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Research Article

Tight K-g-Frame and Its Novel Characterizations via Atomic Systems

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K-g-frame is a generalization of g-frame. We generalize the tight g-frame to K-g-frame via atomic systems. In this paper, the definition of tight K-g-frame is put forward; equivalent characterizations and necessary conditions of tight K-g-frame are given. In particular, the necessary and sufficient condition for tight K-g-frame being tight g-frame is obtained. Finally, by means of methods and techniques of frame theory, several properties of tight K-g-frame are given.

1. Introduction

Frame in Hilbert space was first introduced in 1952 by Duffin and Schaeffer [1] to deal with nonharmonic Fourier series and reintroduced in 1986 by Daubechies et al. [2]. Since then the frame theory began to be more wildly studied. Today, frame theory has been widely used in filter theory [3], image processing [4], numerical analysis, and other areas. We refer to [5–8] for an introduction to frame theory in Hilbert space and its application.

With the deepening of research on frame theory, various generalizations of frames have been proposed; see [9–12]. Atomic systems for subspaces were first introduced by Feichtinger and Werther in [13] based on examples arising in sampling theory. In 2011, Găvruţa [14] introduced K-frame in Hilbert spaces to study atomic decomposition systems and discussed some properties of them. In [15–18], some conclusions of K-frame were given. With the extensive research of K-frame and g-frame in Hilbert space, Zhu et al. [19, 20] began to study K-g-frame, which was limited to the range of a bounded linear operator in Hilbert space and had gained greater flexibility in practical application relative to g-frame. K-g-frame, as a more general frame than g-frame and K-frame, has become one of the most active fields in frame theory in recent years. In [20, 21], several properties and characterizations of K-g-frame were obtained. However, many problems of K-g-frame have not been studied. Based on these important results of K-g-frame, we extend tight g-frame to K-g-frame and put forward the concept of tight K-g-frame. In this paper, we give equivalent characterizations and necessary conditions of tight K-g-frame for Hilbert space. We also obtain the necessary and sufficient condition of tight K-g-frame to be tight g-frame. Finally, we present several properties of tight K-g-frame for Hilbert space.

Throughout this paper, H is separable Hilbert space and I is the identity operator. $L(H_1, H_2)$ is a collection of all bounded linear operators from H_1 to H_2, where H_1 and H_2 are two Hilbert spaces. In particular, $L(H)$ is a collection of all bounded linear operators from H to H. For any $T \in L(H_1, H_2)$, $R(T)$ is the range of T and T^* is the adjoint operator of T. $\{H_j : j \in J\}$ is a sequence of closed subspaces of H, where J is a subset of integers $\mathbb{Z}$. $l^2(\{H_j\}_{j\in J})$ is defined by

$$l^2\left(\{H_j\}_{j\in J}\right) = \left\{\{a_j\}_{j\in J} : a_j \in H_j,\ \ j \in J,\ \sum_{j\in J}\|a_j\|^2 < +\infty\right\}, \tag{1}$$

with the inner product given by

$$\left\langle \{a_j\}_{j\in J}, \{b_j\}_{j\in J} \right\rangle = \sum_{j\in J} \langle a_j, b_j \rangle_{H_j}. \tag{2}$$

It is clear that $l^2(\{H_j\}_{j\in J})$ is a complex Hilbert space.

2. Preliminaries

In this section, some necessary definitions and lemmas are introduced.

Definition 1 (see [9, Definition 1.1]). A sequence $\{\Lambda_j \in L(H, H_j) : j \in J\}$ is called a g-frame for H with respect to $\{H_j\}_{j\in J}$ if there exist two positive constants A and B such that, for all $f \in H$,

$$A\|f\|^2 \le \sum_{j\in J} \|\Lambda_j f\|^2 \le B\|f\|^2. \tag{3}$$

The constants A and B are called the lower and upper bounds of g-frame, respectively. If the right inequality is satisfied, then $\{\Lambda_j\}_{j\in J}$ is said to be a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$. If $A = B$, we call this g-frame a tight g-frame, and if $A = B = 1$, it is called a Parseval g-frame.

