundary measurement pairs $(f_i,\Lambda(f_i))_{i=1,\dots,N}$.

We discuss uniqueness of nonuniqueness of EIT by classifying σ into six cases for a Lipschitz domain D compactly embedded in Ω such that $\Omega\setminus\overline{D}$ is connected:

Case 1: $\sigma=\chi_{\Omega\setminus\overline{D}}+b\chi_D$, $b\neq1$ on ∂D,
Case 2: $\sigma=b(x)$,
Case 3: $\sigma=\chi_{\Omega\setminus\overline{D}}+b(x)\chi_D$, $b(x)\neq1$ on ∂D,
Case 4: $\sigma=I_n\chi_{\Omega\setminus\overline{D}}+B\chi_D$, $B\neq I_n$ on ∂D,
Case 5: $\sigma=B(x)$,
Case 6: $\sigma=I_n\chi_{\Omega\setminus\overline{D}}+B(x)\chi_D$, $B(x)\neq I_n$ on ∂D,

where

(i) b: a positive number,
(ii) $b(x)$: a positive function,

(iii) B: a symmetric positive-definite matrix,

(iv) $B(x)$: a symmetric positive-definite matrix function,

(v) I_n: the $n \times n$ identity matrix.

For Cases 1, 2, 3, and 4, uniqueness of the coefficient σ is known for general conditions on regularity of the conductivity. For Case 5, cloaking is heavily studied recently. Invisibility and cloaking of D are closely related with the nonuniqueness of coefficients not only in EIT [2–6], but also in acoustic scattering [7–9], electromagnetic scattering [10–12], and quantum scattering [13]. The idea of physical devices related to cloaking or invisibility is suggested more concretely than before such as wormhole and metamaterials and these studies draws much attention from physics and material engineering societies [14–16]. However, in Case 6, cloaking of the domain of anomaly D is impossible but the illusion of the property $B(x)$ is possible as in [11].

Acoustic wave propagation is described as follows:

$$\begin{gathered}\nabla \cdot (\sigma \nabla u) + k^2 \tau u = 0 \quad \text{in } \mathbb{R}^n \quad (n = 2, 3) \\ u(x) = e^{ikx\cdot d} + u^s(x) \\ \lim_{r\to\infty} r^{(n-1)/2} \left(\frac{\partial u^s}{\partial r}(r\hat{x}) - iku^s(r\hat{x}) \right) = 0 \\ \text{uniformly for all } \hat{x} = \frac{x}{r} \in S^{n-1},\end{gathered} \tag{3}$$

where σ is a positive-definite matrix function, τ is a positive function, and

$$\sigma - I_n, \qquad \tau - 1 \text{ have compact supports.} \tag{4}$$

It is known that there exists a unique function u^s satisfying (3) and (4) [17]. u^s is called scattering wave. The scattering wave u^s is approximated by far-field pattern u_∞ by the following relation:

$$u^s(x) = r^{(1-n)/2} e^{ikr} \left(u_\infty(\hat{x}) + O\left(\frac{1}{r}\right) \right). \tag{5}$$

The inverse scattering problem (ISP) is defined as follows: given far-field patterns $u_\infty(\cdot, d)$ for all incident directions $d \in S^{n-1}$, $n = 2, 3, \ldots$, identify coefficients σ and τ.

In this paper, we classify σ and τ as the following cases:

Case 7: $\sigma = 1$, $\tau = \chi_{\mathbb{R}^n\setminus\overline{D}} + c\chi_D$, $c \neq 1$ on ∂D,

Case 8: $\sigma = 1$, $\tau = \chi_{\mathbb{R}^n\setminus\overline{D}} + c(x)\chi_D$, $c(x) \neq 1$ on ∂D,

Case 9: $\sigma = 1$, $\tau = c(x)$,

Case 10: $\sigma = I_n\chi_{\mathbb{R}^n\setminus\overline{\Omega}} + B_+(x)\chi_{\Omega\setminus\overline{D}} + B_-(x)\chi_D$, $\tau = \chi_{\mathbb{R}^n\setminus\overline{D}} + c(x)\chi_D$,

Case 11: $\sigma = I_n\chi_{\mathbb{R}^n\setminus\overline{D}} + B(x)\chi_D$, $\tau = \chi_{\mathbb{R}^n\setminus\overline{D}} + c(x)\chi_D$, $c(x) \neq 1$, $B(x) \neq I_n$ on ∂D,

where c is a complex number and $c(x)$ is a complex function. Note that from (2) we have

$$L \leq \frac{y^t B(x) y}{y^t y}, c, c(x) \leq U \tag{6}$$

for all $y \in \mathbb{R}^n \setminus \{\mathbb{O}\}$.

