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Research Article

A Type of Multigrid Method Based on the Fixed-Shift Inverse Iteration for the Steklov Eigenvalue Problem

Feiyan Li and Hai Bi

School of Mathematical Sciences, Guizhou Normal University, Guiyang 550001, China

Correspondence should be addressed to Hai Bi; bihaimath@gznu.edu.cn

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For the Steklov eigenvalue problem, we establish a type of multigrid discretizations based on the fixed-shift inverse iteration and study in depth its a priori/a posteriori error estimates. In addition, we also propose an adaptive algorithm on the basis of the a posteriori error estimates. Finally, we present some numerical examples to validate the efficiency of our method.

1. Introduction

Due to the wide applications in physical and mechanical field (see, e.g., [1–3]), there has been a lot of research on the numerical methods for Steklov eigenvalue problems; for instance, [4] studied the conforming linear finite element approximation, [5, 6] studied the nonconforming finite elements approximation, [7, 8] discussed a two-grid method of the conforming and nonconforming finite element method based on the inverse iteration, respectively, [9] studied multiscale asymptotic method, [10] studied multilevel method, [11] studied the spectral method, and [12] studied an adaptive algorithm based on the shifted inverse iteration.

In this paper we establish a type of multigrid discretizations based on the fixed-shift inverse iteration for the Steklov eigenvalue problem. The multilevel method in [10] made use of the inverse iteration and the extended finite element method. Compared with [10], our method has less computational complexity since we have no correction step in each iteration. On the other hand, compared with [12], we adopt the fixed-shift and thus avoid selecting appropriate shift to ensure the efficiency of shifted inverse iteration; meanwhile, we also do not face the difficulty of solving an almost singular algebraic system in the shifted inverse iteration.

We analyze elaborately the a priori and the a posteriori error estimates of the method proposed in this paper. Then, based on the a posteriori error estimates we design an adaptive algorithm of fixed-shift inverse iteration type. Moreover, we also compare the performance of three types of multigrid methods. Numerical results illustrate that our method is also an efficient method for solving the Steklov eigenvalue problem.

The rest of this paper is organized as follows. In the subsequent section, some preliminaries needed in this paper are presented. In Section 3, a scheme of the inverse iteration with fixed-shift based on multigrid discretizations is established, and the a priori error estimates are also given. The a posteriori error estimates of the inverse iteration with fixed-shift are analyzed in Section 4. Numerical experiments are presented in the final section.

In this paper, C with or without subscript denotes a constant independent of mesh size and iterative times.

2. Preliminaries

Consider the Steklov eigenvalue problem

$$\begin{aligned} -\Delta u + u &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= \lambda u \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain with θ being the largest inner angle of Ω and $\partial u/\partial n$ is the outward normal derivative.

We denote the real order Sobolev spaces with norm $\|\cdot\|_t$ and $\|\cdot\|_{t,\partial\Omega}$ by $H^t(\Omega)$ and $H^t(\partial\Omega)$, respectively; $H^0(\partial\Omega) = L_2(\partial\Omega)$.

The variational form of (1) is given by the following: find $\lambda \in \mathbb{R}$ and $u \in H^1(\Omega)$, $u \neq 0$, such that

$$a(u,v) = \lambda b(u,v), \quad \forall v \in H^1(\Omega), \tag{2}$$

where

$$\begin{aligned} a(u,v) &= \int_\Omega \nabla u \cdot \nabla v + uv\,dx, \\ b(u,v) &= \int_{\partial\Omega} uv\,ds, \\ \|u\|_b &= b(u,u)^{1/2} = \|u\|_{0,\partial\Omega}. \end{aligned} \tag{3}$$

As we know, $a(\cdot,\cdot)$ is a symmetric, continuous, and $H^1(\Omega)$-elliptic bilinear form on $H^1(\Omega)\times H^1(\Omega)$. Thus, we use $a(\cdot,\cdot)$ and $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)} = \|\cdot\|_1$ as the inner product and norm on $H^1(\Omega)$, respectively.

Let $H^{-1/2}(\partial\Omega)$ be the dual space of $H^{1/2}(\partial\Omega)$ with norm given by

$$\|w\|_{-1/2,\partial\Omega} = \sup_{v\in H^{1/2}(\partial\Omega)} \frac{\langle w,v\rangle}{\|v\|_{1/2,\partial\Omega}}, \tag{4}$$

where $\langle w,v\rangle$ is the dual product on $H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)$. When $w \in L^2(\partial\Omega)$, $\langle w,v\rangle = b(w,v)$.

Let $\{\pi_h\}$ be a family of regular triangulations of Ω with the mesh diameter h, and let $V_h \subset H^1(\Omega)$ be a space of piecewise polynomials defined on π_h. For any $w \in H^1(\Omega)$, the following conclusion holds:

$$\lim_{h\to 0} \inf_{v\in V_h} \|w-v\|_a = 0. \tag{5}$$

The conforming finite element approximation of (2) is the following: find $\lambda_h \in \mathbb{R}$ and $u_h \in V_h$, $u_h \neq 0$, such that

$$a(u_h,v) = \lambda_h b(u_h,v), \quad \forall v \in V_h. \tag{6}$$

Define the operators $T : H^1(\Omega) \to H^1(\Omega)$ and $T_h : H^1(\Omega) \to V_h \subset H^1(\Omega)$ satisfying

$$a(Tg,v) = b(g,v), \quad \forall v \in H^1(\Omega), \tag{7}$$

$$a(T_h g,v) = b(g,v), \quad \forall v \in V_h. \tag{8}$$

Define the Ritz projection $P_h : H^1(\Omega) \to V_h$ by

$$a(u - P_h u, v) = 0, \quad \forall v \in V_h. \tag{9}$$

From [13], we know that $\|T - T_h\|_a \to 0$ $(h \to 0)$; (2) and (6) have the equivalent operator forms $Tu = \mu u$ and $T_h u_h = \mu_h u_h$, respectively, where $T_h = P_h T$, $\mu = 1/\lambda$, and $\mu_h = 1/\lambda_h$.

Suppose that λ and λ_h are the kth eigenvalue of (2) and (6), respectively, and the algebraic multiplicity of λ is equal to q, $\lambda = \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+q-1}$. Let $M(\lambda)$ be the space spanned by all eigenfunctions corresponding to λ and let $M_h(\lambda)$ be the direct sum of eigenspaces corresponding to all eigenvalues of (6) that converge to λ. Let $\widehat{M}(\lambda) = \{v : v \in M(\lambda),\ \|v\|_a = 1\}$.

Denote

$$\begin{aligned} \sigma(h) &= \sup_{f\in H^1(\Omega),\|f\|_a=1} \inf_{v\in V_h} \|Tf - v\|_a, \\ \rho(h) &= \sup_{f\in L_2(\partial\Omega),\|f\|_{0,\partial\Omega}=1} \inf_{v\in V_h} \|Tf - v\|_a, \\ \delta_h(\lambda) &= \sup_{u\in\widehat{M}(\lambda)} \inf_{v\in V_h} \|u - v\|_a. \end{aligned} \tag{10}$$

It is obvious that $\delta_h(\lambda) \le \lambda\sigma(h) \le C\rho(h)$. It follows from Lemma 3.3 in [14] that

$$\sigma(h) \longrightarrow 0 \quad (h \longrightarrow 0). \tag{11}$$

By using the trace theorem we have

$$\|v\|_{0,\partial\Omega} \le \|v\|_{1/2,\partial\Omega} \le C_1 \|v\|_a, \quad \forall v \in H^1(\Omega). \tag{12}$$

Moreover, if $v \in H^1(\Omega)$ and $w \in H^{1/2}(\partial\Omega)$ we know that $\langle v,w\rangle = b(v,w) \le \|v\|_{0,\partial\Omega}\|w\|_{0,\partial\Omega}$ and, consequently,

$$\|v\|_{-1/2,\partial\Omega} \le \|v\|_{0,\partial\Omega} \le C_1 \|v\|_a, \quad \forall v \in H^1(\Omega). \tag{13}$$

For any $g \in H^1(\Omega)$, $T_h g \in H^1(\Omega)$. Taking $v = T_h g$ in (8) we deduce

$$\begin{aligned} a(T_h g, T_h g) = b(g, T_h g) &\le \|g\|_{-1/2,\partial\Omega} \|T_h g\|_{1/2,\partial\Omega} \\ &\le C_1 \|g\|_{-1/2,\partial\Omega} \|T_h g\|_a, \end{aligned} \tag{14}$$

and thus we get

$$\|T_h g\|_a \le C_1 \|g\|_{-1/2,\partial\Omega}. \tag{15}$$

The following lemmas are needed in our analysis.

