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Research Article

Asymptotic Expansion of the Solutions to Time-Space Fractional Kuramoto-Sivashinsky Equations

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This paper is devoted to finding the asymptotic expansion of solutions to fractional partial differential equations with initial conditions. A new method, the residual power series method, is proposed for time-space fractional partial differential equations, where the fractional integral and derivative are described in the sense of Riemann-Liouville integral and Caputo derivative. We apply the method to the linear and nonlinear time-space fractional Kuramoto-Sivashinsky equation with initial value and obtain asymptotic expansion of the solutions, which demonstrates the accuracy and efficiency of the method.

1. Introduction

The Kuramoto-Sivashinsky (KS) equation in one space dimension,

$$D_t u(x,t) + D_x^4 u(x,t) + D_x^2 u(x,t) + u(x,t) D_x u(x,t) = 0, \tag{1}$$

has attracted a great deal of interest as a model for complex spatiotemporal dynamics in spatially extended systems and as a paradigm for finite-dimensional dynamics in a partial differential equation. $D_x^2 u$ term in (1) is responsible for an instability at large scales; dissipative term $D_x^4 u$ provides damping at small scales; and the nonlinear term $uD_x u$ stabilizes by transferring energy between large and small scales. The KS equation dates back to the mid-1970s and was first introduced by Kuramoto [1] in the study of phase turbulence in the Belousov-Zhabotinsky reaction-diffusion systems. An extension of this equation to two or more spatial dimensions was given by Sivashinsky [2, 3] in modelling small thermal diffusive instabilities in laminar flame fronts and in small perturbations from a reference Poiseuille flow of a flame layer on an inclined plane. In one space dimension it is also used as model for the problem of Bénard convection in an elongated box, and it may be used to describe long waves on the interface between two viscous fluids and unstable drift wave in plasmas. As a dynamical system, KS equation is known for its chaotic solutions and complicated behavior due to the terms that appear. Because of this fact, KS equation was studied extensively as a paradigm of finite dynamics in a partial differential equation. Its multimodal, oscillatory, and chaotic solutions have been investigated [4–8]; its nonintegrability was established via Painlevé analysis [9] and due to its bifurcation behavior a connection to low finite-dimensional dynamical systems is established [10, 11]. The KS equation is nonintegrable; therefore the exact solution of this equation is not obtainable and only numerical schemes have been proposed [12, 13].

Partial differential equations (PDEs) which arise in real-world physical problems are often too complicated to be solved exactly and even if an exact solution is obtainable, the required calculations may be too complicated to be practical or difficult to interpret the outcome. Very recently, some practical approximate analytical solutions are proposed to solve KS equation, such as Chebyshev spectral collocation scheme [14], lattice Boltzmann technique [15], local discontinuous Galerkin method [16], tanh function method [17], variational iteration method [18], perturbation methods [19], classical and nonclassical symmetries method for the KS equation dispersive effects [20], Riccati expansion method [21], and Lie symmetry method [22], and see also [23–27]. In the last few decades, fractional-order models are found to be more adequate than integer-order models for some real-world problems. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and so forth involves derivatives of fractional order. In particular, for the construction of the approximate solutions of the fractional PDEs, various methods are proposed: finite difference method [28], the finite element method [29], the differential transformation method [30], the fractional subequation method [31], the fractional complex transform method [32], the modified simple equation method [33], the variational iteration method [34], the Lagrange characteristic method [35], the iteration method [36], and so on. For the time-fractional KS equation, in [37] the authors constructed the analytical exact solutions via fractional complex transform, and they obtained new types of exact analytical solutions. In [38], Rezazadeh and Ziabary found travelling wave solutions by the general time-space KS equation by a subequation method.

In this work, we apply residual power series (RPS) method to construct the asymptotic expansion of the solution to the more general linear KS equation

$$D_t^{\alpha} u(x,t) + \beta D_x^{4\tau} u(x,t) + \gamma D_x^{2\sigma} u(x,t) + \delta D_x^{\eta} u(x,t) = 0 \tag{2}$$

and the nonlinear KS equation

$$D_t^{\alpha} u(x,t) + \beta D_x^{4\tau} u(x,t) + \gamma D_x^{2\sigma} u(x,t) + \delta u(x,t) D_x^{\eta} u(x,t) = 0 \tag{3}$$

