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Research Article

New Periodic Solutions for a Class of Zakharov Equations

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Through applying the Jacobian elliptic function method, we obtain the periodic solution for a series of nonlinear Zakharov equations, which contain Klein-Gordon Zakharov equations, Zakharov equations, and Zakharov-Rubenchik equations.

1. Introduction

For most of nonlinear evolution equations, we have many methods to obtain their exactly solutions, such as hyperbolic function expansion method [1], the transformation method [2], the trial function method [3], the automated method [4], and the extended tanh-function method [5]. But these methods can only obtain solitary wave solution and cannot be used to deduce periodic solutions. The Jacobian function method provides a way to find periodic solutions for some nonlinear evolution equations. In particular, in the research of plasma physics theory, quantum mechanics, fluid mechanics, and optical fiber communication, we frequently meet kinds of Zakharov equations.

Hence, in this paper, inspired by Angulo Pava's work [6], we are concerned to obtain exact periodic solutions of a series of Zakharov equations,

$$\begin{gathered} u_{tt} - u_{xx} + u + nu = 0, \\ n_{tt} - n_{xx} = \left(|u|^2\right)_{xx}, \\ u_{tt} - u_{xx} - |v|_{xx} = 0, \\ iv_t + \alpha v_{xx} - uv = 0, \end{gathered} \tag{1}$$

and Zakharov-Rubenchik equations,

$$\begin{gathered} iB_t + \omega B_{xx} - k\left(u - \frac{1}{2}\lambda\rho + q|B|^2 B\right) = 0, \\ \theta\rho_t + (u - \lambda\rho)_x = -k|B|_x^2, \\ \theta u_t + (\beta\rho - \lambda u)_x = \frac{1}{2}k\lambda|B|_x^2. \end{gathered} \tag{2}$$

2. The Periodic Solution for Klein-Gordon Zakharov Equations

The Klein-Gordon Zakharov equations are used to show the interaction between langmuir wave and ion wave in the plasma, which has the following form:

$$\begin{gathered} u_{tt} - u_{xx} + u + nu = 0, \\ n_{tt} - n_{xx} = \left(|u|^2\right)_{xx}, \end{gathered} \tag{3}$$

where $u(x,t)$ denotes the biggest moment scale component produced by electron in electric field. $n(x,t)$ denotes the speed of deviations between the ions at any position and that at equilibrium position.

Now, we suppose that it possesses solitary wave solutions of the following form:

$$\begin{gathered} u(x,t) = e^{i\theta}\varphi(\xi), \\ n(x,t) = n(\xi), \\ \xi = x - ct, \end{gathered} \tag{4}$$

where c is a traveling wave speed and $c^2 < 1$ is a constant.

By substituting (4) into (3), we can obtain

$$c^2\varphi'' - \varphi'' + \varphi + \varphi n = 0, \tag{5}$$

$$\left(c^2 - 1\right)n'' = \left(\varphi^2\right)''. \tag{6}$$

By (6), we have

$$\left(c^2-1\right)n-\varphi^2=c_1\xi+m, \tag{7}$$

where c_1, m are integration constants.

In what follows, we are concerned with the periodic solution of (3); thus we need to require $c_1=0$.

Therefore, (7) implies

$$n=\frac{\varphi^2+m}{c^2-1}. \tag{8}$$

Moreover, through (8) and (5), we have that

$$\left(c^2-1\right)\varphi''+\varphi+\frac{\varphi^3+\varphi m}{c^2-1}=0. \tag{9}$$

