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Research Article

Numerical Simulation of Entropy Growth for a Nonlinear Evolutionary Model of Random Markets

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In this communication, the generalized continuous economic model for random markets is revisited. In this model for random markets, agents trade by pairs and exchange their money in a random and conservative way. They display the exponential wealth distribution as asymptotic equilibrium, independently of the effectiveness of the transactions and of the limitation of the total wealth. In the current work, entropy of mentioned model is defined and then some theorems on entropy growth of this evolutionary problem are given. Furthermore, the entropy increasing by simulation on some numerical examples is verified.

1. Introduction

In the last years, it has been reported [1, 2] that, in western societies, around 95% of the population, the middle and lower economic classes of society, arrange their incomes in an exponential wealth distribution. The incomes of the rest of the population, around 5% of individuals, fit a power law distribution [3]. A kind of models considering the randomness associated to markets is the gas-like models [4]. These random models interpret economic exchanges of money between agents similarly to collisions in a gas where particles share their energy [5]. In this communication, the increasing of the entropy when these systems evolve toward the asymptotic equilibrium is checked. This is associated with the existence of an H-theorem for all these economic models [6, 7]. Exponential distribution is ubiquitous in the framework of multiagent systems. Usually, it appears as an equilibrium state in the asymptotic time evolution of statistical systems. It has been explained from very different perspectives. In statistical physics, it is obtained from the principle of maximum entropy [8]. In the same context, it can also be derived without any consideration about information theory, only from geometrical arguments under the hypothesis of equiprobability in phase space [9]. Also, several multiagent economic models based on mappings, with random, deterministic, or chaotic interactions, can give rise to the asymptotic appearance of the exponential wealth distribution [10–13]. Concretely, the continuous economic model for random markets is given by

$$Ty(x) = \iint_{u+v\geq x} \frac{y(u)\,y(v)}{u+v}\,du\,dv. \tag{1}$$

It is found that the exponential distribution is a stable fixed point of this type of system. In this work, first entropy of mentioned model is defined and then we give some theorems on entropy growth of this evolutionary problem in the next section. Moreover, we illustrate the entropy increasing by simulation on some numerical examples in Section 3. Finally, some concluding remarks are given in Section 4.

2. Entropy Growth

Definition 1. We define space S as follows:

$$S = \left\{ p(x) : p(x) \in L_1^+[0,+\infty),\ \|p(x)\| = 1,\ \int_0^{+\infty} x p(x)\,dx = \frac{1}{\alpha} \right\}. \tag{2}$$

Definition 2. For each $p(x) \in S$, we define $H(p) = -\int_0^{+\infty} p(x)\log[p(x)]dx$ as entropy of $p(x)$.

Definition 3. Assume $y(x) \in S$; we introduce a sequence of real numbers as

$$H_n(y) = -\int_0^{+\infty} T^n y(x) \log\left[T^n y(x)\right] dx, \tag{3}$$

where $Ty(x) = \iint_{u+v\geq x} (y(u)y(v)/(u+v))du\,dv$.

Lemma 4. *Suppose that $p(x) \in L_1^+[0,+\infty)$ is decreasing function; then there exists $\beta > 0$, so that $\forall x \in [\beta,+\infty)$: $p(x) \ll |1/\log[p(x)]|$.*

Proof. It is clearly a consequence of this fact that for each $\varepsilon > 0$ there exists $\delta > 0$ so that $\forall 0 < x < \delta : |x\log x| < \varepsilon$. Now from this point that $p(x)$ is decreasing, function $p(x)$ tends to zero. Therefore, there exists surely a large number as β so that we have $p(x) \ll |1/\log[p(x)]|$ in the interval $[\beta,+\infty)$. □

Theorem 5. *Entropy function $H : S \to \mathscr{R}$ is bounded. In other words, there exist M so that $\forall y(x) \in S : |H(y)| < M$.*

