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Research Article

The Bi-Integrable Couplings of Two-Component Casimir-Qiao-Liu Type Hierarchy and Their Hamiltonian Structures

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A new type of two-component Casimir-Qiao-Liu type hierarchy (2-CQLTH) is produced from a new spectral problem and their bi-Hamiltonian structures are constructed. Particularly, a new completely integrable two-component Casimir-Qiao-Liu type equation (2-CQLTE) is presented. Furthermore, based on the semidirect sums of matrix Lie algebras consisting of 3×3 block matrix Lie algebra, the bi-integrable couplings of the 2-CQLTH are constructed and their bi-Hamiltonian structures are furnished.

1. Introduction

Solitons and integrable systems play an important role in nonlinear wave and dynamics systems. It has been significant in soliton theory to find more new integrable systems. In 1996, Olver and Rosenau obtained a Casimir equation [1]:

$$\rho_t = \left(D \pm D^3\right)\rho^{-2}, \tag{1}$$

which is an integrable case of the general class of equations $\rho_t = D(1 \pm D^2)\rho^k$. In 2009, Qiao and Liu proposed a third-order integrable peakon equation [2, 3]:

$$q_t = \left(\frac{1}{2q^2}\right)_{xxx} - \left(\frac{1}{2q^2}\right)_x \tag{2}$$

which possesses Lax representation and bi-Hamiltonian structures. Here we can find that (2) can be obtained from the Casimir equation (1) by setting $\rho = \sqrt[3]{2}q$. We will use Casimir-Qiao-Liu equation (CQLE) to denote (2) in this paper. The CQLE (2) represents a third-order approximation of long wavelength, small amplitude waves of inviscid and incompressible fluids. Moreover, it may be reduced from the two-dimensional Euler equation by an approximation procedure, and its solutions may be useful to construct new solvable potentials in Newtonian dynamics and to model electrophysiological phenomena in neuroscience. It has attracted many scholars' attention in recent years [4–6].

In recent years, the construction of soliton hierarchies and integrable couplings have become important research fields in soliton theory [7–10]. The soliton hierarchies are generated from the zero curvature equations [11, 12] which are based on semisimple Lie algebras, while the integrable couplings are generated from the zero curvature equations based on semidirect sums of Lie algebras [13–15].

The trace identity proposed by Tu is a useful tool for constructing the Hamiltonian structures for both continuous and discrete integrable systems [11, 12]. Many integrable Hamiltonian systems of infinite dimensions with various physics and mathematical backgrounds have been obtained [16–18]. But when it comes to the integrable couplings, since they are based on semidirect sums of Lie algebras, the trace identity cannot be used properly. In order to solve this problem, Ma proposes the variational identity [19] while Guo and Zhang propose the quadratic-form identity [20]. By using them, the Hamiltonian structures of many integrable couplings systems have been furnished [21–23].

Integrable couplings correspond to nonsemisimple Lie algebras $\overline{g}$, and such Lie algebras can be written as semidirect sums [24]:

$$\overline{g} = g \oplus g_c, \quad g\text{-semisimple}, \ g_c\text{-solvable}. \tag{3}$$

The notion of semidirect sums $\overline{g} = g \oplus g_c$ means that the two Lie subalgebras g and g_c satisfy

$$[g, g_c] \subseteq g_c, \tag{4}$$

where $[g, g_c] = \{[A, B] \mid A \in g,\ B \in g_c\}$, with $[\cdot,\cdot]$ denoting the Lie bracket of $\overline{g}$. We also require the closure property between g and g_c under the matrix multiplication:

$$g g_c, g_c g \subseteq g_c, \tag{5}$$

where $g_1 g_2 = \{[A, B] \mid A \in g_1,\ B \in g_2\}$.

Now we make the following assumptions:

$$\begin{aligned} \overline{U} &= U + U_c, \\ \overline{V} &= V + V_c, \end{aligned} \tag{6}$$

where $U, V \in g$, $U_c, V_c \in g_c$ and $\overline{U}, \overline{V}$ satisfying (3), (4), and (5). In this condition, we can construct the enlarged spectral problem as

$$\begin{aligned} \overline{\phi}_x &= \overline{U\phi} = \overline{U}(\overline{u}, \lambda)\overline{\phi}, \\ \overline{\phi}_t &= \overline{V\phi} = \overline{V}\left(\overline{u}, \overline{u}_x, \ldots, \frac{\partial^{m_0}\overline{u}}{\partial x^{m_0}}; \lambda\right)\overline{\phi}. \end{aligned} \tag{7}$$

From the enlarged zero curvature equations

$$\overline{U} - \overline{V}_x + \left[\overline{U}, \overline{V}\right] = 0, \tag{8}$$

we have

$$\begin{aligned} U - V_x + [U, V] &= 0, \\ U_{c,t} - V_{c,x} + [U, V_c] + [U_c, V] + [U_c, V_c] &= 0. \end{aligned} \tag{9}$$

The first equation of (9) is the original soliton hierarchy while the second equation is the integrable couplings.