Multiplying (9) by φ' and integrating once, we obtain

$$\frac{c^2-1}{2}\left(\varphi'\right)^2+\frac{\varphi^2}{2}+\frac{\varphi^4+2m\varphi^2}{c^2-1}\frac{1}{4}=h, \tag{10}$$

where h is a nonzero integration constant. Furthermore, h, m, and c satisfy the following condition:

$$\begin{gathered}\left(1-c^2-m\right)^2-4h\left(1-c^2\right)\geq 0,\\ \left(1-c^2-m\right)-\sqrt{\left(1-c^2-m\right)^2-4\left(1-c^2\right)h}\geq 0,\\ 0<h<\frac{\left(1-c^2-m\right)^2}{4\left(1-c^2\right)},\end{gathered} \tag{11}$$

so that

$$\left(\varphi'\right)^2=\frac{1}{2}\frac{1}{\left(c^2-1\right)^2}\left(\eta_1^2-\varphi^2\right)\left(\varphi^2-\eta_2^2\right), \tag{12}$$

where $-\eta_1$, η_1, $-\eta_2$, and η_2 are the zeros of the polynomial $F(\lambda)=-((c^2-1+m)/(c^2-1)^2)\lambda^2-(1/2)(\lambda^4/(c^2-1)^2)+2h/(c^2-1)$. Without losing generality, we assume that $\eta_1>\eta_2>0$. Therefore, we can deduce that $\eta_1\leq\varphi\leq\eta_2$; η_1 and η_2 satisfy

$$\begin{gathered}\eta_1^2+\eta_2^2=2\left(1-c^2\right)-2m,\\ \eta_1^2\eta_2^2=4h\left(1-c^2\right).\end{gathered} \tag{13}$$

Let $\phi=\varphi/\eta_1$, $k^2=(\eta_1^2-\eta_2^2)/\eta_1^2$. Hence, (12) can be written as

$$\left(\phi'\right)^2=\frac{1}{2}\frac{1}{\left(c^2-1\right)^2}\eta_1^2\left(1-\phi^2\right)\left(\phi^2-1+k^2\right). \tag{14}$$

Moreover, we define a new variable ψ, which satisfies $\psi'\geq 0$, $\psi(0)=0$, by the relation $\phi^2=1-k^2\sin^2\psi$; through a tedious computation, we obtain that

$$\left(\psi'\right)^2=\frac{1}{2}\frac{1}{\left(c^2-1\right)^2}\eta_1^2\left(1-k^2\sin^2\psi\right). \tag{15}$$

Then we obtain

$$\int_0^{\psi(\xi)}\frac{d\tau}{\sqrt{1-k^2\sin^2\tau}}=\frac{\sqrt{2}}{2}\frac{1}{1-c^2}\eta_1\xi. \tag{16}$$

According to the definition of the Jacobian elliptic function $y=\operatorname{sn}(u;k)$, we can obtain that $\sin\psi=\operatorname{sn}(\eta_1\sqrt{\beta}\xi,k)$. Here, $\beta=\sqrt{2}/2(1-c^2)$. So

$$\begin{aligned}\phi(\xi)&=\sqrt{1-k^2\operatorname{sn}^2\left(\eta_1\frac{\sqrt{2}}{2}\frac{1}{1-c^2}\xi,k\right)}\\ &=\operatorname{dn}\left(\eta_1\frac{\sqrt{2}}{2}\frac{1}{1-c^2}\xi,k\right).\end{aligned} \tag{17}$$

By returning to initial variable, we obtain that

$$\varphi(\xi)=\eta_1\operatorname{dn}\left(\eta_1\frac{\sqrt{2}}{2}\frac{1}{1-c^2}\xi,k\right) \tag{18}$$

is a dnoidal solution of (10).

Furthermore, dn has fundamental period $2K(k)$; that is, $\operatorname{dn}(u+2k;k)=\operatorname{dn}(u;k)$, and $K(k)$ is the complete elliptic integral of first kind. So the dnoidal wave solution φ has fundamental period, T, given by

$$T=\frac{2\sqrt{2}K(k)\left(1-c^2\right)}{\eta_1}. \tag{19}$$

So, by applying the method of the Jacobian elliptic function and inspired by Angulo Pava's ideas, we obtain that (3) has the periodic traveling solution of the following form:

$$\begin{aligned}&u(x,t)=e^{i\theta}\varphi(\xi)=e^{i\theta}\eta_1\operatorname{dn}\left(\eta_1\frac{\sqrt{2}}{2}\frac{1}{1-c^2}\xi,k\right),\\ &n(x,t)\\ &\quad=\frac{\left(\eta_1\operatorname{dn}\left(\eta_1\left(\sqrt{2}/2\right)\left(1/\left(1-c^2\right)\right)\xi,k\right)\right)^2+m}{c^2-1}.\end{aligned} \tag{20}$$