with initial value

$$u(x,0) = a_0(x) \in C^{\infty}(\mathbb{R}), \tag{4}$$

where $0 < \alpha, \eta \le 1$, $3/4 < \tau \le 1$, $1/2 < \sigma \le 1$, β, γ, and δ are any arbitrary constants, and $(x,t) \in \mathbb{R} \times \mathbb{R}$. The general response expression contains different parameters describing the order of the fractional derivative that can be varied to obtain various responses. The fractional power series solutions can be obtained by the RPS method. Particularly, if we take special parameters $\alpha = 1$, $\tau = 1$, $\sigma = 1$, $\eta = 1$, $\beta = 1$, $\gamma = 1$, and $\delta = 1$ (here the equation is integer-order) and special initial condition $a_0(x) = x$, the exact solution of linear KS equation is

$$u(x,t) = x - t, \tag{5}$$

and the exact solution of nonlinear KS equation also can be obtained:

$$u(x,t) = \frac{x}{1+t}. \tag{6}$$

Here the solution of nonlinear KS equation we obtained is different from Porshokouhi and Ghanbari's work in [39] for the integer-order KS equation. They take the travel wave initial condition

$$a_0(x) = c + \frac{5}{19}\sqrt{\frac{11}{19}}\left(11\tanh^3\left(k\left(x-x_0\right)\right) - 9\tanh\left(k\left(x-x_0\right)\right)\right) \tag{7}$$

and get the exact traveling wave solution

$$u(x,t) = c + \frac{5}{19}\sqrt{\frac{11}{19}}\left(11\tanh^3\left(k\left(x-ct-x_0\right)\right) - 9\tanh\left(k\left(x-ct-x_0\right)\right)\right) \tag{8}$$

of (2) with the same special parameters $\alpha = 1$, $\tau = 1/4$, $\sigma = 1$, $\eta = 1$, $\beta = 1$, $\gamma = 1$, and $\delta = 1$. Their skills mainly depend on the variational iteration method and obtain the different numerical examples with different parameters. To the best of information of the authors, no previous research work has been done using proposed technique for solving time-space fractional KS equation. Our method can be applied to the time-space linear and nonlinear fractional KS equations. The main advantage of the RPS method is that it can be applied directly for all types of differential equation, because it depends on the recursive differentiation of time-fractional derivative and uses given initial conditions to calculate coefficients of the multiple fractional power series solution with minimal calculations. Another important advantage is that this method does not require linearization, perturbation, or discretization of the variables; it is not affected by computational round-off errors and does not require large computer memory and extensive time.

The rest of this paper is organised as follows. In Section 2, some necessary concepts on the theory of fractional calculus are presented. The main steps of the PRS method are proposed in Section 3. Section 4 is the application of RPS method to construct analytical solution of linear and nonlinear time-space fractional KS equation with initial value. The paper is concluded with some general remarks in Section 5.

2. Some Concepts on the Theory of Fractional Calculus

There are several definitions of the fractional integral and fractional derivative, which are not necessarily equivalent to each other (see [40–42]). Riemann-Liouville integral and Caputo derivative are the two most used forms which have been introduced in [43–45]. In this section, we give some notions we need in this paper.

Definition 1 (see [40, 41, 43]). A real function $f(t)$, $t > 0$, is said to be in the space C_{μ}, $\mu \in \mathbb{R}$, if there exists a real number

$p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, +\infty)$, and it is said to be in the space C_μ^n if $f^{(n)}(t) \in C_\mu$, $n \in \mathbb{N}$.

Definition 2 (see [40, 41, 43]). The Riemann-Liouville fractional integral operator of order $\alpha \geqslant 0$ of a function $f \in C_\mu$, $\mu \geqslant -1$, is defined as

$$
{}_{t_0}I_t^\alpha f(t) := \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, & \alpha > 0,\ t > \tau > t_0, \\ f(t), & \alpha = 0, \end{cases} \tag{9}
$$

where the symbol ${}_{t_0}I_t^\alpha$ represents the αth Riemann-Liouville fractional integral of f of t between the limits t_0 and t.

Property 1 (see [40, 41, 43]). Here the properties of the operator ${}_{t_0}I_t^\alpha$ are given: for $f \in C_\mu$, $\mu \geqslant -1$, $\alpha, \beta, C \in \mathbb{R}$, and $\gamma \geqslant -1$,

$$
\begin{aligned}
&(\text{Pro1}) \quad {}_{t_0}I_t^\alpha\, {}_{t_0}I_t^\beta f(t) = {}_{t_0}I_t^{\alpha+\beta} f(t) = {}_{t_0}I_t^\beta\, {}_{t_0}I_t^\alpha f(t), \\
&(\text{Pro2.}) \quad {}_{t_0}I_t^\alpha C = \frac{C}{\Gamma(\alpha+1)} (t-t_0)^\alpha, \\
&(\text{Pro3.}) \quad {}_{t_0}I_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} \left. t^{\gamma+\alpha} \right|_{t_0}^t .
\end{aligned} \tag{10}
$$