Lemma 1. *Let (λ,u) be an eigenpair of (2); then for any $v \in H^1(\Omega)$ with $\|v\|_a = 1$, the Rayleigh quotient $R(v) = a(v,v)/\|v\|_b^2$ satisfies*

$$R(v) - \lambda = \frac{\|v-u\|_a^2}{\|v\|_b^2} - \lambda\frac{\|v-u\|_b^2}{\|v\|_b^2}. \tag{16}$$

Proof. See page 699 of [13]. □

Lemma 2. *For any nonzero $u, v \in H^1(\Omega)$,*

$$\begin{aligned} \left\| \frac{u}{\|u\|_a} - \frac{v}{\|v\|_a} \right\|_a &\le 2\frac{\|u-v\|_a}{\|u\|_a}, \\ \left\| \frac{u}{\|u\|_a} - \frac{v}{\|v\|_a} \right\|_a &\le 2\frac{\|u-v\|_a}{\|v\|_a}. \end{aligned} \tag{17}$$

Proof. See [15]. □

Lemma 3. *Let λ and λ_h be the kth eigenvalue of (2) and (6), respectively. Then for any eigenfunction u_h corresponding to λ_h with $\|u_h\|_a = 1$, there exist $u \in M(\lambda)$ and $h_0 > 0$ such that if $h \le h_0$,*

$$\|u_h - u\|_a \le C_2 \delta_h(\lambda), \tag{18}$$

$$\|u_h - u\|_{0,\partial\Omega} \le C_2 \rho(h)\, \delta_h(\lambda), \tag{19}$$

$$\|u_h - u\|_{-1/2,\partial\Omega} \le C_2 \sigma(h)\, \delta_h(\lambda); \tag{20}$$

for any $u \in \widehat{M}(\lambda)$, there exists $u_h \in M_h(\lambda)$ such that if $h \le h_0$,

$$\begin{aligned} \|u - u_h\|_a &\le C_3 \delta_h(\lambda), \\ \|u - u_h\|_{-1/2,\partial\Omega} &\le C_3 \sigma(h) \delta_h(\lambda), \end{aligned} \tag{21}$$

where constants C_2 and C_3 are positive and only depend on λ.

Proof. See page 699 of [13] and Lemma 3.7 and (3.29b) of [14]. □

If $u \in M(\lambda)$, $v \in H^1(\Omega)$, $\|v\|_a = 1$, and $\|v - u\|_a \le (4\sqrt{\lambda}C_1)^{-1}$, then by Lemma 2 we have

$$\begin{aligned} \left\| v - \frac{u}{\|u\|_a} \right\|_a &\le 2\|v - u\|_a \le \left(2\sqrt{\lambda}C_1\right)^{-1}, \\ \left\| v - \frac{u}{\|u\|_a} \right\|_{0,\partial\Omega} &\le C_1 \left\| v - \frac{u}{\|u\|_a} \right\|_a \le \frac{1}{2\sqrt{\lambda}}. \end{aligned} \tag{22}$$

From (2) we have $\|u/\|u\|_a\|_{0,\partial\Omega} = 1/\sqrt{\lambda}$; then

$$\|v\|_{0,\partial\Omega} \ge \left\| \frac{u}{\|u\|_a} \right\|_{0,\partial\Omega} - \left\| v - \frac{u}{\|u\|_a} \right\|_{0,\partial\Omega} \ge \frac{1}{2\sqrt{\lambda}}. \tag{23}$$

Hence, from Lemma 1 we get

$$|R(v) - \lambda| \le 4\lambda\left(1 + \lambda C_1^2\right)\|v - u\|_a^2. \tag{24}$$

Denote

$$C_4 = 4\lambda\left(1 + \lambda C_1^2\right), \tag{25}$$

and then when $\|v\|_a = 1$ and $\|v - u\|_a \le (4\sqrt{\lambda}C_1)^{-1}$, (24) becomes

$$|R(v) - \lambda| \le C_4 \|v - u\|_a^2. \tag{26}$$

Since (6) implies $\lambda_h = R(u_h)$, then combining (26) and (18) we deduce that

$$0 \le \lambda_h - \lambda \le C_4 \|u_h - u\|_a^2 \le C_4 C_2^2 \delta_h^2(\lambda). \tag{27}$$

3. A Priori Error Estimates of the Inverse Iteration with Fixed-Shift

Let $\{V_{h_i}\}_0^\infty$ be a family of conforming finite element spaces that satisfy $V_{h_0} = V_H$, $V_{h_i} \subset V_{h_{i+1}} \subset H^1(\Omega)$ $(i = 0, 1, \ldots)$, and $\sigma(h_i) \to 0$ $(i \to \infty)$. Referring to [16], we establish the following scheme of the inverse iteration with fixed-shift based on multigrid discretizations.

Scheme 4 (the inverse iteration with fixed-shift based on multigrid discretizations). Given the iterative times l and $i0$. Execute the following.

Step 1. Solve (2) on V_H: find $(\lambda_H, u_H) \in \mathbb{R} \times V_H$ such that $\|u_H\|_a = 1$ and

$$a(u_H, v) = \lambda_H b(u_H, v), \quad \forall v \in V_H. \tag{28}$$

Step 2. Let $u^{h_0} \Leftarrow u_H$, $\lambda^{h_0} \Leftarrow \lambda_H$, $i \Leftarrow 1$.

Step 3. Solve a linear system on V_{h_i}: find $u' \in V_{h_i}$ such that

$$a(u', v) - \lambda^{h_{i-1}} b(u', v) = b\left(u^{h_{i-1}}, v\right), \quad \forall v \in V_{h_i}; \tag{29}$$

set $u^{h_i} = u'/\|u'\|_a$.

Step 4. Compute the Rayleigh quotient

$$\lambda^{h_i} = \frac{a\left(u^{h_i}, u^{h_i}\right)}{b\left(u^{h_i}, u^{h_i}\right)}. \tag{30}$$

Step 5. If $i > i0$, then $\lambda^{h_{i0}} \Leftarrow \lambda^{h_{i-1}}$, $i \Leftarrow i + 1$; turn to Step 6; else, $i \Leftarrow i + 1$, and return to Step 3.

Step 6. Solve a linear system on V_{h_i}: find $u' \in V_{h_i}$ such that

$$a(u', v) - \lambda^{h_{i0}} b(u', v) = b\left(u^{h_{i-1}}, v\right), \quad \forall v \in V_{h_i}; \tag{31}$$

set $u^{h_i} = u'/\|u'\|_a$.

Step 7. Compute the Rayleigh quotient

$$\lambda^{h_i} = \frac{a\left(u^{h_i}, u^{h_i}\right)}{b\left(u^{h_i}, u^{h_i}\right)}. \tag{32}$$

Step 8. If $i = l$, then output (λ^{h_l}, u^{h_l}) and stop; else, $i \Leftarrow i + 1$, and return to Step 6.

Let (λ_H, u_H) be the kth eigenpair of (28); then (λ^{h_l}, u^{h_l}) derived from Scheme 4 is the kth eigenpair approximation of (2).

In the following analysis, we also denote $(\lambda_H, u_H) = (\lambda_{k,H}, u_{k,H})$ and $(\lambda^{h_l}, u^{h_l}) = (\lambda_k^{h_l}, u_k^{h_l})$.

Now, we will analyze the a priori error estimates of Scheme 4.

Denote $\operatorname{dist}(u, S) = \inf_{v \in S}\|u - v\|_a$.

Our analysis makes use of the following lemma (see Lemma 4.1 in [16]) for the shifted inverse iteration method. Let (λ_k, u_k) and $(\lambda_{k,h}, u_{k,h})$ denote the kth eigenpair of (2) and (6), respectively, and $\mu_k = 1/\lambda_k$, $\mu_{k,h} = 1/\lambda_{k,h}$, $M(\mu_k) = M(\lambda_k)$, and $M_h(\mu_k) = M_h(\lambda_k)$.

Lemma 5. *Let (μ_0, u_0) be an approximation for (μ_k, u_k), where μ_0 is not an eigenvalue of T_h, and $u_0 \in V_h$ with $\|u_0\|_a = 1$. Suppose that*

(C1) $\operatorname{dist}(u_0, M_h(\mu_k)) \le 1/2$;

(C2) $|\mu_0 - \mu_k| \le \rho/4$ *and* $|\mu_{j,h} - \mu_j| \le \rho/4$, *for* $j = k-1, k, k+q$ $(j \ne 0)$, *where* $\rho = \min_{\mu_j \ne \mu_k}|\mu_j - \mu_k|$ *is the separation constant of the eigenvalue* μ_k;

(C3) $u' \in V_h$, $u_k^h \in V_h$ *satisfy*

$$\begin{aligned} (\mu_0 - T_h) u' &= u_0, \\ u_k^h &= \frac{u'}{\|u'\|_a}. \end{aligned} \tag{33}$$

Then

$$\begin{aligned}&\operatorname{dist}\left(u_k^h, M_h(\mu_k)\right)\\&\quad\le \frac{4}{\rho}\max_{k\le j\le k+q-1}\left|\mu_0-\mu_{j,h}\right|\operatorname{dist}\left(u_0, M_h(\mu_k)\right).\end{aligned}\tag{34}$$