The bi-Hamiltonian structures of (9)

$$\overline{u}_{t_m} = \overline{J}\frac{\delta \mathscr{H}_{m+1}}{\delta \overline{u}}, \quad m \geq 0, \tag{10}$$

can be obtained by using the following variational identity:

$$\begin{aligned} \frac{\delta}{\delta u}\int \left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle dx &= \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\left\langle V, \frac{\partial U}{\partial u}\right\rangle, \\ \gamma &= -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\langle V, V\rangle|, \end{aligned} \tag{11}$$

where $\langle\cdot,\cdot\rangle$ is a bilinear form [19, 20, 25].

In this paper, starting from a new eigenvalue problem, the two-component Casimir-Qiao-Liu type hierarchy (2-CQLTH) is worked out. It is proved that the 2-CQLTH has Lax pairs and bi-Hamiltonian structures, so it is completely integrable. Particularly, the two-component Casimir-Qiao-Liu type equation (2-CQLTE) is given. Then, by constructing the enlarged spectral problem, we obtain the the bi-integrable couplings of the 2-CQLTH. Similarly, we have proved the integrability of the bi-integrable couplings by constructing their Lax pairs and bi-Hamiltonian structures.

This paper is organized as follows. In Section 2, a new 2-CQLTH is given. In Section 3, the bi-Hamiltonian structures of the 2-CQLTH are constructed. In Section 4, we obtain the bi-integrable couplings of the 2-CQLTH. In Section 5, we generate the bi-Hamiltonian structures of the integrable couplings. The conclusion is given in Section 6.

2. The Two-Component Casimir-Qiao-Liu Type Hierarchy

Consider a spatial matrix isospectral problem:

$$\begin{aligned} &\phi_x = U\phi = U(u, \lambda)\phi, \\ &U(u, \lambda) = \begin{bmatrix} -p & \frac{1}{2}\lambda q \\ -\frac{1}{2}\lambda q & p \end{bmatrix}, \quad u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \end{aligned} \tag{12}$$

where p, q are potential functions and λ is the spectral parameter.

Assuming that W has the form

$$W = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, \tag{13}$$

then the stationary zero curvature equation

$$W_x = [U, W] \tag{14}$$

gives rise to

$$\begin{aligned} a_x &= \frac{1}{2}bq\lambda + \frac{1}{2}cq\lambda, \\ b_x &= -aq\lambda - 2bp, \\ c_x &= -aq\lambda + 2cp. \end{aligned} \tag{15}$$

Further, let a, b, c possess the Laurent expansions:

$$\begin{aligned} a &= \sum_{i\geq 0} a_i \lambda^{-2i}, \\ b &= \sum_{i\geq 0} b_i \lambda^{-2i-1}, \\ c &= \sum_{i\geq 0} c_i \lambda^{-2i-1}, \\ & i \geq 0, \end{aligned} \tag{16}$$

where the initial values are as follows:

$$\begin{aligned} a_0 &= 0, \\ b_0 &= \frac{1}{4}, \\ c_0 &= -\frac{1}{4}. \end{aligned} \tag{17}$$

In this case, system (15) becomes

$$\begin{aligned} a_{i,x} &= \frac{q}{2}\left(b_i + c_i\right), \\ b_{i,x} &= -2pb_i - qa_{i+1}, \\ c_{i,x} &= 2pc_i - qa_{i+1}, \\ & \qquad i \geq 0. \end{aligned} \tag{18}$$

Imposing the following conditions on constants of integration:

$$a_i\big|_{u=0} = b_i\big|_{u=0} = c_i\big|_{u=0} = 0, \quad i \geq 1, \tag{19}$$

the sequence of $\{a_i, b_i, c_i \mid i \geq 1\}$ can be uniquely determined and the first two sets are as follows:

$$\begin{aligned} a_1 &= -\frac{p}{2q}, \\ b_1 &= -\frac{1}{2q}\left(\frac{p}{q}\right)_x + \frac{1}{2}\left(\frac{p}{q}\right)^2, \\ c_1 &= -\frac{1}{2q}\left(\frac{p}{q}\right)_x - \frac{1}{2}\left(\frac{p}{q}\right)^2, \\ a_2 &= -\frac{p^3}{q^3} + \frac{p_{xx}}{2q^3} - \frac{3p_xq_x}{2q^4} - \frac{pq_{xx}}{2q^4} + \frac{3pq_x^2}{2q^5}, \\ b_2 &= \frac{a_{2,x}}{q} - 2\int \frac{p}{q}a_{2,x}dx, \\ c_2 &= \frac{a_{2,x}}{q} + 2\int \frac{p}{q}a_{2,x}dx. \end{aligned} \tag{20}$$