Definition 3 (see [40, 41, 43]). The Caputo fractional derivative of order $\alpha > 0$ of $f \in C_{-1}^n$, $n \in \mathbb{N}$, is defined as

$$
{}_{t_0}D_t^\alpha f(t) := \begin{cases} {}_{t_0}I_t^{n-\alpha} f^{(n)}(t), & n-1 < \alpha < n,\ t > t_0, \\ \left. \dfrac{d^n f(t)}{dt^n} \right|_{t=t_0}, & \alpha = n, \end{cases} \tag{11}
$$

where the symbol ${}_{t_0}D_t^\alpha f(t)$ represents the αth Caputo fractional derivative of f with respect to t at t_0.

Property 2 (see [40, 41, 43]). Here the properties of the operator ${}_{t_0}D_t^\alpha$ are given: for $\gamma > -1$, $t > s \geqslant 0$, and $C \in \mathbb{R}$,

$$
\begin{aligned}
(\text{Pro1.}) \quad & {}_{t_0}D_t^\alpha\, {}_{t_0}D_t^\beta f(t) = {}_{t_0}D_t^{\alpha+\beta} f(t) \\
& = {}_{t_0}D_t^\beta\, {}_{t_0}D_t^\alpha f(t), \\
(\text{Pro2.}) \quad & {}_{t_0}D_t^\alpha C = 0, \\
(\text{Pro3.}) \quad & {}_{t_0}D_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} \left. t^{\gamma-\alpha} \right|_{t=t_0} .
\end{aligned} \tag{12}
$$

Remark 4 (see [41, 46]). The Caputo time-fractional derivative operator of order α of function $u(x,t)$ with respect to t at t_0 is defined as

$$
{}_{t_0}D_t^\alpha u(x,t) := \begin{cases} {}_{t_0}I_t^{n-\alpha} \dfrac{\partial^n u(x,t)}{\partial t^n}, & n-1 < \alpha < n,\ t > t_0, \\ \left. \dfrac{d^n f(t)}{dt^n} \right|_{t=t_0}, & \alpha = n \in \mathbb{N}, \end{cases} \tag{13}
$$

where $x \in \mathbb{R}$ and $t > 0$.

Remark 5 (see [41, 46]). The Caputo space-fractional derivative operator of order β of function $u(x,t)$ with respect to x at x_0 is defined as

$$
{}_{x_0}D_x^\beta u(x,t) := \begin{cases} {}_{x_0}I_x^{m-\beta} \dfrac{\partial^m u(x,t)}{\partial t^m}, & m-1 < \alpha < m,\ x > x_0, \\ \left. \dfrac{\partial^m u(x,t)}{\partial x^m} \right|_{x=x_0}, & \alpha = m \in \mathbb{N}, \end{cases} \tag{14}
$$

where $x \in \mathbb{R}$ and $t > 0$.

Definition 6 (see [41, 46]). A power series representation of the form

$$
\sum_{n=0}^\infty c_n (t-t_0)^{n\alpha} := c_0 + c_1 (t-t_0)^\alpha + c_2 (t-t_0)^{2\alpha} + \cdots, \tag{15}
$$

where $m-1 < \alpha \leq m$ and $t \geqslant t_0$, is called a fractional power series (FPS) about t_0, where t is a variable and c_n are constants called the coefficients of the series.

Theorem 7. *Suppose that f has a FPS representation at t_0 of the form*

$$
f(t) = \sum_{n=0}^\infty c_n (t-t_0)^{n\alpha}, \quad 0 \leqslant m-1 < \alpha \leqslant m,\ t_0 \leqslant t < t_0 + R, \tag{16}
$$

and R is the radius of convergence of the FPS. If $f(t) \in C[t_0, t_0 + R)$ and ${}_{t_0}D_t^{n\alpha} f(t) \in C(t_0, t_0 + R)$ for $n = 0, 1, 2, \ldots$, then the coefficients c_n will take the form of

$$
c_n = \frac{{}_{t_0}D_t^{n\alpha} f(t)}{\Gamma(n\alpha+1)}, \tag{17}
$$

where ${}_{t_0}D_t^{n\alpha} = {}_{t_0}D_t^\alpha \cdot {}_{t_0}D_t^\alpha \cdot \cdots \cdot {}_{t_0}D_t^\alpha$ (n-times).