For a g-Bessel sequence $\{\Lambda_j\}_{j\in J}$, $T : l^2(\{H_j\}_{j\in J}) \to H$ defines a bounded linear operator, that is,

$$T\left(\{g_j\}_{j\in J}\right) = \sum_{j\in J} \Lambda_j^* g_j, \quad \forall \{g_j\}_{j\in J} \in l^2\left(\{H_j\}_{j\in J}\right). \tag{4}$$

The adjoint operator $T^* : H \to l^2(\{H_j\}_{j\in J})$ is given by

$$T^* f = \{\Lambda_j f\}_{j\in J}, \quad \forall f \in H. \tag{5}$$

By composing T with its adjoint T^*, we obtain the bounded linear operator

$$\begin{aligned} &S : H \longrightarrow H, \\ &Sf = TT^* f = T\{\Lambda_j f\}_{j\in J} = \sum_{j\in J} \Lambda_j^* \Lambda_j f, \quad \forall f \in H. \end{aligned} \tag{6}$$

We call T, T^*, and S the preframe operator, analysis operator, and frame operator of g-Bessel sequence, respectively.

Definition 2 (see [22, Definition 2.6]). We say $\{\Lambda_j \in L(H, H_j)\}_{j\in J}$ is g-orthonormal basis for H with respect to $\{H_j\}_{j\in J}$, if it is g-biorthonormal with itself, that is, $\langle \Lambda_j^* g_j, \Lambda_i^* g_i \rangle = \delta_{j,i} \langle g_j, g_i \rangle$, $\forall j, i \in J$, $g_j \in H_j$, $g_i \in H_i$, and for any $f \in H$ one has $\sum_{j\in J} \|\Lambda_j f\|^2 = \|f\|^2$.

Definition 3 (see [21, Theorem 2.5]). Let $K \in L(H)$ and $\Lambda_j \in L(H, H_j)$ for any $j \in J$. A sequence $\{\Lambda_j\}_{j\in J}$ is called a K-g-frame for H with respect to $\{H_j\}_{j\in J}$ if there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \le \sum_{j\in J} \|\Lambda_j f\|^2 \le B\|f\|^2, \quad \forall f \in H. \tag{7}$$

The constants A and B are called the lower and upper bounds of K-g-frame, respectively.

Remark 4. Every K-g-frame is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$. If $K = I$, then K-g-frame is just the ordinary g-frame.

Motivated by the definition of tight g-frame, we give the following definition of tight K-g-frame.

Definition 5. Let $K \in L(H)$ and $\Lambda_j \in L(H, H_j)$ for any $j \in J$. A sequence $\{\Lambda_j\}_{j\in J}$ is called a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$ if there exists constant $A > 0$ such that

$$A\|K^* f\|^2 = \sum_{j\in J} \|\Lambda_j f\|^2, \quad \forall f \in H. \tag{8}$$

The constant A is called the bound of tight K-g-frame. If $A = 1$, we call this tight K-g-frame a Parseval K-g-frame for H with respect to $\{H_j\}_{j\in J}$.

Remark 6. If $K = I$, then tight K-g-frame and Parseval K-g-frame are tight g-frame and Parseval g-frame, respectively.

Definition 7. Let $K \in L(H)$. An operator K is said to be left-invertible if there exists an operator $P \in L(H)$ such that $PK = I$. The operator P is called a left-inverse of K; that is, $K_l^{-1} = P$. Similarly, an operator K is said to be right-invertible if there exists an operator $P \in L(H)$ such that $KP = I$. The operator P is called a right-inverse of K; that is, $K_r^{-1} = P$. If there exists $P \in L(H)$ such that $KP = PK = I$, then we say that K is invertible and P is an inverse of K; that is, $K^{-1} = P$.

Lemma 8 (see [23, Theorem 1]). *Let $T_1 \in L(H_1, H)$ and $T_2 \in L(H_2, H)$. The following conditions are equivalent:*

(1) $R(T_1) \subset R(T_2)$.

(2) *There exists $\lambda > 0$ such that $T_1 T_1^* \le \lambda T_2 T_2^*$.*

(3) *There exists a bounded operator $X \in L(H_1, H_2)$ so that $T_1 = T_2 X$.*

Lemma 9 (see [21, Theorem 2.5]). *Let $K \in L(H)$. Then the following statements are equivalent:*

(1) *$\{\Lambda_j\}_{j\in J}$ is a K-g-frame for H with respect to $\{H_j\}_{j\in J}$.*

(2) *$\{\Lambda_j\}_{j\in J}$ is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$ and there exists a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$ such that*

$$Kf = \sum_{j\in J} \Lambda_j^* \Gamma_j f, \quad \forall f \in H. \tag{9}$$

3. Properties of Tight K-g-Frame for Hilbert Space

In this section, we first give characterizations of tight K-g-frame and then give several properties of tight K-g-frame.