Moreover, T and k^2 can be also rewritten as the following form:

$$\begin{aligned}T\left(\eta_2\right)&=\frac{2\sqrt{2}K\left(k\left(\eta_2\right)\right)\left(1-c^2\right)}{\sqrt{2\left(1-c^2\right)-2m-\eta_2^2}},\\ k^2&=\frac{2\left(1-c^2\right)-2m-2\eta_2^2}{2\left(1-c^2\right)-2m-\eta_2^2}.\end{aligned} \tag{21}$$

Furthermore, if $\eta_2\rightarrow 0$ then $k(\eta_2)\rightarrow 1$. Hence $K(k(\eta_2))\rightarrow+\infty$, so that $T(\eta_2)\rightarrow+\infty$. If $\eta_2\rightarrow\sqrt{1-c^2-m}$, then $k(\eta_2)\rightarrow 0$. Hence $K(k(\eta_2))\rightarrow\pi/2$, so that $T(\eta_2)\rightarrow\sqrt{2}\pi(1-c^2)/\sqrt{1-c^2-m}$.

Next we will show that, for an arbitrary but fixed L, $\sqrt{1-(L^2+L\sqrt{L^2-8\pi m})/4\pi}<|c_0|<\sqrt{1+(L^2+L\sqrt{L^2-8\pi m})/4\pi}$. We can obtain that there is a unique $\eta_{2,0} = \eta_{2,0}(c_0) \in (0, \sqrt{1-c_0^2-m})$ such that $T(\eta_2) = T(\eta_2(c)) = L$ is a fundamental period for the dnoidal wave solution (18).

Theorem 1. *Let $L > 0$ be arbitrary but fixed. Consider $\sqrt{1-(L^2+L\sqrt{L^2-8\pi m})/4\pi} < |c_0| < \sqrt{1+(L^2+L\sqrt{L^2-8\pi m})/4\pi}$ and unique $\eta_{2,0} = \eta_{2,0}(c_0) \in (0, \sqrt{1-c_0^2-m})$, such that $T(\eta_2) = L$. Then, there exist an interval $U(c_0)$, an interval $I(\eta_{2,0})$, and a unique smooth function $F : U(c_0) \to I(\eta_{2,0})$ such that $F(c) = \eta_2$ and*

$$L = \frac{2\sqrt{2}K(k(\eta_2))(1-c^2)}{\sqrt{2(1-c^2)-2m-\eta_2^2}}, \tag{22}$$

where $c \in U(c_0)$, $\eta_2 \in U(\eta_{2,0})$.

Proof. Based on the ideas establish in Angulo Pava's work [6], we will give a brief proof. Now, we consider the open set

$$\begin{aligned}\Omega = \Bigg\{ &(\eta_2, c) : \eta_2 \in \left(0, \sqrt{1-c_0^2-m}\right), \ c \\ &\in \left(-\sqrt{1+\frac{L^2+L\sqrt{L^2-8\pi m}}{4\pi}},\right. \\ &\left.\sqrt{1+\frac{L^2+L\sqrt{L^2-8\pi m}}{4\pi}}\right) \cup \left(-\infty,\right. \\ &\left.-\sqrt{1-\frac{L^2+L\sqrt{L^2-8\pi m}}{4\pi}}\right) \\ &\cup \left(\sqrt{1-\frac{L^2+L\sqrt{L^2-8\pi m}}{4\pi}}, +\infty\right)\Bigg\} \in R^2.\end{aligned} \tag{23}$$

We define $\Phi : \Omega \to R$ by

$$\Phi(\eta_2, c) = \frac{2\sqrt{2}K(k(\eta_2))(1-c^2)}{\sqrt{2(1-c^2)-2m-\eta_2^2}}. \tag{24}$$

Here, $k^2 = (2(1-c^2)-2m-2\eta_2^2)/(2(1-c^2)-2m-\eta_2^2)$. Hypothesise $\Phi(\eta_{2,0}, c_0) = L$. In what follows, we will prove that $\partial\Phi/\partial\eta_2 < 0$.