This result is similar to [46, Theorem 2.2] and [47, Theorem 3.4]. It is convenient to give the details for the following applications; thus we write the process of the proof in the form of function with one variable.

Proof. First of all, notice that if we put $t = t_0$ into (16), it yields

$$
c_0 = f(t_0) = \frac{{}_{t_0}D_t^{0\alpha} f(t)}{\Gamma(0\alpha+1)}. \tag{18}
$$

Applying the operator ${}_{t_0}D_t^\alpha$ one time on (16) leads to

$$
c_1 = \frac{{}_{t_0}D_t^\alpha f(t)}{\Gamma(\alpha+1)}. \tag{19}
$$

Again, by applying the operator ${}_{t_0}D_t^\alpha$ two times on (16), one can obtain

$$
c_2 = \frac{{}_{t_0}D_t^{2\alpha} f(t)}{\Gamma(2\alpha+1)}. \tag{20}
$$

By now, the pattern is clearly found, if we continue applying recursively the operator ${}_{t_0}D_t^{\alpha}$ n-times on (16); then it is easy to discover the following form for c_n:

$$c_n = \frac{{}_{t_0}D_t^{n\alpha} f(t)}{\Gamma(n\alpha+1)}, \quad n = 0, 1, 2, \ldots; \tag{21}$$

that is, it has the same form as (17), which completes the proof. □

Following the similar result as [46, Theorem 2.3] or [48, Remark 8], we can obtain the following corollary.

Corollary 8. *If $f(t) = u(x,t)$, then we have*

$$u(x,t) = \sum_{n=0}^{\infty} c_n(x)(t-t_0)^{n\alpha}, \qquad 0 \leqslant m-1 < \alpha \leqslant m,\ x \in \mathbb{R},\ t_0 \leqslant t < t_0 + R, \tag{22}$$

and R is the radius of convergence of the FPS; if $D_t^{n\alpha} \in C(\mathbb{R} \times (t_0, t_0 + R))$, $n = 0, 1, 2, \ldots$, then the coefficients are given by $c_n(x) = {}_{t_0}D_t^{n\alpha} u(x,t)/\Gamma(n\alpha+1)$.

3. Algorithm of RPS Method

Let us consider the higher-order time-space fractional differential equation with initial values as follows:

$$\begin{aligned} &D_t^{m\alpha} u(x,t) + G(x,t) = 0, \\ &u(x,0) = a_0(x),\ D_t^{\alpha} u(x,t)\big|_{t=0} \\ &\quad = a_1(x), \ldots, D_t^{(m-1)\alpha} u(x,t)\big|_{t=0} = a_{m-1}(x), \end{aligned} \tag{23}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, and

$$\begin{aligned} G(x,t) = F\big(&u, D_t^{\alpha}u, D_t^{2\alpha}u, \ldots, D_t^{(m-1)\alpha}u, D_x^{\beta_1}u, D_x^{\beta_2}u, \\ &\ldots, D_x^{\beta_l}u\big). \end{aligned} \tag{24}$$

$u(x,\cdot)$ and $G(x,\cdot)$ are analytical function about t, $(m-1)/m < \alpha \leqslant 1$, $m \in \mathbb{N}$, $p-1 < \beta_p \leqslant p$, and $p = 1, 2, \ldots, l$; $a_q(x) \in C^{\infty}(\mathbb{R})$, $q = 0, 1, 2, \ldots, m-1$.

Assume that $u(x,t)$ is analytical about t; the solution of the system can be written in the form of

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{25}$$

where $u_n(x,t)$ are terms of approximations and are given as

$$u_n(x,t) = C_n(x) t^{n\alpha}, \qquad x \in \mathbb{R},\ |t| < R,\ n = 0, 1, 2, \ldots, \tag{26}$$

where R is the radius of convergence of above series. Obviously, when $i = 0, 1, 2, \ldots, m-1$, using the terms $D_t^{i\alpha} u(x,t)$ which satisfy the initial condition, we can get

$$\begin{aligned} a_i(x) &= D_t^{i\alpha} u(x,t)\big|_{t=0} = C_i(x)\Gamma(i\alpha+1) \Longrightarrow \\ u_i(x,t) = C_i(x) t^{i\alpha} &= \frac{D_t^{i\alpha} u(x,t)\big|_{t=0}}{\Gamma(i\alpha+1)} t^{i\alpha} \\ &= \frac{a_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha}, \quad i = 0, 1, \ldots, m-1. \end{aligned} \tag{27}$$