Theorem 10. *Let $K \in L(H)$; $\Lambda_j \in L(H, H_j)$, and let T be the preframe operator of $\{\Lambda_j\}_{j\in J}$. Then the following statements are equivalent:*

(1) *$\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A.*

(2) *There exists constant* $A > 0$ *such that* $A\|K^* f\|^2 = \|T^* f\|^2$ *for any* $f \in H$.

(3) *There exists constant* $A > 0$ *such that* $AKK^* = TT^*$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A. By the definition of tight K-g-frame, we get

$$A\|K^* f\|^2 = \sum_{j\in J}\|\Lambda_j f\|^2, \quad \forall f \in H. \tag{10}$$

Since T is the preframe operator of $\{\Lambda_j\}_{j\in J}$, we have

$$\begin{aligned}\sum_{j\in J}\|\Lambda_j f\|^2 &= \left\langle \sum_{j\in J}\Lambda_j^*\Lambda_j f, f\right\rangle = \langle TT^* f, f\rangle \\ &= \|T^* f\|^2, \quad \forall f \in H.\end{aligned} \tag{11}$$

This implies that $A\|K^* f\|^2 = \|T^* f\|^2$ for any $f \in H$.

(2) $\Rightarrow$ (3). If there exists constant $A > 0$ such that $A\|K^* f\|^2 = \|T^* f\|^2$ for any $f \in H$, then we obtain

$$\begin{aligned}\langle AKK^* f, f\rangle &= A\langle KK^* f, f\rangle = A\|K^* f\|^2 = \|T^* f\|^2 \\ &= \langle TT^* f, f\rangle, \quad \forall f \in H.\end{aligned} \tag{12}$$

Hence $AKK^* = TT^*$.

(3) $\Rightarrow$ (1). If there exists constant $A > 0$ such that $AKK^* = TT^*$, then $\langle AKK^* f, f\rangle = \langle TT^* f, f\rangle$, $\forall f \in H$. That is,

$$A\|K^* f\|^2 = \|T^* f\|^2 = \sum_{j\in J}\|\Lambda_j f\|^2, \quad \forall f \in H. \tag{13}$$

Therefore, $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A. The proof of Theorem 10 is completed. □

Corollary 11. *Suppose that* $\{\Lambda_j\}_{j\in J}$ *is a tight* K-g-*frame for* H *with respect to* $\{H_j\}_{j\in J}$ *with bound* A, T *is the preframe operator of* $\{\Lambda_j\}_{j\in J}$, *and* S *is the frame operator of* $\{\Lambda_j\}_{j\in J}$; *then*

(1) $R(K) = R(T)$;

(2) $S = AKK^*$;

(3) $\|T\| = \sqrt{A}\|K\|$.

Proof. Theorem 10 together with Lemma 8 shows that (1) and (2) are satisfied. We only need to prove that (3) holds. Assume that $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A. By Theorem 10, we have $A\|K^* f\|^2 = \|T^* f\|^2$, $\forall f \in H$. Therefore,

$$\begin{aligned}\|T\| = \|T^*\| &= \sup_{\|f\|=1, f\in H}\|T^* f\| = \sup_{\|f\|=1, f\in H}\sqrt{A}\|K^* f\| \\ &= \sqrt{A}\|K\|.\end{aligned} \tag{14}$$

The proof of Corollary 11 is completed. □

Lemma 9 gives an equivalent characterization of K-g-frame; does the tight K-g-frame have the similar characterization? Clearly, if $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$, then there exists a g-Bessel sequence $\{\Gamma_j\}_{j\in J}$ for H with respect to $\{H_j\}_{j\in J}$ such that $Kf = \sum_{j\in J}\Lambda_j^*\Gamma_j f$ for any $f \in H$. The theorem below gives a necessary condition of tight K-g-frame.