From (24), we can obtain that

$$\begin{aligned}\frac{\partial\Phi}{\partial\eta_2} &= \frac{2\sqrt{2}\eta_2 K(k(\eta_2))(1-c^2)}{(2(1-c^2)-2m-\eta_2^2)^{3/2}} \\ &\quad + \frac{2\sqrt{2}(1-c^2)}{\sqrt{2(1-c^2)-2m-\eta_2^2}} \frac{dK}{dk}\frac{\partial k}{\partial\eta_2}.\end{aligned} \tag{25}$$

From k^2, we can deduce $k(\eta_2, c)$ is a strictly decreasing function of η_2.

According to Jacobian elliptic function theory [7], we have

$$\frac{dK}{dk} = \frac{E-(1-k^2)K}{k(1-k^2)}, \tag{26}$$

$$\frac{dE}{dk} = \frac{E-K}{k}. \tag{27}$$

Here, E is the complete elliptic integral of second kind.

Next, we adopt the reduction to absurdity to prove $\partial\Phi/\partial\eta_2 < 0$. Now, we assume that $\partial\Phi/\partial\eta_2 \geq 0$. So, we have the following inequality:

$$\begin{aligned}&k\left(2(1-c^2-m)-\eta_2^2\right)K(k(\eta_2)) \\ &\quad \geq 2(1-c^2-m)\frac{dK}{dk}.\end{aligned} \tag{28}$$

Indeed, substituting (26) into (25) and using the method of enlarging and reducing, we obtain that

$$\begin{aligned}&(1-k^2)\left[4(1-c^2-m)-2\eta_2^2\right]K(k) \\ &\quad \geq 2(1-c^2-m)E(k).\end{aligned} \tag{29}$$

From k^2, we can easily deduce that

$$\begin{aligned}1-k^2 &= \frac{\eta_2^2}{2(1-c^2)-2m-\eta_2^2}, \\ 2-k^2 &= \frac{2(1-c^2)-2m}{2(1-c^2)-2m-\eta_2^2}.\end{aligned} \tag{30}$$

Hence,

$$2(1-k^2)K(k) \geq (2-k^2)E(k). \tag{31}$$

Let $\gamma^2(\eta_2, c) = 1-k^2$, $\gamma' = -k(\partial k/\partial\eta_2)(1/\gamma) > 0$. So, γ is an increasing function of $\eta_2 \in (0, \sqrt{1-c^2-m})$. Moreover, $\gamma(0) = 1$, $\gamma(\sqrt{1-c^2-m}) = 1$.

Define

$$f(\gamma) = (1+\gamma^2)E\left(\sqrt{1-\gamma^2}\right) - 2\gamma^2 K\left(\sqrt{1-\gamma^2}\right). \tag{32}$$

Due to $\partial\Phi/\partial\eta_2 \geq 0$, $f(\gamma) \leq 0$.

However, by (27) and a simple computation,

$$f'(\gamma) = 3\gamma\left(E\left(\sqrt{1-\gamma^2}\right) - K\left(\sqrt{1-\gamma^2}\right)\right) < 0, \tag{33}$$

so $f(\gamma)$ is a decreasing function. Furthermore, $f(1) = 0$, so for $\gamma \in (0,1)$, $f(\gamma) > 0$.