So we have the initial guess approximation of $u(x,t)$ in the following form:

$$\begin{aligned} u^{\text{initial}}(x,t) &\coloneqq u_0(x,t) + u_1(x,t) + \cdots + u_{m-1}(x,t) \\ &= a_0(x) + \frac{a_1(x)}{\Gamma(\alpha+1)} t^{\alpha} + \cdots \\ &\quad + \frac{a_{m-1}(x)}{\Gamma((m-1)\alpha+1)} t^{(m-1)\alpha}. \end{aligned} \tag{28}$$

On the other aspect as well, if we choose $u^{\text{initial}}(x,t)$ as initial guess approximation $u_i(x,t)$ for $i = m, m+1, m+2, \ldots$, the approximate solutions of $u(x,t)$ of (23) by the kth truncated series are

$$u^k(x,t) \coloneqq u^{\text{initial}}(x,t) + \sum_{i=m}^{k} C_i(x) t^{i\alpha}, \qquad k = m, m+1, m+2, \ldots. \tag{29}$$

Before applying RPS method for solving (23), we first give some notations:

$$\operatorname{Res}(x,t) \coloneqq D_t^{m\alpha} u(x,t) + G(x,t). \tag{30}$$

Substituting the kth truncated series $u^k(x,t)$ into (23), we can obtain the following definition for kth residual function:

$$\operatorname{Res}^k(x,t) \coloneqq D_t^{m\alpha} u^k(x,t) + G^k(x,t), \tag{31}$$

where

$$\begin{aligned} G^k(x,t) = F\big(&u^k, D_t^{\alpha}u^k, D_t^{2\alpha}u^k, \ldots, D_t^{(m-1)\alpha}u^k, D_x^{\beta_1}u^k, \\ &D_x^{\beta_2}u^k, \ldots, D_x^{\beta_{1n}}u^k, D_{x_1}^{\beta_l}u^k\big). \end{aligned} \tag{32}$$

Then, we have following facts:

$$\begin{aligned} &(1)\quad \lim_{k\to\infty} u^k(x,t) = u(x,t); \\ &(2)\quad \operatorname{Res}(x,t) = 0; \\ &(3)\quad \operatorname{Res}^{\infty}(x,t) = \lim_{k\to+\infty} \operatorname{Res}^k(x,t) = \operatorname{Res}(x,t) = 0, \\ &\qquad\qquad x \in \mathbb{R},\ |t| < R. \end{aligned} \tag{33}$$

These show that the residual function $\text{Res}^{\infty}(x,t)$ is infinitely many times differentiable at $t=0$. On the other hand, we can show that

$$\begin{aligned} 0 &= D_t^{(k-m)\alpha}\ \text{Res}^{\infty}(x,t)\big|_{t=0} \\ &= C_k(x)\,\Gamma(k\alpha+1) + D_t^{(k-m)\alpha} G^k(x,t)\big|_{t=0} \Longrightarrow \\ C_k(x) &= -\frac{D_t^{(k-m)\alpha} G^k(x,t)\big|_{t=0}}{\Gamma(k\alpha+1)}, \\ & \qquad k = m, m+1, m+2, \ldots. \end{aligned} \tag{34}$$

Since $-D_t^{(k-m)\alpha} G^k(x,t)|_{t=0}$ is not dependent on t, denoting it by $f_k(x)$, by Theorem 7, (34) can be written as

$$\begin{aligned} C_k(x) &= \frac{f_k(x)}{\Gamma(k\alpha+1)}, \\ f_k(x) &= -D_t^{(k-m)\alpha}\ G^k(x,t)\big|_{t=0}, \\ & \qquad k = m, m+1, m+2, \ldots. \end{aligned} \tag{35}$$

In fact, the relation of (35) is a fundamental rule in the RPS method and its applications. So the fractional power series solution of (23) is

$$\begin{aligned} u(x,t) &= u^{\text{initial}}(x,t) + \sum_{i=m}^{\infty} C_i(x)\, t^{i\alpha} \\ &= \sum_{i=0}^{m-1} \frac{a_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha} + \sum_{i=m}^{\infty} \frac{f_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha}, \end{aligned} \tag{36}$$

where $a_i(x)$ $(i = 0,1,2,\ldots,m-1)$ are given by the initial conditions and $f_i(x)$ $(i = m, m+1, m+2, \ldots)$ have been constructed by RPS method in the form in (35).

4. Application of RPS Method to Time-Space Fractional KS Equation

In this section, we apply the RPS method to the linear and nonlinear time-space fractional KS equation with the initial conditions. The fractional power series solutions can be obtained by the recursive equation (35) with time-fractional derivative, while it will use the given initial conditions. And the fractional power series solutions we consider in the following examples are all in the convergence radius of the series.