Theorem 12. *Let* $K \in L(H)$. *Suppose that* $\{\Lambda_j\}_{j\in J}$ *is a tight* K-g-*frame for* H *with respect to* $\{H_j\}_{j\in J}$ *with bound* A. *Then there exists a* g-*Bessel sequence* $\{\Gamma_j\}_{j\in J}$ *for* H *with respect to* $\{H_j\}_{j\in J}$ *with bound* B *such that* $Kf = \sum_{j\in J}\Lambda_j^*\Gamma_j f$, $\forall f \in H$, *and* $AB \geq 1$.

Proof. Let $\{\Lambda_j\}_{j\in J}$ be a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$. By Lemma 9, there exists a g-Bessel sequence $\{\Gamma_j\}_{j\in J}$ for H with respect to $\{H_j\}_{j\in J}$ such that $Kf = \sum_{j\in J}\Lambda_j^*\Gamma_j f$, $\forall f \in H$. For any $f, g \in H$, we have

$$\begin{aligned}\langle K^* f, g\rangle = \langle f, Kg\rangle &= \left\langle f, \sum_{j\in J}\Lambda_j^*\Gamma_j g\right\rangle \\ &= \left\langle \sum_{j\in J}\Gamma_j^*\Lambda_j f, g\right\rangle.\end{aligned} \tag{15}$$

Via (15),

$$K^* f = \sum_{j\in J}\Gamma_j^*\Lambda_j f, \quad \forall f \in H. \tag{16}$$

Since $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A, we get

$$A\|K^* f\|^2 = \sum_{j\in J}\|\Lambda_j f\|^2, \quad \forall f \in H. \tag{17}$$

Furthermore,

$$\begin{aligned}\sum_{j\in J}\|\Lambda_j f\|^2 = A\|K^* f\|^2 &= A\left\|\sum_{j\in J}\Gamma_j^*\Lambda_j f\right\|^2 \\ &= A\sup_{\|g\|=1, g\in H}\left\|\sum_{j\in J}\langle\Gamma_j^*\Lambda_j f, g\rangle\right\|^2 \\ &= A\sup_{\|g\|=1, g\in H}\left\|\sum_{j\in J}\langle\Lambda_j f, \Gamma_j g\rangle\right\|^2 \\ &\leq A\sup_{\|g\|=1, g\in H}\sum_{j\in J}\|\Lambda_j f\|^2\sum_{j\in J}\|\Gamma_j g\|^2 \\ &\leq A\sup_{\|g\|=1, g\in H}\sum_{j\in J}\|\Lambda_j f\|^2 B\|g\|^2 \\ &= AB\sum_{j\in J}\|\Lambda_j f\|^2.\end{aligned} \tag{18}$$

Therefore, $AB \geq 1$. The proof of Theorem 12 is completed. □

Note that when $K = I$, tight K-g-frame is tight g-frame. One may wonder whether $K = I$ when tight K-g-frame is tight g-frame as well. In fact, the answer is negative. The following example demonstrates this.

Example 13. Suppose that $H = \mathbb{R}^3$; $J = \{1, 2, 3\}$. Let $\{e_j\}_{j\in J}$ be an orthonormal basis of H, and let $H_j = \overline{\operatorname{span}}\{e_j\}$. Now define $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ as follows:

$$\begin{aligned}
&\Lambda_j : H \longrightarrow H_j,\\
&\Lambda_1 f = \langle f, e_1\rangle e_1,\\
&\Lambda_2 f = \langle f, e_2\rangle e_2,\\
&\Lambda_3 f = \langle f, e_3\rangle e_3;\\
&\Gamma_j : H \longrightarrow H_j,\\
&\Gamma_1 f = \frac{1}{\sqrt{2}}\left(\langle f, e_1\rangle + \langle f, e_2\rangle\right) e_1,\\
&\Gamma_2 f = \frac{1}{\sqrt{2}}\left(\langle f, e_1\rangle - \langle f, e_2\rangle\right) e_2,\\
&\Gamma_3 f = \langle f, e_3\rangle e_3.
\end{aligned} \tag{19}$$