It is in conflict with our assumption. So we obtain our affirmation that $\partial\Phi/\partial\eta_2 < 0$. Hence, by the implicit function theorem, there exists a unique smooth function F, defined in a neighborhood, $U(c_0)$ of c_0, so that $\Phi(\eta_2(c), c) = L$ for $c \in U(c_0)$. Hence, we obtain (22). □

3. The Periodic Solution for Zakharov Equations

Now, we consider the following Zakharov equations:

$$u_{tt} - u_{xx} - (|v|)^2_{xx} = 0,$$
$$iv_t - \alpha v_{xx} - uv = 0, \tag{34}$$

which describe the high frequency moment of plasma, where $u(x,t)$ denotes the ion number density variation, $v(x,t)$ denotes the electric field intensity of slowly varying amplitude, and $\alpha \in R$.

We seek the solitary solutions of (34) in the form

$$u = u(\xi),$$
$$v = \varphi(\xi) e^{i\xi}, \tag{35}$$
$$\xi = x - ct,$$

where u and φ are real functions, c is a traveling speed, and $c^2 \neq 1$.

Substituting (35) into (34), we can obtain that

$$\left(c^2 - 1\right) u'' - \left(\varphi^2\right)'' = 0, \tag{36}$$

$$-\alpha\varphi'' - (c + 2\alpha) i\varphi' + (c + \alpha)\varphi - u\varphi = 0. \tag{37}$$

By (36), we can deduce

$$\left(c^2 - 1\right) u - \varphi^2 = c_2\xi + r. \tag{38}$$

Here, c_2, r are integration constants.

Next, we consider periodic solutions of (34). So, we need to require $c_2 = 0$. Therefore, by (38)

$$u = \frac{\varphi^2 + r}{c^2 - 1}. \tag{39}$$

Furthermore, by (37), we can obtain that

$$c = -2\alpha. \tag{40}$$

So, by (39) and (40), (37) can be rewritten:

$$\varphi'' + \varphi\left(1 + \frac{2r}{c - c^3}\right) + \frac{2\varphi^3}{c - c^3} = 0. \tag{41}$$

Multiplying (41) by φ' and integrating once, we obtain

$$\frac{1}{2}\left(\varphi'\right)^2 + \frac{1}{2}\varphi^2\left(1 + \frac{2r}{c - c^3}\right) + \frac{1}{2}\varphi^4 = h, \tag{42}$$

where h is a nonzero integration constant.

Hence,

$$\left(\varphi'\right)^2 = \left(a^2 + \varphi^2\right)\left(b^2 - \varphi^2\right). \tag{43}$$

Here, $-ai$, ai, $-b$, and b are polynomial roots of $F(t) = -t^4 - t^2(a^2 - b^2) + 2h$, $b > 0$, and

$$a^2 - b^2 = 1 + \frac{2r}{c - c^3},$$
$$a^2 b^2 = 2h. \tag{44}$$

Let $\chi = \varphi/b$ (suppose $\chi(0) = 0$) and $k^2 = b^2/(a^2 + b^2)$. Hence, (43) becomes

$$\left(\chi'\right)^2 = b^2\left(1 - \chi^2\right)\left(\frac{a^2}{b^2} + \chi^2\right). \tag{45}$$

Now, we define $\chi^2 = 1 - \sin^2\psi$, so we get that

$$\left(\psi'\right)^2 = \left(a^2 + b^2\right)\left(1 - k^2\sin^2\psi\right). \tag{46}$$

Let $\tau = \sqrt{a^2 + b^2}$. According to the definition of the Jacobian elliptic function sn, we can obtain $\sin\psi(\xi) = \operatorname{sn}(\tau\xi, k)$. So that, $\chi^2 = 1 - \operatorname{sn}^2(\tau\xi, k)$.

So, we obtain the solution of (41):

$$\varphi(\xi) = b\operatorname{cn}(\tau\xi, k). \tag{47}$$

According to Jacobian elliptic functions theory, cn has period $4K(k)$, and we can obtain that the cnoidal wave solution has period T, which is given by

$$T = \frac{4K(k)}{\tau}. \tag{48}$$

So, we can obtain the periodic solutions of (34):

$$u(x,t) = u(\xi) = \frac{\left(b\operatorname{cn}(\tau\xi, k)\right)^2 + r}{c^2 - 1},$$
$$v(x,t) = v(\xi) = be^{i\xi}\operatorname{cn}(\tau\xi, k). \tag{49}$$

Furthermore, it follows that for $k^2 = b^2/(a^2 + b^2)$, we have $k^2 \in (0, 1/2)$.