4.1. Linear Time-Space Fractional KS Equation. Consider the linear time-space fractional KS equation

$$\begin{aligned} D_t^{\alpha} u(x,t) + \beta D_x^{4\tau} u(x,t) + \gamma D_x^{2\sigma} u(x,t) + \delta D_x^{\eta} u(x,t) \\ = 0. \end{aligned} \tag{37}$$

In (37), if $u(x,t)$ is analytical about t, then $u(x,t)$ can be written as the fractional power series:

$$u(x,t) = \sum_{n=0}^{\infty} \frac{f_n(x)}{\Gamma(n\alpha+1)} t^{n\alpha}. \tag{38}$$

If we can take the initial value,

$$u(x,0) = a_0(x) \in C^{\infty}(\mathbb{R}). \tag{39}$$

According to the initial condition, it is obvious that $f_0(x) = a_0(x)$. Before getting the coefficients $f_n(x)$ $(n = 1,2,\ldots)$, we first present some notations as in Section 3:

$$\begin{aligned} G(x,t) &= \beta D_x^{4\tau} u(x,t) + \gamma D_x^{2\sigma} u(x,t) \\ &\quad + \delta D_x^{\eta} u(x,t); \\ \text{Res}(x,t) &= D_t^{\alpha} u(x,t) + G(x,t); \\ u^k(x,t) &= \sum_{j=0}^{k} \frac{f_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha} \\ &= a_0(x) + \sum_{j=1}^{k} \frac{f_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha}; \\ G^k(x,t) &= \beta D_x^{4\tau} u^k(x,t) + \gamma D_x^{2\sigma} u^k(x,t) \\ &\quad + \delta D_x^{\eta} u^k(x,t); \\ \text{Res}^k(x,t) &= D_t^{\alpha} u_k(x,t) + G^k(x,t). \end{aligned} \tag{40}$$

It follows from the facts in (33) that the residual function $\text{Res}^{\infty}(x,t)$ is infinitely many times differentiable at $t=0$. On the other hand, it follows from Definition 3 and (40) that

$$\begin{aligned} 0 &= D_t^{(k-1)\alpha}\ \text{Res}^{\infty}(x,t)\big|_{t=0} = D_t^{(k-1)\alpha}\ \text{Res}^{k}(x,t)\big|_{t=0} \\ &= D_t^{(k-1)\alpha}\left(D_t^{\alpha} u^k(x,t) + G^k(x,t)\right)\big|_{t=0} \\ &= D_t^{k\alpha} u^k(x,t)\big|_{t=0} + D_t^{(k-1)\alpha} G^k(x,t)\big|_{t=0} \\ &= f_k(x) + D_t^{(k-1)\alpha} G^k(x,t)\big|_{t=0}, \end{aligned} \tag{41}$$

which gives

$$f_k(x) = -D_t^{(k-1)\alpha}\ G^k(x,t)\big|_{t=0}, \quad (k = 1,2,3,\ldots). \tag{42}$$

Equation (42) gives the iterative formula for the coefficients $f_n(x)$ $(n = 1,2,\ldots)$, so we can obtain

$$\begin{aligned} f_0(x) &= a_0(x), \\ f_1(x) &= -\left(\beta D_x^{4\tau} a_0(x) + \gamma D_x^{2\sigma} a_0(x) + \delta D_x^{\eta} a_0(x)\right) \\ &\triangleq a_1(x), \\ f_2(x) &= -\left(\beta D_x^{4\tau} a_1(x) + \gamma D_x^{2\sigma} a_1(x) + \delta D_x^{\eta} a_1(x)\right) \\ &\triangleq a_2(x); \\ f_3(x) &= -\left(\beta D_x^{4\tau} a_2(x) + \gamma D_x^{2\sigma} a_2(x) + \delta D_x^{\eta} a_2(x)\right) \\ &\triangleq a_3(x). \end{aligned} \tag{43}$$

For general $k \in \mathbb{Z}$, we have

$$\begin{aligned} f_k(x) &= -\left(\beta D_x^{4\tau} a_{k-1}(x) + \gamma D_x^{2\sigma} a_{k-1}(x) + \delta D_x^{\eta} a_{k-1}(x)\right) \\ &\triangleq a_k(x). \end{aligned} \tag{44}$$

So the kth approximate solution of (37) with initial value (39) is

$$u^k(x,t) = \sum_{i=0}^{k} \frac{f_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha} = \sum_{i=0}^{k} \frac{a_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha}, \quad k = 1, 2, 3, \ldots, \tag{45}$$

where $a_i(x)$ $(i = 0, 1, 2, \ldots, k)$ are given by (39) and (44).