By a simple calculation, we have

$$\begin{aligned}
&\Lambda_1^* : H_1 \longrightarrow H,\\
&\Lambda_1^*(a_1 e_1) = a_1 e_1,\\
&\Lambda_2^* : H_2 \longrightarrow H,\\
&\Lambda_2^*(a_2 e_2) = a_2 e_2,\\
&\Lambda_3^* : H_3 \longrightarrow H,\\
&\Lambda_3^*(a_3 e_3) = a_3 e_3;\\
&\Gamma_1^* : H_1 \longrightarrow H,\\
&\Gamma_1^*(a_1 e_1) = \frac{1}{\sqrt{2}} a_1 e_1 + \frac{1}{\sqrt{2}} a_1 e_2,\\
&\Gamma_2^* : H_2 \longrightarrow H,\\
&\Gamma_2^*(a_2 e_2) = \frac{1}{\sqrt{2}} a_2 e_1 - \frac{1}{\sqrt{2}} a_2 e_2,\\
&\Gamma_3^* : H_3 \longrightarrow H,\\
&\Gamma_3^*(a_3 e_3) = a_3 e_3.
\end{aligned} \tag{20}$$

For any $f = (c_1, c_2, c_3) \in H$, we have

$$\begin{aligned}
&\sum_{j\in J} \left\|\Lambda_j f\right\|^2 = c_1^2 + c_2^2 + c_3^2 = \|f\|^2;\\
&\sum_{j\in J} \left\|\Gamma_j f\right\|^2 = \frac{1}{2}(c_1 + c_2)^2 + \frac{1}{2}(c_1 - c_2)^2 + c_3^2 = \|f\|^2.
\end{aligned} \tag{21}$$

Obviously, $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are Parseval g-frames for H with respect to $\{H_j\}_{j\in J}$.

Define the bounded linear operator K as follows:

$$\begin{aligned}
&K : H \to H,\\
&Kf = \sum_{j\in J} \Lambda_j^* \Gamma_j f, \quad \forall f \in H.
\end{aligned} \tag{22}$$

Now we prove that $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$. For any $f, g \in H$, we have

$$\begin{aligned}
\langle K^* f, g\rangle = \langle f, Kg\rangle &= \left\langle f, \sum_{j\in J} \Lambda_j^* \Gamma_j g \right\rangle\\
&= \left\langle \sum_{j\in J} \Gamma_j^* \Lambda_j f, g \right\rangle.
\end{aligned} \tag{23}$$

Hence, $K^* f = \sum_{j\in J} \Gamma_j^* \Lambda_j f$. It follows that

$$\begin{aligned}
\|K^* f\|^2 &= \left\|\sum_{j\in J} \Gamma_j^* \Lambda_j f\right\|^2\\
&= \left\|\Gamma_1^* c_1 e_1 + \Gamma_2^* c_2 e_2 + \Gamma_3^* c_3 e_3\right\|^2\\
&= \left\|\frac{1}{\sqrt{2}} c_1 (e_1 + e_2) + \frac{1}{\sqrt{2}} c_2 (e_1 - e_2) + c_3 e_3\right\|^2\\
&= \|f\|^2, \quad \forall f \in H.
\end{aligned} \tag{24}$$

Therefore, for any $f \in H$, we have $\sum_{j\in J} \|\Lambda_j f\|^2 = \|f\|^2 = \|K^* f\|^2$. Via the definition of tight K-g-frame, $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$. For any $f = (c_1, c_2, c_3) \in H$, we have $Kf = \sum_{j\in J} \Lambda_j^* \Gamma_j f = ((1/\sqrt{2})(c_1 + c_2), (1/\sqrt{2})(c_1 - c_2), c_3) \neq f$.

Example 13 shows that if a tight K-g-frame $\{\Lambda_j\}_{j\in J}$ is a tight g-frame, then K cannot be I. In the following theorem, we state a necessary and sufficient condition for a tight K-g-frame being a tight g-frame.

Theorem 14. *Let $K \in L(H)$ and $A_1, A_2 > 0$. Suppose that $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A_1. Then $\{\Lambda_j\}_{j\in J}$ is a tight g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A_2 if and only if K is right-invertible and the right-invertible operator is $K_r^{-1} = (A_1/A_2)K^*$.*

Proof. First, we prove the sufficient condition. Since $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A_1, we have

$$A_1 \|K^* f\|^2 = \sum_{j\in J} \left\|\Lambda_j f\right\|^2, \quad \forall f \in H. \tag{25}$$

Assume that $\{\Lambda_j\}_{j\in J}$ is a tight g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A_2. Then, for any $f \in H$, we get

$$\sum_{j\in J} \left\|\Lambda_j f\right\|^2 = A_2 \|f\|^2. \tag{26}$$

By (25) and (26), $A_1\|K^*f\|^2 = A_2\|f\|^2$ for any $f \in H$, implying that $\|K^*f\|^2 = (A_2/A_1)\|f\|^2$. Then, for any $f \in H$, we have $\langle KK^*f, f\rangle = \langle (A_2/A_1)f, f\rangle$. This implies that $K((A_1/A_2)K^*) = I$. So K is right-invertible and the right-invertible operator is $K_r^{-1} = (A_1/A_2)K^*$.