Proof. Suppose that $y(x) \in S$; then it is clear that there exists $\varphi \in \mathscr{R}^+$ so that $y(x)$ is decreasing function in the interval $[\varphi,+\infty)$, because in the case of the fact that function $y(x)$ is not decreasing we have $\int_\varphi^{+\infty} y(x)dx = +\infty$ which is contradiction by definition of the set S. Now, it is obvious from Lemma 4 for decreasing function $y(x)$ to say that there exist $\beta > \varphi$ and $\gamma, k > 1$ and $c > 0$ so that $|1/\log[y(x)]| > 1/c(1+x^k)$ and $y(x) < 1/x^{k+\gamma}$; therefore

$$\begin{aligned}
|H(y)| &= \left|\int_0^{+\infty} y(x)\log\left[y(x)\right]dx\right| \\
&= \left|\int_0^{\beta} y(x)\log\left[y(x)\right]dx + \int_\beta^{+\infty} y(x)\log\left[y(x)\right]dx\right| \\
&\leq \left|\int_0^{\beta} y(x)\log\left[y(x)\right]dx\right| \\
&\quad + \left|\int_\beta^{+\infty} y(x)\log\left[y(x)\right]dx\right| \\
&< M_1 + \int_\beta^{+\infty} y(x)\left|\log\left[y(x)\right]\right|dx \leq M_1 \\
&\quad + \int_\beta^{+\infty} \frac{c\left(1+x^k\right)}{x^{k+\gamma}}dx \\
&= M_1 + \int_\beta^{+\infty}\frac{c}{x^{k+\gamma}}dx + \int_\beta^{+\infty}\frac{c}{x^{\gamma}}dx \leq M_1 + M_2 + M_3 \\
&= M,
\end{aligned} \tag{4}$$

so the proof is completed. □

Corollary 6. *The sequence of real numbers $H_n(y)$, $y(x) \in S$, is bounded sequence.*

Proof. It is obviously the consequence of Theorem 5. □

Theorem 7. *Suppose that $\lim_{n\to\infty} T^n y(x)$, $y(x) \in S$ exists, and then $\lim_{n\to\infty} H_n(y) = 1 - \log\alpha$.*

Proof. We know that $\lim_{n\to\infty} T^n y(x) = \alpha e^{-\alpha x}$. Now, since the function $\log(\cdot)$ is continuous in the interval $(0,+\infty)$, then we have

$$\begin{aligned}
\lim_{n\to\infty} H_n(y) &= -\lim_{n\to\infty}\left[\int_0^{+\infty} T^n y(x)\log\left[T^n y(x)\right]dx\right] \\
&= -\int_0^{+\infty}\lim_{n\to\infty}\left[T^n y(x)\right]\log\left[\lim_{n\to\infty}\left[T^n y(x)\right]\right]dx \\
&= -\int_0^{+\infty}\alpha e^{-\alpha x}\log\left[\alpha e^{-\alpha x}\right]dx \\
&= -\int_0^{+\infty}\alpha e^{-\alpha x}\log\left[\alpha\right]dx + \int_0^{+\infty}\alpha^2 x e^{-\alpha x}dx \\
&= -\log\left[\alpha\right] + 1.
\end{aligned} \tag{5}$$

□

Theorem 8. *Suppose that $H_n(y)$, $y(x) \in S$, is increasing sequence; then the entropy of $\alpha e^{-\alpha x}$ is the supremum among entropies of all distributions in the set S.*

Proof. Assume that $H_n(y)$ is increasing sequence; on the other hand by Corollary 6 we know that $H_n(y)$ is bounded. Therefore there is a limit for sequence $H_n(y)$; from Theorem 5 this limit should be $1 - \log\alpha$; that is,

$$\lim_{n\to\infty} H_n(y) = 1 - \log\alpha = H\left(\alpha e^{-\alpha x}\right). \tag{6}$$

Then the entropy of $\alpha e^{-\alpha x}$ is the supremum. □

Theorem 9. *Suppose that $\lim_{n\to\infty} T^n y(x)$, $y(x) \in S$, exists and the entropy of $\alpha e^{-\alpha x}$ is the supremum among the entropies of all distributions in the set S; then $H_n(y)$, $y(x) \in S$, is increasing sequence for enough large $n \in \mathbb{N}$.*