Let δ_0 be a positive constant satisfying the following inequalities:

$$\max\{1, C_2, C_3\}\,\delta_0 \le \min\left\{\frac{1}{2}, \left(4\sqrt{\lambda_j}C_1\right)^{-1}\right\},\tag{35}$$

$$\begin{aligned}&4C_1\left(C_1C_4\delta_0^2+\lambda_kC_1\delta_0+\lambda_kC_3\delta_0^2\right.\\&\quad\left.+C_1C_5q^{1/2}C_4C_2^2\delta_0^2\right)\le\frac{1}{2},\end{aligned}\tag{36}$$

$$\frac{\delta_0}{(\lambda_k-\delta_0)\lambda_k}\le\frac{\rho}{4},\quad \delta_0\le\frac{\lambda_k}{2},\tag{37}$$

$$\frac{C_4C_2^2}{\lambda_j^2}\delta_0^2\le\frac{\rho}{4},\quad j=k-1,k,\ldots,k+q,\ j\ne 0.\tag{38}$$

Condition 6. There exists $\overline{u}\in\widehat{M}(\lambda_k)$ for $j=k-1,k,k+q$ ($j\ne 0$) such that

$$\begin{aligned}\left\|u_k^{h_{l-1}}-\overline{u}\right\|_a&\le\delta_0,\\\left|\lambda_0-\lambda_k\right|&\le\delta_0,\\\delta_{h_l}\left(\lambda_j\right)&\le\delta_0,\\\sigma\left(h_l\right)&\le\delta_0,\end{aligned}\tag{39}$$

where λ_0 is an approximate eigenvalue of λ_k, $u_k^{h_{l-1}}$ is an approximate eigenfunction obtained by Scheme 4, and ρ is the separation constant of the eigenvalue $\mu_k=1/\lambda_k$.

Let the eigenvectors $\{u_{j,h_l}\}_k^{k+q-1}$ be an orthonormal basis of $M_{h_l}(\lambda_k)$ with respect to $a(\cdot,\cdot)$, and denote

$$u^*=\sum_{j=k}^{k+q-1}a\left(u_k^{h_l},u_{j,h_l}\right)u_{j,h_l}.\tag{40}$$

From Lemma 3, we know that there exist eigenvectors $\{u_j^0\}_k^{k+q-1}\subset M(\lambda_k)$ making $u_{j,h_l}-u_j^0$ satisfy (18), (19), and (20). Let

$$u_k=\sum_{j=k}^{k+q-1}a\left(u_k^{h_l},u_{j,h_l}\right)u_j^0,\tag{41}$$

and then $u_k\in M(\lambda_k)$ and

$$u_k-u^*=\sum_{j=k}^{k+q-1}a\left(u_k^{h_l},u_{j,h_l}\right)\left(u_j^0-u_{j,h_l}\right).\tag{42}$$

To estimate the error, we split

$$u_k^{h_l}-u_k=\left(u_k^{h_l}-u^*\right)-\left(u_k-u^*\right).\tag{43}$$

Now, we will analyze the first term $u_k^{h_l}-u^*$.

Theorem 7. *Let $(\lambda_k^{h_l},u_k^{h_l})$ be an approximate eigenpair obtained by Scheme 4 with $\lambda_0=\lambda_k^{h_{i0}}$. Assume that Lemma 3 and Condition 6 hold; then*

$$\begin{aligned}\left\|u_k^{h_l}-u^*\right\|_a&\le\frac{C_0}{4}\left|\lambda_0-\lambda_k\right|\left\{\left|\lambda_k^{h_{l-1}}-\lambda_k\right|\right.\\&\quad\left.+\left\|u_k^{h_{l-1}}-\overline{u}\right\|_{-1/2,\partial\Omega}+\sigma\left(h_l\right)\delta_{h_l}\left(\lambda_k\right)\right\},\quad l\ge 1,\end{aligned}\tag{44}$$

where C_0 is independent of mesh parameters and l.

Proof. We use Lemma 5 to complete the proof. First, we will verify that the conditions of Lemma 5 are satisfied.

From Lemma 3, we know that, for any given $\overline{u}\in\widehat{M}(\lambda_k)$, there exists $\widetilde{u}_{k,h_l}\in M_{h_l}(\lambda_k)$ such that

$$\left\|\overline{u}-\widetilde{u}_{k,h_l}\right\|_a\le C_3\delta_{h_l}\left(\lambda_k\right),\tag{45}$$

$$\left\|\overline{u}-\widetilde{u}_{k,h_l}\right\|_{-1/2,\partial\Omega}\le C_3\sigma\left(h_l\right)\delta_{h_l}\left(\lambda_k\right),\tag{46}$$

where

$$\begin{aligned}\widetilde{u}_{k,h_l}&=\sum_{i=k}^{k+q-1}\alpha_iu_{i,h_l},\\\left\|\widetilde{u}_{k,h_l}\right\|_a&\le C_5.\end{aligned}\tag{47}$$

Select $\mu_0=1/\lambda_0$ and $u_0=\lambda_k^{h_{l-1}}T_{h_l}u_k^{h_{l-1}}/\|\lambda_k^{h_{l-1}}T_{h_l}u_k^{h_{l-1}}\|_a$. Then, by (15) and (13) we have

$$\begin{aligned}&\left\|\lambda_k^{h_{l-1}}T_{h_l}u_k^{h_{l-1}}-\widetilde{u}_{k,h_l}\right\|_a=\left\|\lambda_k^{h_{l-1}}T_{h_l}u_k^{h_{l-1}}\right.\\&\quad\left.-\sum_{i=k}^{k+q-1}\lambda_{i,h_l}T_{h_l}\alpha_iu_{i,h_l}\right\|_a\le C_1\left\|\lambda_k^{h_{l-1}}u_k^{h_{l-1}}\right.\\&\quad\left.-\sum_{i=k}^{k+q-1}\lambda_{i,h_l}\alpha_iu_{i,h_l}\right\|_{-1/2,\partial\Omega}=C_1\left\|\lambda_k^{h_{l-1}}u_k^{h_{l-1}}-\lambda_ku_k^{h_{l-1}}\right.\\&\quad+\lambda_ku_k^{h_{l-1}}-\lambda_k\overline{u}+\lambda_k\overline{u}-\lambda_k\widetilde{u}_{k,h_l}+\lambda_k\widetilde{u}_{k,h_l}\\&\quad\left.-\sum_{i=k}^{k+q-1}\lambda_{i,h_l}\alpha_iu_{i,h_l}\right\|_{-1/2,\partial\Omega}\le C_1\left(C_1\left|\lambda_k^{h_{l-1}}-\lambda_k\right|\right.\\&\quad+\lambda_k\left\|u_k^{h_{l-1}}-\overline{u}\right\|_{-1/2,\partial\Omega}+\lambda_k\left\|\overline{u}-\widetilde{u}_{k,h_l}\right\|_{-1/2,\partial\Omega}\\&\quad\left.+\left\|\sum_{i=k}^{k+q-1}\left(\lambda_k-\lambda_{i,h_l}\right)\alpha_iu_{i,h_l}\right\|_{-1/2,\partial\Omega}\right),\end{aligned}\tag{48}$$

noting that $\|\widetilde{u}_{k,h_l}\|_a\ge\|\overline{u}\|_a-\|\overline{u}-\widetilde{u}_{k,h_l}\|_a\ge 1-C_3\delta_{h_l}(\lambda_k)\ge 1-C_3\delta_0\ge 1/2$; then using Lemma 2, (46), the Cauchy-Schwartz inequality, (26), Condition 6, and (36) we obtain

$$\begin{aligned}\operatorname{dist}\left(u_0,M_{h_l}\left(\lambda_k\right)\right)&\le\left\|u_0-\frac{\widetilde{u}_{k,h_l}}{\left\|\widetilde{u}_{k,h_l}\right\|_a}\right\|_a\\&\le\frac{2}{\left\|\widetilde{u}_{k,h_l}\right\|_a}\left\|\lambda_k^{h_{l-1}}T_{h_l}u_k^{h_{l-1}}-\widetilde{u}_{k,h_l}\right\|_a\end{aligned}$$

$$\begin{aligned}
&\le 4C_1\left(C_1\left|\lambda_k^{h_{l-1}}-\lambda_k\right|+\lambda_k\left\|u_k^{h_{l-1}}-\overline{u}\right\|_{-1/2,\partial\Omega}\right.\\
&\left.+\lambda_k C_3\sigma(h_l)\delta_{h_l}(\lambda_k)+C_1C_5q^{1/2}C_4C_2^2\delta_{h_l}^2(\lambda_k)\right)\\
&\le 4C_1\left(C_1C_4\delta_0^2+\lambda_kC_1\delta_0+\lambda_kC_3\delta_0^2\right.\\
&\left.+C_1C_5q^{1/2}C_4C_2^2\delta_0^2\right)\le\frac{1}{2},
\end{aligned}\tag{49}$$

and then Condition $(C1)$ in Lemma 5 holds.

By using the same arguments in [16], it is clear that the other two conditions in Lemma 5 are valid.

Hence, we see that the conditions of Lemma 5 hold.