Next, we prove the necessary condition. Suppose that K is right-invertible and the right-invertible operator is $K_r^{-1} = (A_1/A_2)K^*$. Then $KK_r^{-1} = K((A_1/A_2)K^*) = I$; that is, $KK^* = (A_2/A_1)I$. So

$$\langle KK^*f, f\rangle = \left\langle \frac{A_2}{A_1}f, f\right\rangle, \quad \forall f \in H. \tag{27}$$

That is,

$$\|K^*f\|^2 = \frac{A_2}{A_1}\|f\|^2. \tag{28}$$

Since $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A_1, we have

$$\sum_{j\in J}\|\Lambda_j f\|^2 = A_1\|K^*f\|^2 = A_2\|f\|^2. \tag{29}$$

This implies that $\{\Lambda_j\}_{j\in J}$ is a tight g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A_2. The proof of Theorem 14 is completed. □

In the following, we will verify whether the K_r^{-1} in Example 13 is equal to K^*. For any $f = (c_1, c_2, c_3) \in H$, we have

$$\begin{aligned} KK^*f &= \sum_{j\in J}\Lambda_j^*\Gamma_j K^*f = \sum_{j\in J}\Lambda_j^*\Gamma_j\left(\sum_{j\in J}\Gamma_j^*\Lambda_j f\right) \\ &= \sum_{j\in J}\Lambda_j^*\Gamma_j\left(\frac{1}{\sqrt{2}}(c_1+c_2), \frac{1}{\sqrt{2}}(c_1-c_2), c_3\right) \\ &= \Gamma_1^*\left(\frac{1}{2}(c_1+c_2)+\frac{1}{2}(c_1-c_2)\right)e_1 \\ &\quad + \Gamma_2^*\left(\frac{1}{2}(c_1+c_2)-\frac{1}{2}(c_1-c_2)\right)e_2 \\ &\quad + \Gamma_3^*(c_3e_3) \\ &= \Gamma_1^*(c_1e_1)+\Gamma_2^*(c_2e_2)+\Gamma_j^*(c_3e_3) = f. \end{aligned} \tag{30}$$

It follows that $KK^* = I$. This implies that K is right-invertible and the right-invertible operator is $K_r^{-1} = K^*$.

Corollary 15. *Let $K \in L(H)$. Suppose that $\{\Lambda_j\}_{j\in J}$ is a Parseval K-g-frame for H with respect to $\{H_j\}_{j\in J}$; then $\{\Lambda_j\}_{j\in J}$ is a Parseval g-frame for H with respect to $\{H_j\}_{j\in J}$ if and only if K is right-invertible and the right-invertible operator is $K_r^{-1} = K^*$.*

To enrich the theory of atomic systems, we give several important properties of tight K-g-frame in the following.

Theorem 16. *Let $K \in L(H)$. If $\{\Lambda_j\}_{j\in J}$ is a tight g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A, then $\{\Lambda_j K^*\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A.*

Proof. Since $\{\Lambda_j\}_{j\in J}$ is a tight g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A, we have

$$\sum_{j\in J}\|\Lambda_j f\|^2 = A\|f\|^2, \quad \forall f \in H. \tag{31}$$

Again, for any $f \in H$, we have $K^*f \in H$; then

$$\sum_{j\in J}\|\Lambda_j K^*f\|^2 = A\|K^*f\|^2, \quad \forall f \in H. \tag{32}$$

Therefore, $\{\Lambda_j K^*\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A. The proof of Theorem 16 is completed. □

Corollary 17. *Let $K \in L(H)$. If $\{\Lambda_j\}_{j\in J}$ is a g-orthonormal basis for H with respect to $\{H_j\}_{j\in J}$, then $\{\Lambda_j K^*\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$.*