ISP for (4) with σ, τ given in Cases 7, 8, 9, 10, and 11 is equivalently formulated as an inverse problem to find σ and τ from $\Lambda_{\sigma,\tau}$ in (1a) and (1b) for some Lipschitz domain Ω compactly embedded in $\mathbb{R}^n$ and containing $\overline{D}$ and $q = 0$. By Theorem 6.1.3 in [18], we can take Ω as a ball centered at the origin with radius R being chosen such that k is not a Dirichlet eigenvalue of (1a) with $q = 0$.

In this paper, DOT is explained as an inverse problem with respect to a forward problem formulated as an elliptic partial differential equation. Propagation of light in biological tissues is usually described by diffusion approximation equation in the frequency domain, the simplest but nontrivial approximation of the Boltzmann equation, as follows:

$$-\nabla \cdot (\kappa \nabla \Phi) + \left(\mu_a + \frac{im\omega}{l} \right) \Phi = q \quad \text{in } \Omega, \tag{7a}$$

$$\Phi + 2a\nu \cdot (\kappa \nabla \Phi) = 0 \quad \text{on } \partial\Omega, \tag{7b}$$

where Φ is photon density distribution, μ_a absorption coefficient, μ_s' reduced scattering coefficient, $\kappa = 1/3(\mu_a + \mu_s')$ a diffusion coefficient, and m refractive index. Usually, we assume $m = 1$ and a is boundary reflection coefficient, l the speed of light, ω modulation frequency of light, and ν outer unit normal vector.

DOT is to find the optical coefficients μ_a and μ_s' from the measurement informations $\Phi_{i,j}$ which is the value of the solution of (7a) and (7b) at $r_i \in \partial\Omega$ when $q(r) = \delta(r, r_j)$, $r_j \in \partial\Omega$. The r_i and r_j are usually called source and detector point, respectively.

Near infrared light is known to be deepest in the penetration depth to the tissue, compared to visible or near-visible lights. Therefore, near infrared light is used in DOT. DOT is known to be of low cost, portable, nonionized, and nonmagnetized. And DOT has higher temporal resolution and more functional information than conventionl structural medical imaging modalities such as magnetic resonance imaging (MRI) and computerized tomography (CT). For the comparison to other functional imaging modalities such as functional MRI (fMRI), photon emission tomography (PET), and electroencephalogram (EEG), see [19]. DOT is used in the area of breast imaging [20–22], functional neuroimaging [23, 24], brain computer interface (BCI) [25, 26], and the study about seizure [27, 28], new born infants [29, 30], osteoarthritis [31], and rat brain [32, 33].

We interpret DOT also as an inverse problem for (1a) and (1b) in isotropic coefficient and the uniqueness is discussed in the following cases:

Case 12: $m = m_0$ is given and $\omega \neq 0$, space dimension $n \geq 3$,

Case 13: $m = m_0$ is given and $\omega \neq 0$, space dimension $n = 2$,

Case 14: $\omega = 0$ or n is to be determined.

In Sections 2, 3, and 4, uniqueness and nonuniqueness of EIT, ISP, and DOT are treated, respectively. The relation between nonuniqueness and cloaking or illusion is also explained.

2. EIT

Consider $k = q = 0$ for the Dirichlet elliptic problem (1a) and (1b). EIT is formulated as follows:

(i) find σ such that $\Lambda_\sigma (:= \Lambda_{\sigma,0}) = \Lambda$ for given or measured Dirichlet-to-Neumann map Λ.

2.1. Uniqueness in Cases 1, 2, 3, and 4. The uniqueness studies of EIT in Cases 1, 2, 3, and 4 are summarized in the following way.