Then, by the same proof method in [16], we derive that

$$\begin{aligned}
\left\|u_k^{h_l}-u^*\right\|_a &\le \frac{16}{\rho}C_1\left(\frac{2}{\lambda_k^2}\left|\lambda_0-\lambda_k\right|\right.\\
&\left.+\frac{2}{\lambda_k^2}C_4C_2^2\delta_{h_l}^2(\lambda_k)\right)\left(C_1\left|\lambda_k^{h_{l-1}}-\lambda_k\right|\right.\\
&+\lambda_k\left\|u_k^{h_{l-1}}-\overline{u}\right\|_{-1/2,\partial\Omega}+\lambda_kC_3\sigma(h_l)\delta_{h_l}(\lambda_k)\\
&\left.+C_1C_5q^{1/2}C_4C_2^2\delta_{h_l}^2(\lambda_k)\right).
\end{aligned}\tag{50}$$

Noting that the constants C_1, C_2, C_3, C_4, C_5, and ρ are independent of mesh parameters and l and Condition 6 holds, then based on the above inequality we conclude that there exists a positive constant C_0 that is independent of mesh parameters and l such that (44) holds. And we can have $\min\{C_0/4, C_0C_1/4\} > q^{1/2}C_2$. The proof is completed. □

Next, we will analyze the error $u_k - u^*$.

Theorem 8. *The error $u_k - u^*$ satisfies*

$$\left\|u_k-u^*\right\|_a\le\frac{C_0}{4}\delta_{h_l}(\lambda_k),\tag{51}$$

$$\left\|u_k-u^*\right\|_{-1/2,\partial\Omega}\le\frac{C_0C_1}{4}\sigma(h_l)\delta_{h_l}(\lambda_k).\tag{52}$$

Proof. The estimates (51) and (52) can be obtained by the proof arguments in [16]. □

Based on the above two theorems, we now analyze the a priori error estimates of Scheme 4.

Condition 9. For any given $\beta_0, \beta_0' \in (0,1)$, there exist $0 < \beta_0 \le \beta_i < 1$ and $0 < \beta_0' \le \beta_i' < 1$ $(i = 1, 2, \ldots)$ such that $\delta_{h_i}(\lambda_k) = \beta_i\delta_{h_{i-1}}(\lambda_k)$ and $\sigma(h_i) = \beta_i'\sigma(h_{i-1})$, respectively, $\sigma(h_i) \to 0$ $(i \to \infty)$.

In the practice Condition 9 is not a restrictive condition. For example, let π_{h_i} be obtained from $\pi_{h_{i-1}}$ via regular refinement (producing 4 congruent elements) such that $h_i = (1/2)h_{i-1}$; then, when $M(\lambda_k) \subset H^{1+\gamma}(\Omega)$ and $\{Tf : f \in H^1(\Omega)\} \subset H^{1+\gamma}(\Omega)$ we have $\delta_{h_i}(\lambda_k) \approx (1/2)^\gamma\delta_{h_{i-1}}(\lambda_k)$ and $\sigma(h_i) \approx (1/2)^\gamma\sigma(h_{i-1})$ (see [10]), where $\gamma = 1$ if Ω is convex and $0 < \gamma < 1$ if Ω is concave.

Theorem 10. *Let $(\lambda_k^{h_l}, u_k^{h_l})$ be an approximate eigenpair obtained by Scheme 4. Suppose that Condition 9 holds; then there exist $u_k \in M(\lambda_k)$ and $H_0 > 0$ such that if $H \le H_0$ it is valid that*

$$\left\|u_k^{h_l}-u_k\right\|_a\le C_0\delta_{h_l}(\lambda_k),\tag{53}$$

$$\left\|u_k^{h_l}-u_k\right\|_{-1/2,\partial\Omega}\le C_0C_1\sigma(h_l)\delta_{h_l}(\lambda_k),\tag{54}$$

$$\left|\lambda_k^{h_l}-\lambda_k\right|\le C_4C_0^2\delta_{h_l}^2(\lambda_k),\quad l\ge i0.\tag{55}$$

Proof. We only prove the result (54) since (53) and (55) can be proved analogously by referring to [16].

The proof is completed by using induction, Theorems 7 and 8. Note that $\delta_H(\lambda_k) \le \lambda_k\sigma(H) \to 0(H \to 0)$; then there exists a proper small $H_0 > 0$ such that if $H \le H_0$, Lemma 3 and the following inequalities hold:

$$\begin{aligned}
C_0\delta_H(\lambda_k)&\le\delta_0,\\
C_4C_0^2\delta_H^2(\lambda_k)&\le\delta_0,\\
\delta_H(\lambda_j)&\le\delta_0,\\
\sigma(H)&\le\delta_0,
\end{aligned}\tag{56}$$

$$C_4^2C_0^4\delta_H^2(\lambda_k)\lambda_k\frac{1}{\beta_0'}\frac{1}{\beta_0}+C_4C_0^3C_1\delta_H^2(\lambda_k)\frac{1}{\beta_0'}\frac{1}{\beta_0}\le 1,\tag{57}$$

where $j = k-1, k, k+q$ $(j \ne 0)$.

When $l = i0$, it is easy to know that (53)–(55) are valid (see [12, 16]). Suppose that Theorem 10 holds for $l-1$; that is, there exists $\overline{u} \in M(\lambda_k)$ such that

$$\begin{aligned}
\left\|u_k^{h_{l-1}}-\overline{u}\right\|_a&\le C_0\delta_{h_{l-1}}(\lambda_k),\\
\left\|u_k^{h_{l-1}}-\overline{u}\right\|_{-1/2,\partial\Omega}&\le C_0C_1\sigma(h_{l-1})\delta_{h_{l-1}}(\lambda_k),\\
\left|\lambda_k^{h_{l-1}}-\lambda_k\right|&\le C_4C_0^2\delta_{h_{l-1}}^2(\lambda_k).
\end{aligned}\tag{58}$$

Then we infer from (56) that the conditions of Theorem 7 hold.

From Theorems 7 and 8 we get

$$\begin{aligned}
\left\|u_k^{h_l}-u_k\right\|_{-1/2,\partial\Omega}&\le\frac{C_0C_1}{2}\left\{\left|\lambda_0-\lambda_k\right|\right.\\
&\cdot\left(\left|\lambda_k^{h_{l-1}}-\lambda_k\right|+\left\|u_k^{h_{l-1}}-\overline{u}\right\|_{-1/2,\partial\Omega}\right)+\sigma(h_l)\\
&\left.\cdot\delta_{h_l}(\lambda_k)\right\},\quad l\ge 1.
\end{aligned}\tag{59}$$

Therefore, for l, from (59) we derive that

$$\begin{aligned}
\left\|u_k^{h_l}-u_k\right\|_{-1/2,\partial\Omega}&\le\frac{C_0C_1}{2}\left\{C_4^2C_0^4\delta_{h_{i0}}^2(\lambda_k)\delta_{h_{l-1}}^2(\lambda_k)\right.\\
&+C_4C_0^3C_1\delta_{h_{i0}}^2(\lambda_k)\sigma(h_{l-1})\delta_{h_{l-1}}(\lambda_k)\\
&\left.+\sigma(h_l)\delta_{h_l}(\lambda_k)\right\}
\end{aligned}$$

$$\leq \frac{C_0C_1}{2}\left\{C_4^2C_0^4\delta_{h_{i0}}^2(\lambda_k)\lambda_k\frac{1}{\beta_l'}\frac{1}{\beta_l}\right.$$
$$\left.+C_4C_0^3C_1\delta_{h_{i0}}^2(\lambda_k)\frac{1}{\beta_l'}\frac{1}{\beta_l}+1\right\}\sigma(h_l)\delta_{h_l}(\lambda_k)$$
$$\leq \frac{C_0C_1}{2}\left\{C_4^2C_0^4\delta_H^2(\lambda_k)\lambda_k\frac{1}{\beta_0'}\frac{1}{\beta_0}\right.$$
$$\left.+C_4C_0^3C_1\delta_H^2(\lambda_k)\frac{1}{\beta_0'}\frac{1}{\beta_0}+1\right\}\sigma(h_l)\delta_{h_l}(\lambda_k), \tag{60}$$

which together with (57) we get (54) immediately. □

4. A Posteriori Error Estimates of the Inverse Iteration with Fixed-Shift

Based on the work of [4, 12, 17–19], in this section, we will discuss the a posteriori error estimates of Scheme 4 for the Steklov eigenvalue problem.

Consider the boundary value problem corresponding to (2): find $w \in H^1(\Omega)$ such that

$$a(w,v) = b(f,v), \quad \forall v \in H^1(\Omega), \tag{61}$$

and its finite element approximation states: find $w_h \in V_h$ such that

$$a(w_h,v) = b(f,v), \quad \forall v \in V_h. \tag{62}$$

For any element $T \in \pi_h$ with diameter h_T, we denote by $\mathscr{E}_T$ the set of edges, and

$$\mathscr{E} = \bigcup_{T\in\pi_h}\mathscr{E}_T. \tag{63}$$

We decompose $\mathscr{E} = \mathscr{E}_\Omega \cup \mathscr{E}_\Gamma$, where $\mathscr{E}_\Omega$ and $\mathscr{E}_\Gamma$ refer to interior edges and edges on the boundary $\Gamma = \partial\Omega$, respectively. For each $\ell \in \mathscr{E}_\Omega$, we choose an arbitrary unit normal vector n_ℓ and denote the two triangles sharing this edge by T_{in} and T_{out}, where n_ℓ points outwards T_{in}.