Theorem 18. *Let $T, K \in L(H)$. If $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A, then $\{\Lambda_j T^*\}_{j\in J}$ is a tight TK-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A.*

Proof. Since $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A, we have

$$A\|K^*f\|^2 = \sum_{j\in J}\|\Lambda_j f\|^2, \quad \forall f \in H. \tag{33}$$

And, for any $f \in H$, we have $T^*f \in H$; then

$$\sum_{j\in J}\|\Lambda_j T^*f\|^2 = A\|K^*T^*f\|^2 = A\|(TK)^*f\|^2, \quad \forall f \in H. \tag{34}$$

So $\{\Lambda_j T^*\}_{j\in J}$ is a tight TK-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A. The proof of Theorem 18 is completed. □

Corollary 19. *Let $K \in L(H)$. If $\{\Lambda_j\}_{j\in J}$ is a tight K-g-frame for H with respect to $\{H_j\}_{j\in J}$, then $\{\Lambda_j(K^*)^N\}_{j\in J}$ is a tight K^{N+1}-g-frame for H with respect to $\{H_j\}_{j\in J}$, where N is a given positive integer.*

Theorem 20. *Let $K_1, K_2 \in L(H)$, and let $gF_K(H)$ be the collection of all tight K-g-frames for H with respect to $\{H_j\}_{j\in J}$. Then $gF_{K_1}(H) \subset gF_{K_2}(H)$ if and only if there exists $A > 0$ such that $K_1K_1^* = AK_2K_2^*$.*

Proof. $\Rightarrow$. If $\{\Lambda_j\}_{j\in J}$ is a g-orthonormal basis for H with respect to $\{H_j\}_{j\in J}$, by Corollary 17, we get that $\{\Lambda_j K_1^*\}_{j\in J}$ is a tight K_1-g-frame for H with respect to $\{H_j\}_{j\in J}$. Since $gF_{K_1}(H) \subset gF_{K_2}(H)$, we have that $\{\Lambda_j K_1^*\}_{j\in J}$ is a tight K_2-g-frame for H with respect to $\{H_j\}_{j\in J}$. Assume that the bound

of tight K_2-g-frame $\{\Lambda_j K_1^*\}_{j\in J}$ is A. By the definition of tight K-g-frame, we obtain

$$\sum_{j\in J}\left\|\Lambda_j K_1^* f\right\|^2 = A\left\|K_2^* f\right\|^2, \quad \forall f\in H. \tag{35}$$

Since $\{\Lambda_j\}_{j\in J}$ is a g-orthonormal basis for H with respect to $\{H_j\}_{j\in J}$, we have $K^* f\in H$ for any $f\in H$. By the definition of g-orthonormal basis, we get

$$\sum_{j\in J}\left\|\Lambda_j K_1^* f\right\|^2 = \left\|K_1^* f\right\|^2, \quad \forall f\in H. \tag{36}$$

By (35) and (36), we get $\|K_1^* f\|^2 = A\|K_2^* f\|^2$ for any $f\in H$. So $K_1K_1^* = AK_2K_2^*$.

$\Leftarrow$. Suppose that $\{\Lambda_j\}_{j\in J}$ is a tight K_1-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A_1; then

$$\sum_{j\in J}\left\|\Lambda_j f\right\|^2 = A_1\left\|K_1^* f\right\|^2, \quad \forall f\in H. \tag{37}$$

Under this assumption, there exists $A>0$ such that $K_1K_1^* = AK_2K_2^*$. So for any $f\in H$, we have $\|K_1^* f\|^2 = A\|K_2^* f\|^2$. Hence

$$A_1A\left\|K_2^* f\right\|^2 = A_1\left\|K_1^* f\right\|^2 = \sum_{j\in J}\left\|\Lambda_j f\right\|^2, \quad \forall f\in H. \tag{38}$$

Therefore, $\{\Lambda_j\}_{j\in J}$ is a tight K_2-g-frame for H with respect to $\{H_j\}_{j\in J}$ with bound A_1A. The proof of Theorem 20 is completed. □

Corollary 21. *Let $K_1, K_2 \in L(H)$ and let $PgF_K(H)$ be the collection of all Parseval K-g-frames for H with respect to $\{H_j\}_{j\in J}$. Then $PgF_{K_1}(H) = PgF_{K_2}(H)$ if and only if $K_1K_1^* = K_2K_2^*$.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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