If $\tau \to 0$, then $k \to 0$, and $K(k) \to \pi/2$, $T(\tau) \to +\infty$. Furthermore, if $\tau \to +\infty$, $T(\tau) \to 0$.

Next, we will prove that for an arbitrary but fixed $L > 0$, there exists a unique $\alpha = \alpha(c) \in (0, +\infty)$ such that $T(\alpha(c)) = L$ is a fundamental period of the cnoidal wave solution (47). So, we have the following theorem.

Theorem 2. *Let $L > 0$ but fixed. Consider $c_0 > 0$ and the unique $\tau_0 = \tau_0(c_0) \in (0, +\infty)$ such that $T(\tau_0) = L$. Then, there exists an internal $A(c_0)$ around c_0, an internal $B(b_0)$ around b_0, and a unique smooth function Υ such that $\Upsilon(c_0) = \alpha_0$ and*

$$L = \frac{4K(k)}{\tau}, \tag{50}$$

where $c \in A(c_0)$, $\tau = \Upsilon(c)$.

Proof. The proof is similar to that of Theorem 1. For details see Theorem 1 (see also Angulo Pava [6, 8, 9]). □

4. The Periodic Solution for Zakharov-Rubenchik Equations

Now, we consider the Zakharov-Rubenchik equations:

$$iB_t + \omega B_{xx} - \ell\left(u - \frac{1}{2}\lambda\rho + q|B|^2 B\right) = 0,$$
$$\theta\rho_t + \left(u - \lambda\rho\right)_x = -\ell|B|^2_x, \tag{51}$$
$$\theta u_t + \left(\beta\rho - \lambda u\right)_x = \frac{1}{2}\ell\lambda|B|^2_x,$$

which describe the dynamics of small amplitude Alfvén waves propagating in a plasma. Here, $B(x,t)$ denotes the magnetic field, $u(x,t)$ the fluid speed, and $\rho(x,t)$ the density of mass. Moreover, ω, ℓ, λ, θ, β, q are real constants.

Motivated by Oliveira [10], we look for the solitary waves of (51), as follows:

$$\begin{aligned} B(x,t) &= e^{i\xi} A(\xi), \\ u(x,t) &= aA^2(\xi), \\ \rho(x,t) &= bA^2(\xi), \\ \xi &= x - ct, \end{aligned} \tag{52}$$

where a, b, and A are real functions and c denotes traveling speed.

Putting (52) into (51), from the second and third equations of (51), we can deduce

$$\begin{aligned} a = a(c) &= \frac{\ell\left[(\lambda/2)(\lambda + c\theta) - \beta\right]}{\beta - (c\theta + \lambda)^2}, \\ b = b(c) &= \frac{\ell(-c\theta - \lambda/2)}{\beta - (c\theta + \lambda)^2}. \end{aligned} \tag{53}$$

Moreover, from the first equation of (51), we can obtain that

$$\begin{aligned} &(c - \omega) A + i(2\omega - c) A' + \omega A'' \\ &\quad - \ell\left(a - \frac{1}{2}\lambda b + q\right) A^3 = 0. \end{aligned} \tag{54}$$

So, by (54), it implies

$$c = 2\omega, \tag{55}$$

$$A'' + A - \frac{2\ell(a - (\lambda/2)b + q)}{c} A^3 = 0. \tag{56}$$

Let $b_1 = -\ell(a - (\lambda/2)b + q)/c > 0$ and multiplying (56) by A' and integrating once, we obtain

$$\left[A'\right]^2 = b_1\left(A^2 + a_1^2\right)\left(b_2^2 - A^2\right), \tag{57}$$

where $-a_1 i$, $a_1 i$, $-b_2$, b_2 are the polynomial roots of $F(r) = -b_1 r^4 - r^2 + h$. Moreover,

$$\begin{aligned} b_2^2 - a_1^2 &= -\frac{1}{b_1}, \\ b_2^2 a_1^2 &= \frac{h}{b_1}. \end{aligned} \tag{58}$$