For $v_h \in V_h$ we set

$$\left[\left[\frac{\partial v_h}{\partial n_\ell}\right]\right]_\ell = \nabla\left(v_h|_{T_{\text{out}}}\right)\cdot n_\ell - \nabla\left(v_h|_{T_{\text{in}}}\right)\cdot n_\ell. \tag{64}$$

Let

$$\widehat{\lambda}_{k,h_l} = \frac{1}{q}\sum_{j=k}^{k+q-1}\lambda_{j,h_l}. \tag{65}$$

For each $\ell \in \mathscr{E}$ we define the jump residual:

$$J_\ell\left(u_k^{h_l}\right) = \begin{cases}\dfrac{1}{2}\left[\left[\dfrac{\partial u_k^{h_l}}{\partial n_\ell}\right]\right]_\ell & \ell \in \mathscr{E}_\Omega,\\ \lambda_k^{h_l}u_k^{h_l} - \dfrac{\partial u_k^{h_l}}{\partial n_\ell} & \ell \in \mathscr{E}_\Gamma.\end{cases} \tag{66}$$

Now, the local error indicator is defined as

$$\eta_T\left(u_k^{h_l}\right) = \left(h_T^2\left\|u_k^{h_l}\right\|_{0,T}^2 + \sum_{\ell\in\mathscr{E}_T}|\ell|\left\|J_\ell\left(u_k^{h_l}\right)\right\|_{0,\ell}^2\right)^{1/2}, \tag{67}$$

and then the global error estimator is given by

$$\eta_\Omega\left(u_k^{h_l}\right) = \left(\sum_{T\in\pi_h}\eta_T^2\left(u_k^{h_l}\right)\right)^{1/2}. \tag{68}$$

Substituting u^* for $u_k^{h_l}$, we can get the definitions of $J_\ell(u^*)$, $\eta_T(u^*)$, and $\eta_\Omega(u^*)$ similarly.

Now, we will estimate the error $e = u_k - u^*$.

From [4, 12], we give the following two lemmas among which Lemma 11 provides the global upper bound of e, while Lemma 12 provides the local lower bound of e.

Lemma 11. *The error $e = u_k - u^*$ satisfies*

$$\|e\|_a \leq C_6\left\{\eta_\Omega(u^*) + \sigma(h_l)\delta_{h_l}(\lambda_k)\right\}. \tag{69}$$

Lemma 12. *The error $e = u_k - u^*$ satisfies the following:*

(a) For $T \in \pi_h$, if $\partial T \cap \Gamma = \emptyset$, then

$$\eta_T(u^*) \leq C_7\|e\|_{1,T^*}, \tag{70}$$

where T^* denotes the union of T and the triangles sharing an edge with T.

(b) For $T \in \pi_h$, if $\partial T \cap \Gamma \neq \emptyset$, then

$$\eta_T(u^*) \leq C_8\left\{\|e\|_{1,T} + \sum_{\ell\in\mathscr{E}_T\cap\mathscr{E}_\Gamma}|\ell|^{1/2}\left\|\lambda_k u_k - \widehat{\lambda}_{k,h_l}u^*\right\|_{0,\ell}\right\}. \tag{71}$$

Next, we will analyze the error $u_k^{h_l} - u^*$.

Theorem 13. *Suppose that the conditions of Theorem 10 are satisfied; then*

$$\left\|u_k^{h_l} - u^*\right\|_a \leq \frac{C_0}{4}\sigma(h_l)\delta_{h_l}(\lambda_k). \tag{72}$$

Proof. Note that $\delta_H(\lambda_k) \leq \lambda_k\sigma(H) \to 0(H \to 0)$; then there exists a proper small $H_0 > 0$ such that if $H \leq H_0$, the following inequality holds:

$$C_4^2C_0^4\delta_{\text{H}}^2(\lambda_k)\lambda_k\frac{1}{\beta_0'}\frac{1}{\beta_0} + C_4C_0^3C_1\delta_H^2(\lambda_k)\frac{1}{\beta_0'}\frac{1}{\beta_0}$$
$$+C_4C_0^2\delta_H^2(\lambda_k) \leq 1. \tag{73}$$

From (44), Theorem 10, and Condition 9, we have

$$\left\|u_k^{h_l} - u^*\right\|_a \leq \frac{C_0}{4}\left\{C_4^2C_0^4\delta_{h_{i0}}^2(\lambda_k)\delta_{h_{l-1}}^2(\lambda_k)\right.$$
$$+C_4C_0^3C_1\delta_{h_{i0}}^2(\lambda_k)\sigma(h_{l-1})\delta_{h_{l-1}}(\lambda_k)$$
$$\left.+C_4C_0^2\delta_{h_{i0}}^2(\lambda_k)\sigma(h_l)\delta_{h_l}(\lambda_k)\right\}$$

$$\leq \frac{C_0}{4}\left\{C_4^2C_0^4\delta_{h_{i0}}^2(\lambda_k)\lambda_k\frac{1}{\beta_l'}\frac{1}{\beta_l}\right.$$
$$\left.+C_4C_0^3C_1\delta_{h_{i0}}^2(\lambda_k)\frac{1}{\beta_l'}\frac{1}{\beta_l}+C_4C_0^2\delta_{h_{i0}}^2(\lambda_k)\right\}\sigma(h_l)$$
$$\cdot\delta_{h_l}(\lambda_k)\leq\frac{C_0}{4}\left\{C_4^2C_0^4\delta_H^2(\lambda_k)\lambda_k\frac{1}{\beta_0'}\frac{1}{\beta_0}\right.$$
$$\left.+C_4C_0^3C_1\delta_H^2(\lambda_k)\frac{1}{\beta_0'}\frac{1}{\beta_0}+C_4C_0^2\delta_H^2(\lambda_k)\right\}\sigma(h_l)$$
$$\cdot\delta_{h_l}(\lambda_k), \tag{74}$$

which together with (73) yields (72) immediately. □

We give the following lemma by referring to [12] (see Lemma 3.4 in [12]).

Lemma 14. *Suppose that the conditions of Theorem 10 are satisfied; then*

$$\left|\eta_T(u^*)-\eta_T\left(u_k^{h_l}\right)\right| \leq C_9\left\{\delta_{h_l}^2(\lambda_k)\|u^*\|_{1,T}+\left\|u_k^{h_l}-u^*\right\|_{1,T}\right\}, \tag{75}$$

$$\left|\eta_\Omega(u^*)-\eta_\Omega\left(u_k^{h_l}\right)\right| \leq C_{10}\left\{\delta_{h_l}^2(\lambda_k)\|u^*\|_a+\left\|u_k^{h_l}-u^*\right\|_a\right\}. \tag{76}$$

In the following discussion, combining Lemmas 11, 12, and 14 and Theorem 13, we give the global upper bound and the local lower bound of the error.

Theorem 15. *Suppose that the conditions of Theorem 10 are satisfied; then there exists $u_k\in M(\lambda_k)$ such that*

$$\left\|u_k-u_k^{h_l}\right\|_a\leq C_6\eta_\Omega\left(u_k^{h_l}\right)+R_1, \tag{77}$$

where $R_1 = C_6C_{10}\delta_{h_l}^2(\lambda_k)\|u^*\|_a+(C_6+C_0/4+C_6C_{10}(C_0/4))\sigma(h_l)\delta_{h_l}(\lambda_k)$.

Proof. Select $u_k\in M(\lambda_k)$ which is given by (41); then from Lemma 11, Theorem 13, and (76) we get

$$\begin{aligned}\left\|u_k-u_k^{h_l}\right\|_a &\leq \|u_k-u^*\|_a+\left\|u^*-u_k^{h_l}\right\|_a\leq C_6\left\{\eta_\Omega(u^*)\right.\\ &\left.+\sigma(h_l)\delta_{h_l}(\lambda_k)\right\}+\frac{C_0}{4}\sigma(h_l)\delta_{h_l}(\lambda_k)\\ &\leq C_6\left\{\eta_\Omega\left(u_k^{h_l}\right)\right.\\ &\left.+C_{10}\left(\delta_{h_l}^2(\lambda_k)\|u^*\|_a+\left\|u_k^{h_l}-u^*\right\|_a\right)\right.\\ &\left.+\sigma(h_l)\delta_{h_l}(\lambda_k)\right\}+\frac{C_0}{4}\sigma(h_l)\delta_{h_l}(\lambda_k)\\ &\leq C_6\eta_\Omega\left(u_k^{h_l}\right)+R_1.\end{aligned} \tag{78}$$

The proof is completed. □

It is obvious that R_1 is a higher order term. Hence, we obtain that $\eta_\Omega(u_k^{h_l})$ is a global reliable error indicator of $\|u_k-u_k^{h_l}\|_a$.