Proof. Existence of $\lim_{n\to\infty} T^n y(x)$ implies that $\lim_{n\to\infty} H_n(y) = H(\alpha e^{-\alpha x})$; then for each $\varepsilon > 0$ there exist N such that for all $n > N$ we have $|H_n(y) - H(\alpha e^{-\alpha x})| < \varepsilon$. On the other hand we know that $H(\alpha e^{-\alpha x})$ is supremum; therefore for all $n > N$ we have

$$H\left(\alpha e^{-\alpha x}\right) - H_{n+1}(y) \leq H\left(\alpha e^{-\alpha x}\right) - H_n(y) < \varepsilon. \tag{7}$$

Hence, $H_n(y) \leq H_{n+1}(y)$ which means $H_n(y)$ is increasing sequence. □

3. Numerical Simulations

Example 1. Consider $p(x) = xe^{-x} \in S$; then $\int_0^{+\infty} xp(x)dx = 2$ and so $\alpha = 1/2$ and

$$1 - \log(\alpha) = 1 - \log\left(\frac{1}{2}\right) = 1.6931. \tag{8}$$

Table 1

$p(x)$	$H_0(p)$	$H_1(p)$	$1-\log(\alpha)$
$6/(1+2x)^4$	−0.458426	−0.430621	−0.386294
$2\sqrt{2}/\pi(1+x^4)$	0.613654	0.642990	0.653426
$2\,\mathrm{sech}[x]/\pi$	1.144729	1.150917	1.153787
$(1-\tanh[x])/\log[2]$	0.473482	0.476254	0.477918
$4/\pi(1+x^2)^2$	0.531024	0.540870	0.548417

$H_n(p)$ for some primary n have been calculated as follows:

$$\begin{aligned} H_0(p) &= 1.5772 \\ H_1(p) &= 1.6667 \\ H_2(p) &= 1.6839 \\ H_3(p) &= 1.6895 \\ H_4(p) &= 1.6917 \\ H_5(p) &= 1.6925. \end{aligned} \tag{9}$$

Example 2. Consider $p(x) = e^{-2x} + e^{-x}/2 \in S$; then $\int_0^{+\infty} xp(x)dx = 3/4$ and so $\alpha = 4/3$ and

$$1-\log(\alpha) = 1-\log\left(\frac{4}{3}\right) = 0.712318. \tag{10}$$

$H_n(p)$ for some primary n have been calculated as follows:

$$\begin{aligned} H_0(p) &= 0.707208 \\ H_1(p) &= 0.709950 \\ H_2(p) &= 0.711228. \end{aligned} \tag{11}$$

Example 3. Consider $p(x) = 9(\log[10])^2 x/10^{3x} \in S$; then $\int_0^{+\infty} xp(x)dx = 2/3\log[10]$ and so $\alpha = 3\log[10]/2$ and

$$1-\log(\alpha) = 1-\log\left(\frac{3\log[10]}{2}\right) = -0.239498. \tag{12}$$

$H_n(p)$ for some primary n have been calculated as follows:

$$\begin{aligned} H_0(p) &= -0.355429 \\ H_1(p) &- 0.265942 \\ H_2(p) &= -0.248720. \end{aligned} \tag{13}$$

Example 4. In this example we have provided Table 1 of other types of distributions in S and calculated some primary $H_n(p)$ and corresponding $1-\log(\alpha)$ for each function as well.

4. Conclusion

Different versions of a continuous economic model that takes into account idealistic characteristics of the markets have been reconsidered. In these models, the agents interact by pairs and exchange their money in a random way. The asymptotic steady state of these models is the exponential wealth distribution. The system decays to this final distribution with a monotonic increasing of the entropy taking its maximum value just on the equilibrium.

Competing Interests

The authors declare that they have no competing interests.

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