Now we introduce

$$V^{[m]} = \lambda\left(\lambda^{2m+1}W\right)_+ + \Delta_m, \quad m \geq 0, \tag{21}$$

where

$$\Delta_m = \begin{bmatrix} -a_{m+1} & \lambda\dfrac{q}{p}a_{m+1} \\ -\lambda\dfrac{q}{p}a_{m+1} & a_{m+1} \end{bmatrix}. \tag{22}$$

By considering

$$\phi_{t_m} = V^{(m)}\phi, \tag{23}$$

the compatibility of (12) and (23) gives the zero curvature equation:

$$U_{t_m} - V_x^{[m]} + \left[U, V^{[m]}\right] = 0. \tag{24}$$

From the zero curvature equation (24), we can obtain the 2-CQLTH:

$$u_{t_m} = K_m = \begin{bmatrix} a_{m+1,x} \\ 2\left(\dfrac{q}{p}a_{m+1}\right)_x \end{bmatrix} = J\begin{bmatrix} -4a_{m+1} \\ c_{m+1} - b_{m+1} \end{bmatrix}, \tag{25}$$

where

$$J = \begin{bmatrix} -\dfrac{1}{4}\partial & 0 \\ 0 & \dfrac{1}{2}\partial\dfrac{q}{p}\partial^{-1}\dfrac{q}{p}\partial \end{bmatrix}. \tag{26}$$

Remark 1. The Lax pair for the 2-CQLTH (see (25)) is given by (12) and (23). This implies that the 2-CQLTH (see (25)) is integrable in Lax sense.

Particularly, when we take $m = 1$, we obtain

$$\begin{aligned} p_t &= -\left(\frac{3p^3}{q^3} + \frac{2pq_{xx}}{q^4} - \frac{15pq_x^2}{2q^5}\right)\left(\frac{p_x}{p} - \frac{q_x}{q}\right) \\ &\quad - \frac{3pq_x}{q^4}\left(\frac{p_{xx}}{p} - \frac{q_{xx}}{q}\right) + \frac{p}{2q^3}\left(\frac{p_{xxx}}{p} - \frac{q_{xxx}}{q}\right), \\ q_t &= -\left(\frac{4p^2}{q^2} + \frac{p_{xx}}{pq^2} - \frac{3p_xq_x}{pq^3} + \frac{3q_{xx}}{q^3} - \frac{12q_x^2}{q^4}\right) \\ &\quad \cdot\left(\frac{p_x}{p} - \frac{q_x}{q}\right) - \frac{6q_x}{q^3}\left(\frac{p_{xx}}{p} - \frac{q_{xx}}{q}\right) \\ &\quad + \frac{1}{q^2}\left(\frac{p_{xxx}}{p} - \frac{q_{xxx}}{q}\right). \end{aligned} \tag{27}$$

Remark 2. Taking $p = 1/2$, the second equation of (27) can be reduced to the CQLE (2). Thus, (27) is called 2-CQLTE.

3. Bi-Hamiltonian Structures and Liouville Integrability

By direct computation, we can get

$$\begin{aligned} \frac{\partial U}{\partial \lambda} &= \begin{bmatrix} 0 & \dfrac{1}{2}q \\ -\dfrac{1}{2}q & 0 \end{bmatrix}, \\ \frac{\partial U}{\partial p} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \frac{\partial U}{\partial q} &= \begin{bmatrix} 0 & \dfrac{1}{2}\lambda \\ -\dfrac{1}{2}\lambda & 0 \end{bmatrix}, \end{aligned} \tag{28}$$

so we have

$$\begin{aligned} \operatorname{tr}\left(W\frac{\partial U}{\partial \lambda}\right) &= \frac{1}{2}qc - \frac{1}{2}qb, \\ \operatorname{tr}\left(W\frac{\partial U}{\partial p}\right) &= -2a, \\ \operatorname{tr}\left(W\frac{\partial U}{\partial q}\right) &= \frac{1}{2}c\lambda - \frac{1}{2}b\lambda. \end{aligned} \tag{29}$$