Theorem 16. *Under the conditions of Theorem 10, there exists $u_k\in M(\lambda_k)$ such that the following hold:*

(a) For $T\in\pi_{h_l}$, if $\partial T\cap\Gamma=\emptyset$, then

$$\eta_T\left(u_k^{h_l}\right)\leq C_7\left\|u_k-u_k^{h_l}\right\|_{1,T^*}+R_2, \tag{79}$$

where $R_2=(C_7+C_9)\|u_k^{h_l}-u^*\|_{1,T^*}+C_9\delta_{h_l}^2(\lambda_k)\|u^*\|_{1,T}$.

(b) For $T\in\pi_{h_l}$, if $\partial T\cap\Gamma\neq\emptyset$, then

$$\eta_T\left(u_k^{h_l}\right)\leq C_8\left\|u_k-u_k^{h_l}\right\|_{1,T}+R_3, \tag{80}$$

where

$$\begin{aligned}R_3 &= (C_8+C_9)\left\|u_k^{h_l}-u^*\right\|_{1,T}+C_9\delta_{h_l}^2(\lambda_k)\|u^*\|_{1,T}\\ &+C_8\sum_{\ell\in\mathscr{E}_T\cap\mathscr{E}_\Gamma}|\ell|^{1/2}\left\|\lambda_ku_k-\widehat{\lambda}_{k,h_l}u^*\right\|_{0,\ell}.\end{aligned} \tag{81}$$

Proof. We can prove the desired results by using the proof method of Theorem 3.4 in [12]. □

According to Remark 3.1 in [4] and Remark 3.2 in [12] we know that the term $\sum_{\ell\in\mathscr{E}_T\cap\mathscr{E}_\Gamma}|\ell|^{1/2}\|\lambda_ku_k-\widehat{\lambda}_{k,h_l}u^*\|_{0,\ell}$ is a higher order term. From Theorem 13, we know that $\|u_k^{h_l}-u^*\|_{1,T^*}$ and $\|u_k^{h_l}-u^*\|_{1,T}$ are also higher order terms. And it is obvious that $\delta_{h_l}^2(\lambda_k)\|u^*\|_{1,T}$ is a higher order term. Therefore, from (79) and (80) we know that $\eta_T(u_k^{h_l})$ is an efficient local error indicator of $\|u_k-u_k^{h_l}\|_{1,T^*}$ and $\|u_k-u_k^{h_l}\|_{1,T}$.

In the following theorem, we give the estimate for approximate eigenvalue.

Theorem 17. *Suppose that the conditions of Theorem 10 are satisfied; then*

$$\left|\lambda_k^{h_l}-\lambda_k\right|=\mathcal{O}\left(\eta_\Omega^2\left(u_k^{h_l}\right)\right). \tag{82}$$

Proof. From Theorem 10 and Lemma 1 it is easy to prove that

$$\lambda_k^{h_l}-\lambda_k=\mathcal{O}\left(\left\|u_k^{h_l}-u_k\right\|_a^2\right); \tag{83}$$

then combining with Theorem 16 and (77), we can get the desired result (82). □

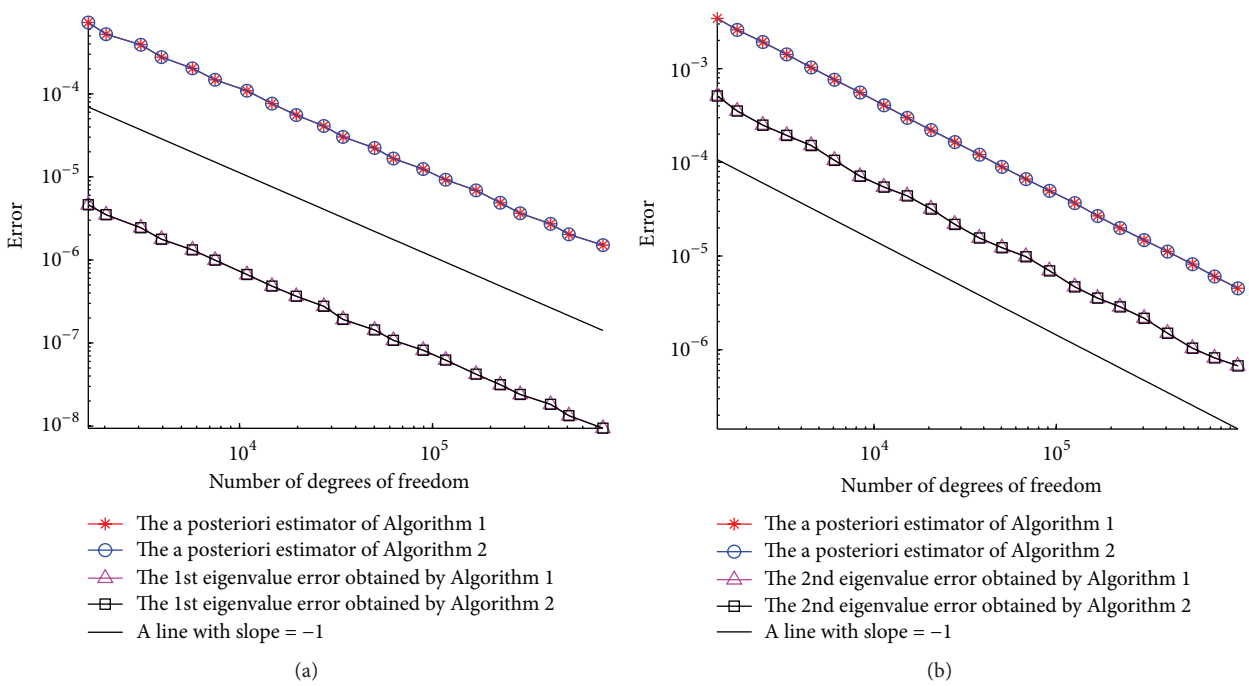
5. Numerical Experiments

In this section we first give an adaptive algorithm of the Rayleigh quotient iteration type and establish an adaptive algorithm of fixed-shift inverse iteration type for the Steklov eigenvalue problem.

The following Algorithm 1 of the Rayleigh quotient iteration type refers to Algorithm 4.3 in [12] or Algorithm 6.1 in [16].

Table 1: The 1st and the 2nd eigenvalues of Example 1 obtained by Algorithms 1 and 2 with $H = \sqrt{2}/32$.

k	l	$N_{k,l}(1)$	$\lambda_k^{h_l}(1)$	$\mathrm{CPU}_{k,l}^{(1)}$	$N_{k,l}(2)$	$\lambda_k^{h_l}(2)$	$\mathrm{CPU}_{k,l}^{(2)}$
1	5	5700	0.24008040	0.54	5700	0.24008040	0.57
1	10	27344	0.24007936	1.60	27343	0.24007936	1.66
1	15	117001	0.24007915	7.67	117001	0.24007915	7.82
1	19	408971	0.24007910	29.06	408971	0.24007910	29.25
1	20	509032	0.24007910	39.88	509032	0.24007910	40.07
1	21	764069	0.24007909	55.90	764069	0.24007909	55.96
2	5	4526	1.49245505	0.51	4529	1.49245448	0.54
2	12	37855	1.49231870	2.31	37941	1.49231871	2.33
2	18	223337	1.49230601	15.77	224277	1.49230600	15.89
2	24	1329617	1.49230362	109.68	1334272	1.49230362	109.90
2	25	1805637	1.49230347	151.36	1812726	1.49230347	151.02
2	26	2439573	1.49230337	211.29	2447212	1.49230337	210.12



(a) (b)

Figure 1: The curves of error and the a posteriori error estimators of two algorithms for the 1st (a) and 2nd (b) eigenvalues on the square domain.

Algorithm 1. Choose parameter $0 < \omega < 1$.

Step 1. Pick any initial mesh π_{h_0}.

Step 2. Solve (2) on π_{h_0} for discrete solution (λ^{h_0}, u^{h_0}).

Step 3. Let $l \Leftarrow 0$, $\lambda_0 \Leftarrow \lambda^{h_0}$.

Step 4. Compute the local indicators $\eta_T(u^{h_l})$.

Step 5. Construct $\widehat{\pi}_{h_l} \subset \pi_{h_l}$ by *Marking Strategy E* and ω.

Step 6. Refine π_{h_l} to get a new mesh $\pi_{h_{l+1}}$ by Procedure REFINE.

Step 7. Find $u' \in V_{h_{l+1}}$ such that

$$a(u', v) - \lambda_0 b(u', v) = b(u^{h_l}, v), \quad \forall v \in V_{h_{l+1}}; \tag{84}$$

denote $u^{h_{l+1}} = u'/\|u'\|_a$ and compute the Rayleigh quotient

$$\lambda^{h_{l+1}} = \frac{a(u^{h_{l+1}}, u^{h_{l+1}})}{b(u^{h_{l+1}}, u^{h_{l+1}})}. \tag{85}$$

Step 8. Let $\lambda_0 \Leftarrow \lambda^{h_{l+1}}$, $l \Leftarrow l + 1$ and go to Step 4.

Marking Strategy E. Give parameter $0 < \omega < 1$.