Case 1. One or two measurements uniqueness results are known when b is assumed to be known [34–36]. The finite measurement uniqueness is known only for Case 1. In this case, the number of measurements, the geometry of the obstacle D, and the choice of suitable Dirichlet or Neumann data minimizing the number of measurements are interesting issues.

Case 2. Many mathematicians conduct extensive works in this case and thus we are able to understand the unique determination of electrical conductivity when $b(x) \in L^\infty(\Omega)$ for two dimensional case [37–39] and $b(x) \in C^{3/2}(\Omega)$ for the case in dimensions higher than two [40–43].

Case 3. In [44], an orthogonality relation between the two solutions of (7a) and (7b) for arbitary obstacles D_1 and D_2 is derived. Based on this orthogonal relation and the Hahn-Banach theorem, the uniqueness for Case 3 is derived.

Case 4. The uniqueness in Case 4 is also proved in [44] with additional condition that $B - A$ is positive-definite: this additional condition is generalized and removed by [45, 46].

2.2. Nonuniqueness in Case 5. The nonuniqueness of EIT in Case 5 is observed early in [47]: if F is a boundary fixing diffeomorphism on Ω and push-forward map F_* is defined by

$$F_*\sigma = \frac{(DF)\,\sigma\,(DF)^t}{\det(DF)}, \tag{8}$$

then we have

$$\Lambda_\sigma = \Lambda_{F_*\sigma}. \tag{9}$$

The proof of (9) is summarized well in [6]: Knowing the Dirichlet-to-Neumann map is equivalent to knowing the quadratic form

$$Q(v) = \int \sigma\nabla v \cdot \nabla v \quad v \in H^1(\Omega) \tag{10}$$

by the polarization identity. Then, (9) follows from the change of variable method for the quadratic form

$$\begin{aligned}&\int_\Omega \sum_{ij} \sigma_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dx \\ &\quad = \int_\Omega \sum_{ij} \sigma_{ij} \frac{\partial u}{\partial y_k}\frac{\partial y_k}{\partial x_i}\frac{\partial u}{\partial y_l}\frac{\partial y_l}{\partial x_j} \det\left(\frac{\partial x}{\partial y}\right) dy.\end{aligned} \tag{11}$$

Many researchers raised questions whether the change of variable (9) is a unique obstruction to the uniqueness. And it is proved that

$$\Lambda_{\sigma_1} = \Lambda_{\sigma_2} \text{ implies } \sigma_2 = F_*\sigma_1 \tag{12}$$

for some boundary fixing diffeomorphism F in the following cases:

(i) $n = 2$ and $\sigma \in C^3(\Omega)$ [48],
(ii) $n = 2$ and $\sigma \in C^{0,1}(\Omega)$ (Lipschitz functions) [49],
(iii) $n = 2$ and $\sigma \in L^\infty(\Omega)$ [50],
(iv) $n \geq 3$: ∂D and B are analytic [51, 52].

2.3. Near-Cloaking in Case 5. Let D be some domain and contained in Ω, $B_+ = B|_{\Omega\setminus\overline{D}}$, $B_- = B|_D$, and

$$\sigma = B_+(x)\,\chi_{\Omega\setminus\overline{D}} + B_-(x)\,\chi_D. \tag{13}$$

Define

$$\sigma_F := F_*\sigma = I_n\chi_{\Omega\setminus\epsilon\overline{D}} + F_*B_-(x)\,\chi_{\epsilon D}(x), \tag{14}$$

for a boundary-fixing diffeomorphism F on Ω transforming D into arbitrary small domain ϵD and $F_*B_+ = I_n$ on $\Omega \setminus \overline{D}$. Then, D is nearly cloaked into ϵD, since $\Lambda_\sigma = \Lambda_{\sigma_F}$ by (9). If ϵ goes to 0, D is called perfectly cloaked. Otherwise, D is called nearly cloaked into ϵD. Conversely, if D is nearly cloaked into ϵD, there is a diffeomorphism F such that $\Lambda_\sigma = \Lambda_{\sigma_F}$ by (12).