Specifically, if taking $\alpha = 1$, $\tau = 1$, $\sigma = 1$, $\eta = 1$, $\beta = 1$, $\gamma = 1$, and $\delta = 1$ (the equation becomes integer order) and special initial value $a_0(x) = x$, we can obtain

$$\begin{aligned} a_1(x) &= -1, \\ a_k(x) &= 0, \quad k = 2, 3, \ldots. \end{aligned} \tag{46}$$

So, we can obtain the kth approximate power series solution of integer-order equation (37):

$$u^k(x,t) = \sum_{i=0}^{k} \frac{a_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha} = x - t. \tag{47}$$

Thus, the exact solution is

$$u(x,t) = \lim_{k\to\infty} u^k(x,t) = x - t, \tag{48}$$

when $\alpha = 1$, $\tau = 1$, $\sigma = 1$, $\eta = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$, and $a_0(x) = x$.

4.2. Nonlinear Time-Space Fractional KS Equation. Let us rewrite the nonlinear time-space fractional KS equation:

$$\begin{aligned} &D_t^{\alpha} u(x,t) + \beta D_x^{4\tau} u(x,t) + \gamma D_x^{2\sigma} u(x,t) \\ &\quad + \delta u(x,t) D_x^{\eta} u(x,t) = 0. \end{aligned} \tag{49}$$

If $u(x,t)$ is analytical about t of (49), then $u(x,t)$ can be written as the fractional power series:

$$u(x,t) = \sum_{n=0}^{\infty} \frac{f_n(x)}{\Gamma(n\alpha+1)} t^{n\alpha}. \tag{50}$$

Here, the initial value satisfies

$$u(x,0) = a_0(x) \in C^{\infty}(\mathbb{R}). \tag{51}$$

According to the initial condition, it is obvious that $f_0(x) = a_0(x)$.

Before getting the coefficients $f_n(x)$ $(n = 1, 2, \ldots)$, we also present some symbols:

$$\begin{aligned} G(x,t) &= \beta D_x^{4\tau} u(x,t) + \gamma D_x^{2\sigma} u(x,t) \\ &\quad + \delta u(x,t) D_x^{\eta} u(x,t); \\ \mathrm{Res}(x,t) &= D_t^{\alpha} u(x,t) + G(x,t); \\ u^k(x,t) &= \sum_{j=0}^{k} \frac{f_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha} \\ &= a_0(x) + \sum_{j=1}^{k} \frac{f_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha}; \\ G^k(x,t) &= \beta D_x^{4\tau} u^k(x,t) + \gamma D_x^{2\sigma} u^k(x,t) \\ &\quad + \delta u^k(x,t) D_x^{\eta} u^k(x,t); \\ \mathrm{Res}^k(x,t) &= D_t^{\alpha} u_k(x,t) + G^k(x,t). \end{aligned} \tag{52}$$

It also follows from the facts in (33) that the residual function $\mathrm{Res}^{\infty}(x,t)$ is infinitely many times differentiable at $t = 0$.

On the other hand, it follows from Definition 3 and the notations above that

$$\begin{aligned} 0 &= D_t^{(k-1)\alpha} \mathrm{Res}^{\infty}(x,t)\big|_{t=0} = D_t^{(k-1)\alpha} \mathrm{Res}^{k}(x,t)\big|_{t=0} \\ &= D_t^{(k-1)\alpha}\left(D_t^{\alpha} u^k(x,t) + G^k(x,t)\right)\big|_{t=0} \\ &= D_t^{k\alpha} u^k(x,t)\big|_{t=0} + D_t^{(k-1)\alpha} G^k(x,t)\big|_{t=0} \\ &= f_k(x) + D_t^{(k-1)\alpha} G^k(x,t)\big|_{t=0}, \end{aligned} \tag{53}$$

which gives

$$f_k(x) = -D_t^{(k-1)\alpha} G^k(x,t)\big|_{t=0}, \quad (k = 1, 2, 3, \ldots). \tag{54}$$