Table 2: The 1st and the 3rd eigenvalues of Example 2 obtained by two algorithms with $H = \sqrt{2}/32$.

k	l	$N_{k,l}(1)$	$\lambda_k^{h_l}(1)$	$\mathrm{CPU}_{k,l}^{(1)}$	$N_{k,l}(2)$	$\lambda_k^{h_l}(2)$	$\mathrm{CPU}_{k,l}^{(2)}$
1	5	4369	0.18296573	0.46	4369	0.18296573	0.45
1	12	35119	0.18296446	2.00	35119	0.18296446	1.97
1	18	227719	0.18296426	14.90	227719	0.18296426	14.80
1	21	557370	0.18296425	39.77	557370	0.18296425	40.10
1	22	738108	0.18296425	55.04	738108	0.18296425	55.43
1	23	1105016	0.18296424	78.27	1105016	0.18296424	78.83
3	5	3690	1.68889444	0.49	3690	1.68889444	0.52
3	12	31089	1.68863207	1.79	31089	1.68863207	1.84
3	18	183089	1.68860633	11.84	183089	1.68860633	12.08
3	24	1098418	1.68860134	81.42	1098418	1.68860134	83.23
3	25	1485695	1.68860109	112.52	1485695	1.68860109	114.26
3	26	2002315	1.68860097	155.64	2002315	1.68860097	156.76

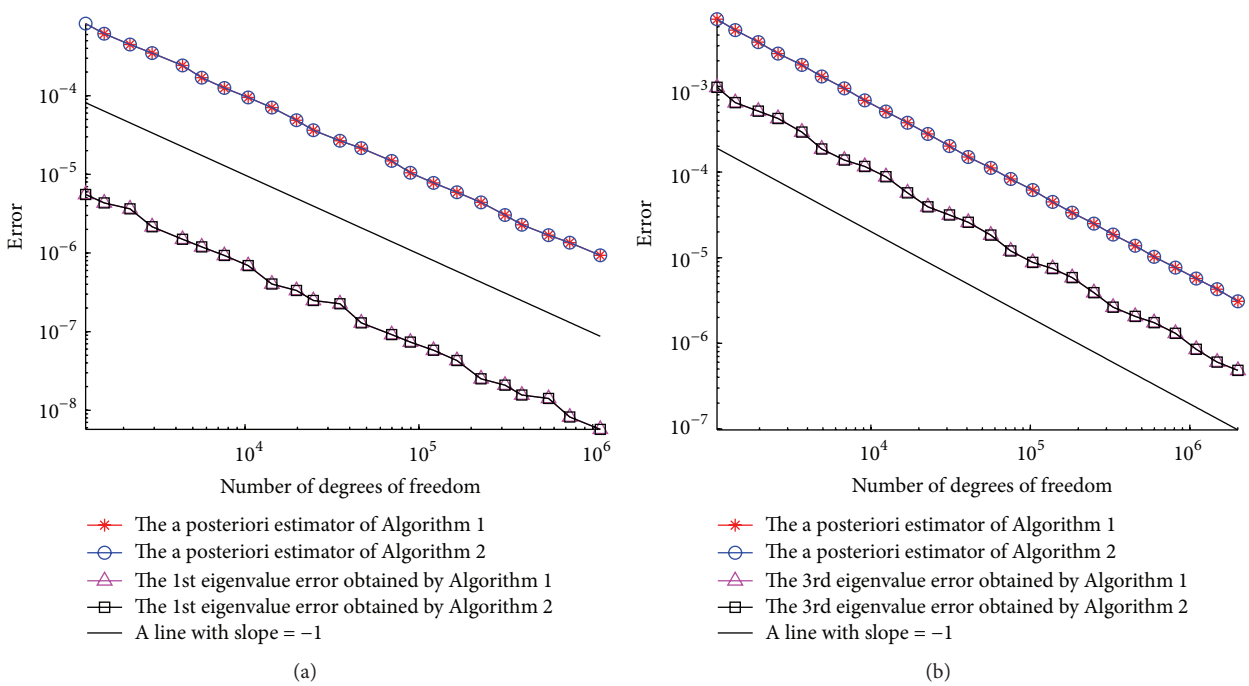


Figure 2: The curves of error and the a posteriori error estimators of two algorithms for the 1st (a) and 3rd (b) eigenvalues on the L-shaped domain.

Step 1. Construct a minimal subset $\widehat{\pi}_{h_l}$ of π_{h_l} by selecting some elements in π_{h_l} such that

$$\sum_{T\in\widehat{\pi}_{h_l}} \eta_T^2\left(u^{h_l}\right) \geq \omega\eta_\Omega^2\left(u^{h_l}\right). \tag{86}$$

Step 2. Mark all the elements in $\widehat{\pi}_{h_l}$.

$\eta_T(u^{h_l})$ and $\eta_\Omega(u^{h_l})$ are defined by (67) and (68) with $u_k^{h_l}$ and $\lambda_k^{h_l}$ replaced by u^{h_l} and λ^{h_l}, respectively.

Note that when $|\lambda_0 - \lambda|$ is too small, (84) is an almost singular linear equation. Although it has no difficulty in solving (84) numerically (see [12]), one would like to think of selecting a proper integer $i0 \geq 0$ to establish the following adaptive algorithm.

Algorithm 2. Choose parameter $0 < \omega < 1$.

Steps 1–7. Execute Steps 1–7 of Algorithm 1.

Step 8. If $l < i_0$, $\lambda_0 \Leftarrow \lambda^{h_{l+1}}$, $l \Leftarrow l+1$, go to Step 4; else $l \Leftarrow l+1$, go to Step 4.

Marking Strategy E in Algorithm 2 is the same as that in Algorithm 1.

Now, we will implement some numerical experiments to validate our theoretical analysis and show the efficiency of Algorithm 2 with $i_0 = 0$. We use MATLAB 2012 together with the package of Chen [20] to solve Examples 1, 2, and 3, and we take $\omega = 0.5$.

Table 3: The 1st and 5th eigenvalues of Example 3 obtained by two algorithms with $H = \sqrt{2}/32$.

k	l	$N_{k,l}(1)$	$\lambda_k^{h_l}(1)$	$\mathrm{CPU}_{k,l}^{(1)}$	$N_{k,l}(2)$	$\lambda_k^{h_l}(2)$	$\mathrm{CPU}_{k,l}^{(2)}$
1	5	9391	0.23957586	0.66	9391	0.23957586	0.75
1	10	42645	0.23957397	2.45	42645	0.23957397	2.56
1	15	189550	0.23957352	12.52	189550	0.23957352	12.72
1	19	634556	0.23957342	48.00	634556	0.23957342	47.09
1	20	860490	0.23957341	66.32	860490	0.23957341	65.38
1	21	1131274	0.23957340	91.04	1131274	0.23957340	90.05
5	5	9842	1.41254843	0.70	9842	1.41254843	0.74
5	12	77005	1.41241115	4.53	77005	1.41241115	4.68
5	18	453543	1.41238410	31.53	453543	1.41238410	31.74
5	21	1092642	1.41238245	82.23	1092642	1.41238245	82.74
5	22	1497488	1.41238165	112.92	1497488	1.41238165	113.82
5	23	1993327	1.41238104	155.36	1993327	1.41238104	156.14

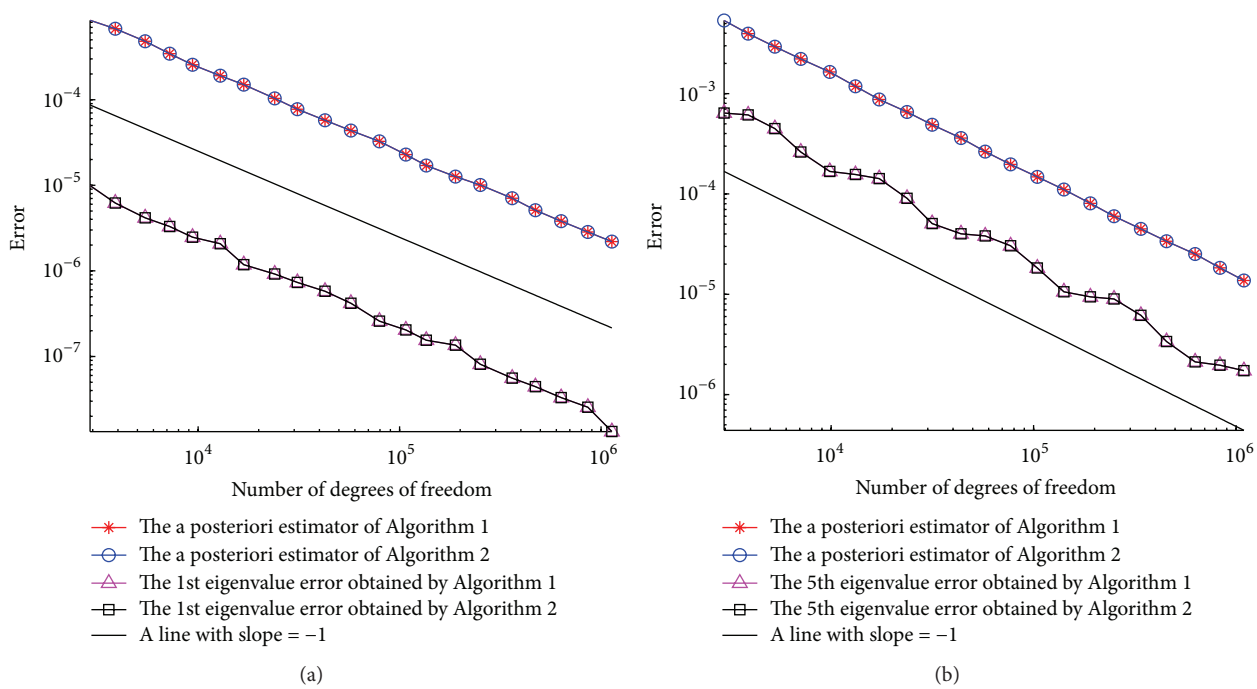


Figure 3: The curves of error and the a posteriori error estimators of two algorithms for the 1st (a) and 5th (b) eigenvalues on slit domain.