Now, the trace identity [11]

$$\frac{\delta}{\delta u}\int \operatorname{tr}\left(W\frac{\partial U}{\partial \lambda}\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\operatorname{tr}\left(W\frac{\partial U}{\partial u}\right), \qquad \gamma = \frac{\lambda}{2}\frac{d}{d\lambda}\ln\left|\operatorname{tr}\left(W^2\right)\right| \tag{30}$$

gives

$$\frac{\delta}{\delta u}\int\left(\frac{1}{2}qc-\frac{1}{2}qb\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{bmatrix}-2a\\ \frac{1}{2}c\lambda-\frac{1}{2}b\lambda\end{bmatrix}. \tag{31}$$

Comparing coefficients of all powers of λ in the equality (31), we have

$$\frac{\delta}{\delta u}\int\frac{qc_m-qb_m}{\gamma-2m}dx = \begin{bmatrix}-4a_m\\ c_m-b_m\end{bmatrix}, \quad m\geq 0. \tag{32}$$

Checking a particular case with $m=1$ in (32), we have $\gamma=1$.

Thus, we obtain

$$\frac{\delta}{\delta u}\mathscr{H}_m = \begin{bmatrix}-4a_m\\ c_m-b_m\end{bmatrix}, \quad m\geq 0, \tag{33}$$

where the Hamiltonian functionals are defined by

$$\mathscr{H}_m = \int\frac{qc_m-qb_m}{1-2m}dx. \tag{34}$$

It now follows that the soliton hierarchy (25) has the following Hamiltonian structures:

$$u_{t_m} = K_m = J\frac{\delta\mathscr{H}_m}{\delta u}, \quad m\geq 0. \tag{35}$$

From the recursion relations (18), we can get

$$\begin{pmatrix}-4a_{m+1}\\ c_{m+1}-b_{m+1}\end{pmatrix} = L\begin{pmatrix}-4a_m\\ c_m-b_m\end{pmatrix}, \quad m\geq 0, \tag{36}$$

where

$$L = \begin{bmatrix}-\frac{1}{q}\partial\frac{1}{q}\partial+4\frac{p}{q}\partial^{-1}\frac{p}{q}\partial & 0\\ 0 & -\partial^{-1}\frac{p}{q}\partial\frac{1}{q}\partial\frac{1}{p}\partial+4\partial^{-1}\frac{p}{q}\partial\frac{p}{q}\end{bmatrix}. \tag{37}$$

It is easy to verify that $JL = L^*J$. Actually, we know that the hierarchy (25) is bi-Hamiltonian:

$$u_{t_m} = K_m = J\frac{\delta\mathscr{H}_m}{\delta u} = M\frac{\delta\mathscr{H}_{m-1}}{\delta u}, \quad m\geq 1, \tag{38}$$

where the second Hamiltonian operator is

$$M = JL = \begin{bmatrix}-\partial\frac{p}{q}\partial^{-1}\frac{p}{q}\partial+\frac{1}{4}\partial\frac{1}{q}\partial\frac{1}{q}\partial & 0\\ 0 & 2\partial-\frac{1}{2}\partial\frac{1}{p}\partial\frac{1}{p}\partial\end{bmatrix} \tag{39}$$

and J and M constitute the Hamiltonian pairs. Thus, the soliton hierarchy (25) is Liouville integrable.

Remark 3. Based on SO(3) [26, 27], a 3×3 matrix isospectral problem is constructed:

$$\phi_x = U\phi = U(u,\lambda)\phi, \quad u = \begin{bmatrix}p\\ q\end{bmatrix}, \ \phi = \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}, \tag{40}$$

with the spectral matrix as

$$U = \begin{bmatrix}0 & -\frac{q\lambda}{2} & -p\\ \frac{q\lambda}{2} & 0 & -\frac{q\lambda}{2}\\ p & \frac{q\lambda}{2} & 0\end{bmatrix}. \tag{41}$$