For example, if Ω, D and ϵD are two-dimensional disks of radii 2, 1, and $\epsilon > 0$, respectively, centered at $\mathbb{O}$, we could take the diffeomorphism F on Ω mapping D into ϵD as follows:

$$F(x) = \epsilon x\chi_D(x) + ((2-\epsilon)|x| - (2-2\epsilon))\frac{x}{|x|}\chi_{\Omega\setminus\overline{D}}(x). \tag{15}$$

2.4. Illusion in Case 6. In Case 6, the uniqueness of D is solved in [44, 45, 53] and the nonuniqueness of $B(x)$ inside D is shown using (12) [53]. This is called illusion of material property $B(x)$ inside the uniquely determined domain D. Note that D is not cloaked into ϵD any more.

In more detail, if

$$\Lambda_{I_n\chi_{\Omega\setminus\overline{D}_1}+B_1(x)\chi_{D_1}} = \Lambda_{I_n\chi_{\Omega\setminus\overline{D}_2}+B_2(x)\chi_{D_2}} \tag{16}$$

then we have

$$D_1 = D_2. \tag{17}$$

And there is a boundary fixing diffeomorphism F on D such that

$$B_2 = F_*B_1. \tag{18}$$

Therefore, in Case 6, the domain D is uniquely determined by the Dirchlet-to-Neumann map; however, the property B is nonunique up to the change of variables inside $D_1 = D_2$. Specifically, the domain of anomaly D cannot be cloaked into a much smaller domain, but the property B_1 could be illuded into other property B_2. We will call (18) property illusion or just illusion.

2.5. Cloaking in Case 5 versus Illusion in Case 6. Define three kinds of conductivities as follows:

$$\sigma_1 = I_n \chi_{\Omega\setminus\overline{D}} + B_1 \chi_D, \tag{19a}$$

$$\sigma_2 = B_{2+} \chi_{\Omega\setminus\overline{D}} + B_{2-} \chi_D, \tag{19b}$$

$$\sigma_3 = I_n \chi_{\Omega\setminus\epsilon\overline{D}} + B_3 \chi_{\epsilon D}. \tag{19c}$$

Cloaking and illusion for three conductivities (19a), (19b), and (19c) are summarized as follows:

(i) $\Lambda_{\sigma_1} = \Lambda_{\sigma_3}$ implies that $\epsilon = 1$ and, with some more assumptions [53], there exists a diffeomorphism F on D such that $\sigma_1 = F_* \sigma_3$. Therefore, only property illusion between B_1 and B_3 is possible and near cloaking of domain is not possible;

(ii) $\Lambda_{\sigma_2} = \Lambda_{\sigma_3}$ implies that there exists a diffeomorphism F on Ω such that $\sigma_2 = F_* \sigma_3$. Therefore, near-cloaking of domain is possible.

Comparing the above two results, we can conclude that if background conductivity is known only on the boundary, near-cloaking of the domain D is possible to any smaller domain ϵD. But if the background conductivity is known in the neighborhood of the boundary, near-cloaking of the domain D is not possible and only property illusion between B_1 and B_2 is possible.

3. ISP

ISP is formulated as follows:

(i) given far-field patterns $u_\infty(\cdot, d)$ for all incident directions $d \in S^{n-1}, n = 2, 3, \ldots$, identify coefficients σ and τ for (4).

Let E_R be a ball centered at the origin with radius R being chosen such that k is not a Dirichlet eigenvalue of (1a) with $q = 0$ and compactly embedded in $\mathbb{R}^n$ and containing $\overline{D}$. Then ISP is also formulated as follows (Theorem 6.1.3 in [54]):

(ii) find σ and τ from $\Lambda_{\sigma,\tau}$ in (1a) and (1b) with $q = 0$ for E_R.

3.1. Uniqueness in Cases 7, 8, and 9

Case 7. The case for positive constant c can be understood as special cases of Case 8. The limiting cases $c = \infty$ and $c = 0$ could be considered as (3) in $\Omega \setminus \overline{D}$ and the boundary condition on ∂D as $\Phi = 0$ (sound-soft case) and $\partial\Phi/\partial\nu = 0$ (sound-hard case) on ∂D. The uniqueness for sound-soft and sound-hard obstacle D is considered in [17, 55].