For reading conveniently, we use the following notations in our tables:

$\lambda_k^{h_l}(m)$: the kth eigenvalue derived from the lth iteration obtained by Algorithm m ($m = 1, 2$).

$|\lambda_k^{h_l}(m) - \lambda_k|$: the error of $\lambda_k^{h_l}(m)$ obtained by Algorithm m ($m = 1, 2$).

$N_{k,l}(m)$: the degrees of freedom of the lth iteration for $\lambda_k^{h_l}(m)$ ($m = 1, 2$).

$\mathrm{CPU}_{k,l}^{(m)}$(s): the CPU time(s) from the program starting to calculate result of the lth iteration appearing by using Algorithm m ($m = 1, 2$).

Example 1. We use Algorithms 1 and 2 to compute the approximations of the 1st and the 2nd eigenvalue of (1) with the triangle linear finite element on $\Omega = [0, 1] \times [0, 1]$. The numerical results are listed in Table 1.

Since the exact eigenvalues are unknown, we use $\lambda_1 \approx 0.24007908542$ and $\lambda_2 \approx 1.49230313453$ obtained by the spectral element method (see [21]) as the reference eigenvalues. We show the error curves and the a posteriori estimators obtained by two algorithms for λ_1 and λ_2 in Figure 1. It can be seen from Figure 1 that the error curves are approximately parallel to the line with slope -1, which indicates that Algorithm 2 achieves the optimal convergence rate of $\mathcal{O}(N_l^{-1})$ as well as Algorithm 1.

Observing the numerical results in Table 1, we can find that when the degrees of freedom are almost the same, the approximate eigenvalues obtained by Algorithm 2 are nearly as accurate as those obtained by Algorithm 1 and their CPU time are roughly the same.

Table 4: The results of Example 4 on $\Omega = [0,1] \times [0,1]$.

k	l	$N_{k,l}$	$\lambda_k^{h_l}(3)$	$\mathrm{CPU}_{k,l}^{(3)}$	$\lambda_k^{h_l}(2)$	$\mathrm{CPU}_{k,l}^{(2)}$
1	1	1089	0.240088481	0.08	0.240088481	0.18
1	2	4225	0.240081438	0.17	0.240081438	0.22
1	3	16641	0.240079674	0.48	0.240079674	0.43
1	4	66049	0.240079233	1.92	0.240079233	1.45
1	5	263169	0.240079122	8.92	0.240079122	6.48
1	6	1050625	0.240079095	38.36	0.240079095	32.00
2	1	1089	1.492905398	0.11	1.492905378	0.19
2	2	4225	1.492454269	0.23	1.492454267	0.24
2	3	16641	1.492340958	0.65	1.492340958	0.46
2	4	66049	1.492312593	2.67	1.492312593	1.47
2	5	263169	1.492305499	12.19	1.492305499	6.57
2	6	1050625	1.492303726	56.05	1.492303726	32.00

Table 5: The results of Example 4 on $\Omega = ([0,1] \times [0,1/2]) \cup ([0,1/2] \times [1/2,1])$.

k	l	$N_{k,l}$	$\lambda_k^{h_l}(3)$	$\mathrm{CPU}_{k,l}^{(3)}$	$\lambda_k^{h_l}(2)$	$\mathrm{CPU}_{k,l}^{(2)}$
1	1	833	0.182975157	0.04	0.182975157	0.19
1	2	3201	0.182966980	0.11	0.182966980	0.22
1	3	12545	0.182964924	0.31	0.182964924	0.36
1	4	49665	0.182964409	1.25	0.182964409	1.06
1	5	197633	0.182964280	5.88	0.182964280	4.60
1	6	788481	0.182964248	26.02	0.182964248	21.61
3	1	833	1.690165085	0.05	1.690165013	0.20
3	2	3201	1.688996545	0.12	1.688996536	0.22
3	3	12545	1.688700132	0.34	1.688700131	0.36
3	4	49665	1.688625481	1.27	1.688625481	1.04
3	5	197633	1.688606742	5.95	1.688606742	4.34
3	6	788481	1.688602046	26.11	1.688602046	20.20

Example 2. We use Algorithms 1 and 2 to compute the approximations of the 1st and the 3rd eigenvalue of (1) with the triangle linear finite element on $\Omega = ([0,1] \times [0,1/2]) \cup ([0,1/2] \times [1/2,1])$. The numerical results are presented in Table 2.

In Figure 2 we depict the error curves and the a posteriori estimators obtained by two algorithms for λ_1 and λ_3. Here we use $\lambda_1 \approx 0.18296423687$ and $\lambda_3 \approx 1.68860048358$ obtained by the spectral element method (see [21]) as the reference eigenvalues. It can be seen from Figure 2 that the error curves are approximately parallel to the line with slope -1, which indicates that Algorithm 2 achieves the optimal convergence rate of $\mathcal{O}(N_l^{-1})$ as well as Algorithm 1.

It also can be seen from Table 2 that when the degrees of freedom are the same, one can use Algorithms 1 and 2 to get the same accurate approximations with nearly the same CPU time.

Example 3. We use Algorithms 1 and 2 to compute the approximations of the 1st and the 5th eigenvalue of (1) with the triangle linear finite element on $\Omega = \{(x_1,x_2) : |x_1| + |x_2| < 1\} \setminus \{(x_1,x_2) : 0 \le x_1 \le 1,\ x_2 = 0\}$. The numerical results are presented in Table 3.

Since the exact eigenvalues are unknown, we compute the approximations of two exact eigenvalues of (1): $\lambda_1 \approx 0.23957338768$ and $\lambda_5 \approx 1.41238071918$ by the standard adaptive algorithm (see, e.g., [22]) with the degrees of freedom of more than 5000000. We show the curves of the error and the a posteriori estimators obtained by two algorithms for λ_1 and λ_5 in Figure 3. We can see from Figure 3 that the error curves are approximately parallel to the line with slope -1, which indicates that Algorithm 2 achieves the optimal convergence rate of $\mathcal{O}(N_l^{-1})$ as well as Algorithm 1.

From the numerical results in Table 3, we can conclude that Algorithm 2 is also an efficient approach like Algorithm 1 for solving the Steklov eigenvalue problem.

Example 4. We use the method in [10] (see Algorithms 4.1 and 7.2 there) to compute the numerical eigenvalues of (1) on $[0,1] \times [0,1]$, $([0,1] \times [0,1/2]) \bigcup ([0,1/2] \times [1/2,1])$, and $\{(x_1,x_2) : |x_1| + |x_2| < 1\} \setminus \{(x_1,x_2) : 0 \le x_1 \le 1,\ x_2 = 0\}$, respectively, and list the associated results in Tables 4–6 which are denoted by $\lambda_k^{h_l}(3)$ and $\mathrm{CPU}_{k,l}^{(3)}$.

Table 6: The results of Example 4 on $\Omega = \{(x_1, x_2) : |x_1| + |x_2| < 1\} \backslash \{(x_1, x_2) : 0 \le x_1 \le 1,\ x_2 = 0\}$.

k	l	$N_{k,l}$	$\lambda_k^{h_l}(3)$	$\mathrm{CPU}_{k,l}^{(3)}$	$\lambda_k^{h_l}(2)$	$\mathrm{CPU}_{k,l}^{(2)}$
1	1	2145	0.239589697	0.12	0.239589697	0.22
1	2	8385	0.239577621	0.28	0.239577621	0.31
1	3	33153	0.239574482	0.96	0.239574482	0.77
1	4	131841	0.239573668	4.03	0.239573668	2.99
1	5	525825	0.239573457	17.43	0.239573457	13.80
1	6	2100225	0.239573403	78.31	0.239573403	71.35
5	1	2145	1.413086665	0.12	1.413086553	0.20
5	2	8385	1.412557485	0.28	1.412557472	0.29
5	3	33153	1.412424609	0.91	1.412424608	0.68
5	4	131841	1.412391332	3.92	1.412391332	2.77
5	5	525825	1.412383004	17.35	1.412383004	12.79
5	6	2100225	1.412380921	79.20	1.412380921	64.99

From Tables 4–6 we can see that, with the same degrees of freedom $N_{k,l}$, our method uses less CPU time to obtain the same accurate approximations, especially for multiple eigenvalue λ_2 on $[0, 1] \times [0, 1]$, comparing with the one in [10].

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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