Let $\chi = A/b_2$ (suppose $\chi(0) = 0$) and $k^2 = b_2^2/(a_1^2 + b_2^2)$. Hence, (57) becomes

$$\left(\chi'\right)^2 = b_1 b_2^2\left(1 - \chi^2\right)\left(\frac{a_1^2}{b_2^2} + \chi^2\right). \tag{59}$$

Now, we define $\chi^2 = 1 - \sin^2\psi$, so we get that

$$\left(\psi'\right)^2 = b_1\left(a_1^2 + b_2^2\right)\left(1 - k^2\sin^2\psi\right). \tag{60}$$

Let $\alpha = \sqrt{b_1(a_1^2 + b_2^2)}$. According to the definition of the Jacobin elliptic function sn, we can obtain $\sin\psi(\xi) = \operatorname{sn}(\alpha\xi, k)$. So, $\chi^2 = 1 - \operatorname{sn}^2(\alpha\xi, k)$.

So, we obtain the solution of (56):

$$A(\xi) = b_2 \operatorname{cn}(\alpha\xi, k). \tag{61}$$

Since cn has fundamental period $4K(k)$, we can obtain that solution (61) has fundamental period, T, which is

$$T = \frac{4K(k)}{\sqrt{b_1\left(a_1^2 + b_2^2\right)}}. \tag{62}$$

Hence, (51) have the periodic solutions of the following form:

$$\begin{aligned} B(\xi) &= e^{i\xi} b_2 \operatorname{cn}(\alpha\xi, k), \\ u(\xi) &= a b_2^2 \operatorname{cn}^2(\alpha\xi, k), \\ \rho(\xi) &= b b_2^2 \operatorname{cn}^2(\alpha\xi, k). \end{aligned} \tag{63}$$

Moreover, from (58) and the definition of k^2, it follows that

$$\begin{aligned} k^2 &= \frac{a_1^2 - 1/b_1}{2a_1^2 - 1/b_1}, \\ T(a_1) &= \frac{4K(k)}{\sqrt{2a_1^2 b_1 - 1}}. \end{aligned} \tag{64}$$

Moreover, if $a_1 \to 0$, $k^2 \to 1$, $T(a_1) \to +\infty$. If $a_1 \to +\infty$, $k^2 \to 1/2$, $T(a_1) \to 0$.

Next, we will show that for an arbitrary but fixed $L > 0$, there exists $a_1 = a_1(c) \in (0, +\infty)$ such that $T(a_1(c)) = L$ is a fundamental period of the cnoidal wave solution (61). Motivated by Angulo Pava's result [6], we have the following theorem.

Theorem 3. *For $L > 0$ arbitrary and fixed, consider $c_0 > 0$ and the unique $a_1 = a_1(c_0) \in (0, +\infty)$. Then, there exist an interval $C(c_0)$ around c_0, an interval $D(a_1)$ around a_1, and a unique smooth function $H : C(c_0) \to D(a_1)$ such that $H(c_0) = a_1$ and*

$$L = \frac{4K(k)}{\sqrt{2a_1^2 b_1 - 1}}, \tag{65}$$

where $c \in C(c_0)$, $a_1 = H(c)$.