Case 8. This case is called "inverse transmission problem" and the uniqueness is solved in [17].

Case 9. Uniqueness with a few subcases is solved in [18].

3.2. Nonuniqueness in Case 10. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism such that $F|_{\mathbb{R}^n\setminus\Omega}$ is an identity map on $\mathbb{R}^n\setminus\Omega$. Then

$$u^\infty_{\sigma,\tau}(\cdot, d) = u^\infty_{F_*(\sigma,\tau)}(\cdot, d) \quad \forall d \in S^{n-1}, \tag{20}$$

where push-forward map is defined by

$$F_*\sigma = \frac{(DF)\,\sigma\,(DF)^t}{\det(DF)}, \qquad F_*\tau = \frac{\tau}{\det(DF)}. \tag{21}$$

This is just change of variable in the weak formulation of the direct problem such that

$$\begin{aligned} &\int_\Omega \sum_{ij} \sigma_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dx \\ &= \int_\Omega \sum_{ij} \sigma_{ij} \frac{\partial u}{\partial y_k}\frac{\partial y_k}{\partial x_i}\frac{\partial u}{\partial y_l}\frac{\partial y_l}{\partial x_j} \det\left(\frac{\partial x}{\partial y}\right) dy, \\ &\int_\Omega \tau u^2 dx = \int_D \tau u^2 \det\left(\frac{\partial x}{\partial y}\right) dy. \end{aligned} \tag{22}$$

We have similar question for EIT. Is the change of variable unique obstruction to the uniqueness of anisotropic ISP? And it is proved that

$$u^\infty_{\sigma_1}(\cdot, d) = u^\infty_{\sigma_2}(\cdot, d) \quad \forall d \in S^{n-1} \text{ implies } \sigma_2 = F_*\sigma_1 \tag{23}$$

for some boundary fixing diffeomorphism F in the following references:

(i) $n = 2$ [56],

(ii) $n \geq 3$ [57].

3.3. Near-Cloaking in Case 10. Let D be some domain and contained in Ω, $B_+ = B|_{\Omega\setminus\overline{D}}$, $B_- = B|_D$, and

$$\sigma = I_n \chi_{\mathbb{R}^n\setminus\overline{\Omega}} + B_+(x)\,\chi_{\Omega\setminus\overline{D}} + B_-(x)\,\chi_D. \tag{24}$$

Define

$$\sigma_F := F_*\sigma = I_n \chi_{\mathbb{R}^n\setminus\epsilon\overline{D}} + F_* B_-(x)\,\chi_{\epsilon D}(x), \tag{25}$$

for a diffeomorphism F on $\mathbb{R}^n$ transforming D into arbitrary small domain ϵD, $F_* B_+ = I_n$ on $\Omega\setminus\overline{D}$, and fixing $\mathbb{R}^n\setminus\overline{\Omega}$. Then, D is nearly cloaked into ϵD, since $u_\sigma(\cdot, d) = u^\infty_{\sigma_F}(\cdot, d)$ for all $d \in S^{n-1}$ by (20). If ϵ goes to 0, D is called perfectly cloaked. Otherwise, D is called nearly cloaked into ϵD. Conversely, if D is nearly cloaked into ϵD, there is a diffeomorphism F such that $u_\sigma(\cdot, d) = u^\infty_{\sigma_F}(\cdot, d)$ for all $d \in S^{n-1}$ by (23).

For example, if Ω, D and ϵD are two-dimensional disks of radii 2, 1, and $\epsilon > 0$, respectively, centered at $\mathbb{O}$, we could take the diffeomorphism F on Ω mapping D into ϵD as follows:

$$\begin{aligned} F(x) = {} & \epsilon x \chi_D(x) + ((2-\epsilon)|x| - (2-2\epsilon))\frac{x}{|x|}\chi_{\Omega\setminus\overline{D}}(x) \\ & + \chi_{\mathbb{R}^n\setminus\overline{\Omega}}(x). \end{aligned} \tag{26}$$