Equation (54) gives the iterative formula for the coefficients $f_n(x)$ $(n = 1, 2, \ldots)$, so we can obtain

$$\begin{aligned} f_0(x) &= a_0(x), \\ f_1(x) &= -\left(\beta D_x^{4\tau} a_0(x) + \gamma D_x^{2\sigma} a_0(x) \right. \\ &\quad \left. + \delta a_0(x) D_x^{\eta} a_0(x)\right) \triangleq a_1(x), \\ f_2(x) &= -\left(\beta D_x^{4\tau} a_1(x) + \gamma D_x^{2\sigma} a_1(x) \right. \\ &\quad \left. + \delta a_0(x) D_x^{\eta} a_1(x) + \delta a_1(x) D_x^{\eta} a_0(x)\right) \triangleq a_2(x); \\ f_3(x) &= -\left(\beta D_x^{4\tau} a_2(x) + \gamma D_x^{2\sigma} a_2(x) \right. \\ &\quad + \delta a_0(x) D_x^{\eta} a_2(x) + \delta \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} a_1(x) D_x^{\eta} a_1(x) \\ &\quad \left. + \delta a_2(x) D_x^{\eta} a_0(x)\right) \triangleq a_3(x). \end{aligned} \tag{55}$$

For general $k \in \mathbb{Z}$, we have

$$f_k(x) = -\left(\beta D_x^{4\tau} a_{k-1}(x) + \gamma D_x^{2\sigma} a_{k-1}(x) + \delta \sum_{j=0}^{k-1} \frac{\Gamma((k-1)\alpha+1)}{\Gamma(j\alpha+1)\Gamma((k-1-j)\alpha+1)} a_j(x) \cdot D_x^{\eta} a_{k-1-j}(x)\right) \triangleq a_k(x). \quad (56)$$

So the kth approximate solution of (51) with initial value (51) is

$$u^k(x,t) = \sum_{i=0}^{k} \frac{f_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha} = \sum_{i=0}^{k} \frac{a_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha}, \quad k = 1, 2, 3, \ldots, \quad (57)$$

where $a_i(x)$ $(i = 0, 1, 2, \ldots, k)$ are given by (51) and (56).

In particular, if taking $\alpha = 1$, $\tau = 1$, $\sigma = 1$, $\eta = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$, and $a_0(x) = x$, we can obtain

$$\begin{aligned}
a_1(x) &= -x, \\
a_2(x) &= 2!x, \\
a_3(x) &= -3!x, \\
&\vdots \\
a_{k-1}(x) &= (-1)^{k-1}(k-1)!x, \\
a_k(x) &= D_x^4 a_{k-1}(x) + D_x^2 a_{k-1}(x) + \sum_{j=0}^{k-1} \frac{\Gamma((k-1)\alpha+1)}{\Gamma(j\alpha+1)\Gamma((k-1-j)\alpha+1)} a_j(x) \\
&\cdot D_x a_{k-1-j}(x) = \sum_{j=0}^{k-1} \frac{\Gamma((k-1)+1)}{\Gamma(j+1)\Gamma((k-1-j)+1)} \\
&\cdot (-1)^j j!x(-1)^{k-1-j}(k-1-j)! \\
&= \sum_{j=0}^{k-1} \frac{(k-1)!}{j!(k-1-j)!}(-1)^{k-1} j!x(k-1-j)! \\
&= (-1)^k (k-1)! \cdot kx = (-1)^k k!x, \quad k = 1, 2, 3, \ldots.
\end{aligned} \quad (58)$$

So the kth approximate FPS solution of (49) with initial value (51) is

$$u^k(x,t) = \sum_{i=0}^{k} \frac{a_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha} = \sum_{i=0}^{k} \frac{(-1)^i i!x}{i!} t^i = x\sum_{i=0}^{k}(-t)^i = x\frac{1-(-t)^k}{1+t}. \quad (59)$$

When $t \in (-1, 1)$, it shows

$$u(x,t) = \lim_{k\to\infty} u^k(x,t) = \lim_{k\to\infty} x\frac{1-(-t)^k}{1+t} = \frac{x}{1+t}, \quad (60)$$

which is the exact solution for (49) with special parameters $\alpha = 1$, $\tau = 1$, $\sigma = 1$, $\eta = 1$, $\beta = 1$, $\gamma = 1$, and $\delta = 1$ and special initial value $a_0(x) = x$.

5. Concluding Remarks

In this paper, we have used a new method: the residual power series method for the general fractional differential equations. The asymptotic expansion of the solutions can be obtained successfully with respect to initial conditions which are infinitely differentiable by the RPS method. We apply RPS method to linear and nonlinear Kuramoto-Sivashinsky equation with infinitely differential initial conditions and obtain the asymptotic expansion of the solutions. Particularly, if taking special parameters and special initial value, the analytical solutions are obtained. These applications show that this method is efficient and does not require linearization or perturbation; it is not affected by computational round-off errors and does not require large computer memory and extensive time.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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