Proof. The idea and method are similar to those of Theorem 1. For details, please see Theorem 1 (see also Angulo Pava [6, 8, 9]). □

5. Conclusion

Inspired by Angulo Pava's idea, by applying Jacobian elliptic function method, we have obtained new period wave solution for Klein-Gordon Zakharov equations, Zakharov equations, and Zakharov-Rubenchik equations. In particular, the solutions of (18), (47), and (58) were not found in the previous work. The method can help to look for periodic solution for a class of nonlinear equations.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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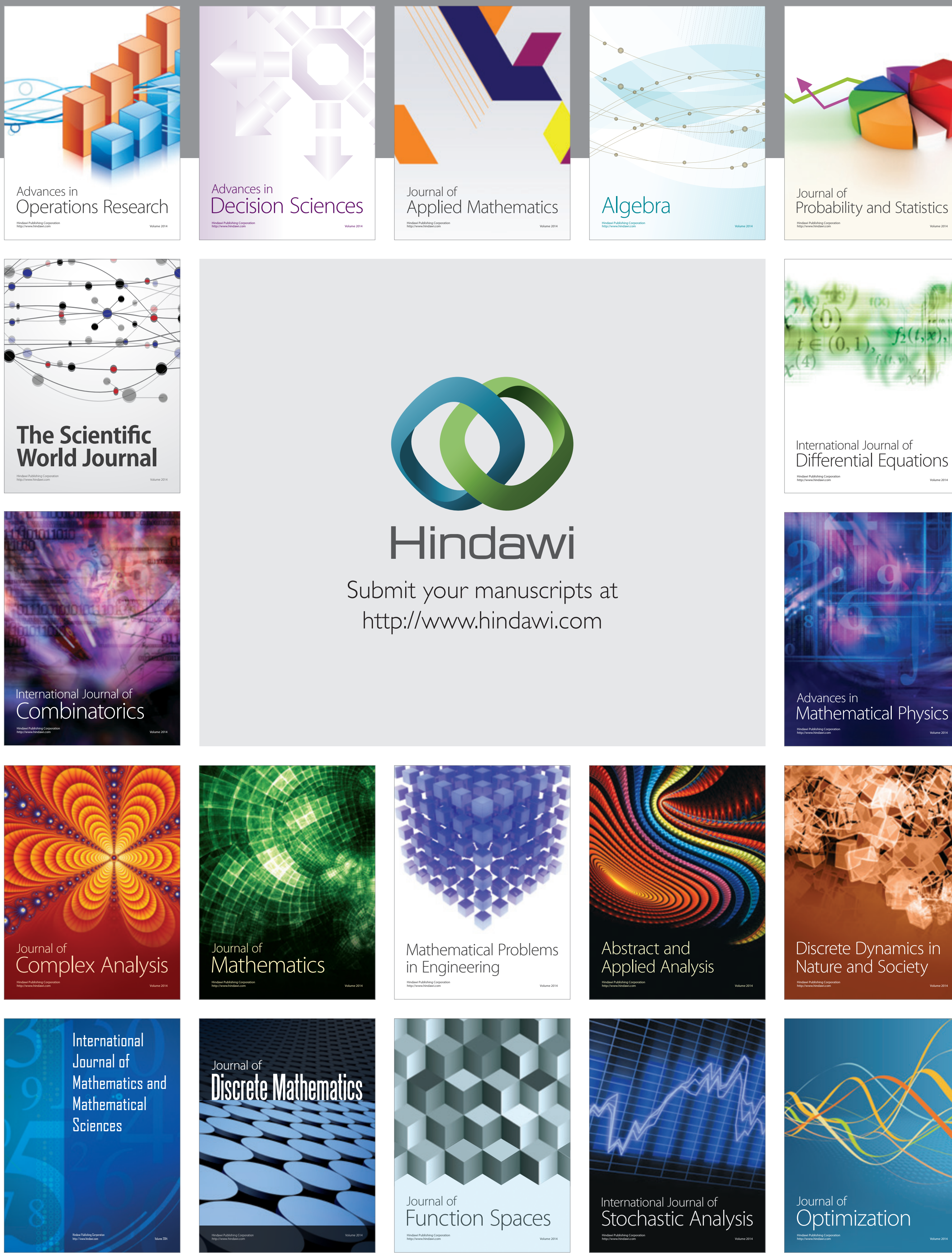
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