3.4. Illusion in Case 11. In case 11, D is uniquely determined and is studied in [18, 53, 58–60] but $B(x)$ is not uniquely determined. That is to say, D is uniquely determined and not cloaked into any smaller domain but $B(x)$ is illuded into another property B' by some diffeomorphism F on $\mathbb{R}^n$. In more detail, if

$$u^{\infty}_{\sigma_1,\tau_1}(\cdot,d) = u^{\infty}_{\sigma_2,\tau_2}(\cdot,d) \quad \forall d \in S^{n-1}, \tag{27}$$

where

$$\begin{aligned} \sigma_i &= I_n\chi_{\mathbb{R}^n\setminus D_i} + B_i(x)\chi_{D_i}, \\ \tau_i &= \chi_{\mathbb{R}^n\setminus D_i} + c(x)\chi_{D_i}, \\ & \qquad i = 1,2, \end{aligned} \tag{28}$$

then we have

$$D_1 = D_2 \tag{29}$$

and there is exterior fixing diffeomorphism F on D such that

$$B_2 = F_* B_1, \qquad c_2 = \frac{c_1}{|\det(DF)|}. \tag{30}$$

In [11], a few interesting property illusions are considered; the optical transformation of an object into another object with different property is considered. In the paper, stereoscopic image of a man is transformed into an illusion image of a woman and dielectric spoon of the electric permeability 2 into an illusion image of metallic cup of electric permeability −1 in an electromagnetic scattering problem.

3.5. Cloaking in Case 10 and Illusion in Case 11. Define three pairs of coefficients as follows:

$$\begin{aligned} (\sigma_1(x), q_1(x)) &\\ = \chi_{\mathbb{R}^n\setminus\overline{D}}(x) &+ (B_1(x), c_1(x))\chi_D(x), \end{aligned} \tag{31a}$$

$$\begin{aligned} (\sigma_2(x), q_2(x)) &\\ = \chi_{\mathbb{R}^n\setminus\overline{\Omega}}(x) &+ (B_{2+}, c_{2+})\chi_{\Omega\setminus\overline{D}}(x) + (B_{2-}, c_{2-})\chi_D(x), \end{aligned} \tag{31b}$$

$$\begin{aligned} (\sigma_3(x), q_3(x)) &\\ = \chi_{\mathbb{R}^n\setminus\overline{\epsilon D}}(x) &+ (B_3(x), c_3(x))\chi_{\epsilon D}(x), \end{aligned} \tag{31c}$$

where D is compactly imbedded in Ω, which is also compactly imbedded in $\mathbb{R}^n$.

Let us denote $u^{\infty}_i := u^{\infty}_{\sigma_i,\tau_i}$. Cloaking and illusion for three coefficients in (31a), (31b), and (31c) are summarized as follows:

(i) $u^{\infty}_1(\cdot,d) = u^{\infty}_2(\cdot,d)$ for all $d \in S^{n-1}$ implies $\epsilon = 1$ and, with additional assumptions in [53], there is a exterior fixing diffeomorphism F on D such that

$$(B_3(x), c_3(x)) = F_*(B_1(x), c_1(x)) \tag{32}$$

[56, 57]. Therefore, only property illusion between B_1 and B_2 is possible and near-cloaking of domain D is not possible:

(ii) $u^{\infty}_2(\cdot,d) = u^{\infty}_3(\cdot,d)$ for all $d \in S^{n-1}$ implies that there is an exterior fixing diffeomorphism F on D such that

$$(B_3(x), c_3(x)) = F_*(B_2(x), c_2(x)). \tag{33}$$

Therefore, near-cloaking of the domain is possible in this case.

Also as in EIT, we can conclude that if background identity coefficient is known only in $\mathbb{R}^n \setminus \Omega$, near-cloaking of the domain is possible to any smaller domain ϵD. But if the background coefficient is known and fixed in $\mathbb{R}^n \setminus D$, near-cloaking of the domain is not possible and only property illusion between B_1 and B_2 is possible.

4. DOT

DOT is formulated using source-to-detector map as follows:

(i) find the optical coefficients μ_a and μ'_s from the measurement informations $\Phi_{i,j}$ which is the value of the solution of (7a) and (7b) at $r_i \in \partial\Omega$ when $q(r) = \delta(r, r_j), r_j \in \partial\Omega$.

r_i and r_j are usually called source and detector point, respectively. We assumed that μ_a, μ'_s are isotropic in this paper.

Suppose that $\partial\kappa/\partial\nu = 0$ on $\partial\Omega$. Then DOT is also formulated using Dirichlet-to-Neumann map as follows:

(ii) find complex valued τ from $\Lambda_{1,\tau}$ in (1a) and (1b) with $q = 0$ and given $k = 1$.

4.1. Source-to-Detector Map, Dirichlet-to-Neumann Map, and Far-Field Map. We summarize the relation between source-to-detector map and Dirichlet-to-Neumann map following the approach used in [61].

By setting $\Psi = \sqrt{\kappa}\Phi$ and $\tau = \sqrt{\Delta\sqrt{\kappa}/\sqrt{\kappa} + \mu_a/\kappa + i(m\omega/l\kappa)}$ with $\operatorname{Im}(\tau) \geq 0$, we have

$$-\Delta\Psi + \tau\Psi = \frac{q}{\sqrt{\kappa}} \quad \text{in } \Omega, \tag{34a}$$

$$\Psi + 2a\nu\cdot(\kappa\nabla\Psi) = 0 \quad \text{on } \partial\Omega. \tag{34b}$$

If $\tau = 1$ and $q(\cdot)/\sqrt{\kappa} = \delta(\cdot, r_s)$ for some source point r_s, we have the following solution of (34a) as follows:

$$\Psi(r) = R(r, r_s) = \frac{e^{ik|r-r_s|}}{4\pi|r-r_s|}. \tag{35}$$

For the fundamental solution with nonconstant function τ, see [62].

When $\mu_a, \kappa, \nabla\kappa$ has upper and lower bound and q is contained in $H^{-1}(\Omega)$ or a Dirac delta function, (7a), (7b), (34a), and (34b) have a unique solution Φ and Ψ contained in $H^1(\Omega)$, respectively [1, 63].

Boundary value problem (7a) and (7b) with $q(r) = \delta(r, r_s)$ is equivalent to boundary value problem with (7a) for $q = 0$ and nonzero Robin boundary condition replacing (7b). This argument can be proved using the function H in [62]. Therefore, DOT is redescribed as to find the optical coefficients from Robin-to-Dirichlet map defined as a map from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$. Using unique solvability of (7a) with Dirichlet or Neumann boundary condition replacing (7b), Robin-to-Dirichlet map is equivalent to Neumann-to-Dirichlet map and to Dirichlet-to-Neumann map.

4.2. Uniqueness and Nonuniqueness of DOT. The research about unique determination of the optical coefficients in DOT is rare except [61], but it is a very important issue for DOT as an inverse problem. The determination of optical coefficients (μ_a, μ_s) in (7a) and (7b) is equivalent to the determination of τ in (34a) and (34b) when $\omega \neq 0$.

Using the relation between source-to-detector map for (7a) and (7b) and Dirichlet-to-Neumamm map for (34a), we add a comment on the result of [42, 54, 61] and reference therein.

Case 12. τ is determined in $L^{\infty}(\Omega)$ (Theorem 5.2.2 in [54]) and μ_a, μ_s' is determined by comparing the real and imaginary part of τ.

Case 13. τ is "almost" determined in $H^1_{\infty(\Omega)}$ (Theorem 5.5.3 in [54]) and μ_a, μ_s' is determined in a similar way for $n \geq 3$.

Case 14. Even though τ is determined, if refractive index m is not known or modulation frequency $\omega = 0$, we cannot determine μ_a and μ_s', simultaneously. If m is not known, we should have at least three equations and we only know at most two equations for real and imaginary part of τ. And if $\omega = 0$, we have no information on the imaginary part and we only have one equation. The detailed nonuniqueness example is given in [61].

In summary, if refractive index is not known or continuous light source case $\omega = 0$ is used, we cannot uniquely determine the optical coefficients and if refractive index is known and frequency domain light source $\omega \neq 0$ is used, we can uniquely determine optical coefficients.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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