By a standard procedure similar to Sections 2 and 3, another 2-QLTH can be worked out:

$$u_{t_m} = \begin{bmatrix}-a_{m+1,x}\\ -2\left(\frac{q}{p}a_{m+1}\right)_x\end{bmatrix} = J\frac{\delta\mathscr{H}_m}{\delta u} = M\frac{\delta\mathscr{H}_{m-1}}{\delta u}, \quad m\geq 1, \tag{42}$$

with the Hamiltonian operators

$$\begin{aligned}
J &= \begin{bmatrix}-\partial\frac{1}{q}\partial\frac{1}{q}\partial-\partial\frac{p}{q}\partial^{-1}\frac{p}{q}\partial & 0\\ 0 & 2\partial\frac{1}{p}\partial\frac{1}{p}\partial+2\partial\end{bmatrix},\\
M &= \begin{bmatrix}M_1 & 0\\ 0 & M_2\end{bmatrix},\\
M_1 &= 2\partial\frac{1}{q}\partial\frac{1}{q}\partial\frac{1}{q}\partial\frac{1}{q}\partial+2\partial\frac{1}{q}\partial\frac{1}{q}\partial\frac{p}{q}\partial^{-1}\frac{p}{q}\partial\\
&\quad+2\partial\frac{p}{q}\partial^{-1}\frac{p}{q}\partial\frac{1}{q}\partial\frac{1}{q}\partial+2\partial\frac{p}{q}\partial^{-1}\frac{p}{q}\partial\frac{p}{q}\partial^{-1}\frac{p}{q}\partial,\\
M_2 &= -4\partial\frac{1}{p}\partial\frac{1}{q}\partial\frac{1}{q}\partial\frac{1}{p}\partial-4\partial\frac{1}{p}\partial\frac{1}{q}\partial\frac{p}{q}-4\frac{p}{q}\partial\frac{1}{q}\partial\frac{1}{p}\partial\\
&\quad-4\frac{p}{q}\partial\frac{p}{q},
\end{aligned} \tag{43}$$

and the Hamiltonian functionals

$$\mathscr{H}_m = \int\frac{-qb_m-qc_m}{1-2m}dx, \quad m\geq 0. \tag{44}$$

Particularly, when we take $m = 1$, a new 2-CQLTE is presented as

$$\begin{aligned} p_t &= \left(\frac{3p^3}{4q^3} - \frac{2pq_{xx}}{q^4} + \frac{15pq_x^2}{2q^5}\right)\left(\frac{p_x}{p} - \frac{q_x}{q}\right) - 3 \\ &\quad \cdot \frac{pq_x}{q^4}\left(\frac{p_{xx}}{p} - \frac{q_{xx}}{q}\right) + \frac{p}{2q^3}\left(\frac{p_{xxx}}{p} - \frac{q_{xxx}}{q}\right), \\ q_t &= \left(\frac{p^2}{q^2} - \frac{p_{xx}}{pq^2} - 3\frac{q_{xx}}{q^3} + 3\frac{p_x q_x}{pq^3} + 12\frac{q_x^2}{q^4}\right) \\ &\quad \cdot \left(\frac{p_x}{p} - \frac{q_x}{q}\right) - 6\frac{q_x}{q^3}\left(\frac{p_{xx}}{p} - \frac{q_{xx}}{q}\right) \\ &\quad + \frac{1}{q^2}\left(\frac{p_{xxx}}{p} - \frac{q_{xxx}}{q}\right). \end{aligned} \tag{45}$$

Equation (45) can be transformed into (27) in complex field, but they are different equations in real field.

4. Bi-Integrable Couplings of the 2-CQLTH

Supposing that the triangular block matrices M have the following form [28, 29]:

$$M(A_1, A_2, A_3) = \begin{pmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 & \delta A_2 \\ 0 & 0 & A_1 \end{pmatrix}, \tag{46}$$

where δ is a constant, and defining $\overline{g} = \{M(A_1, A_2, A_3)\}$, $g = \{M(A_1, 0, 0)\}$, and $g_c = \{M(0, A_2, A_3)\}$, we know that $\overline{g}, g, g_c$ satisfy (3), (4), and (5).

Considering the enlarged spectral matrix

$$\begin{aligned} \overline{U} &= \begin{pmatrix} U & U_1 & U_2 \\ 0 & U & \delta U_1 \\ 0 & 0 & U \end{pmatrix}, \\ \overline{W} &= \begin{pmatrix} W & W_1 & W_2 \\ 0 & W & \delta W_1 \\ 0 & 0 & W \end{pmatrix}, \end{aligned} \tag{47}$$

where

$$\begin{aligned} U &= \begin{pmatrix} -p & \frac{q\lambda}{2} \\ -\frac{q\lambda}{2} & p \end{pmatrix}, \\ U_1 &= \begin{pmatrix} -r_1 & \frac{r_2\lambda}{2} \\ -\frac{r_2\lambda}{2} & r_1 \end{pmatrix}, \\ U_2 &= \begin{pmatrix} -s_1 & \frac{s_2\lambda}{2} \\ -\frac{s_2\lambda}{2} & s_1 \end{pmatrix}, \\ W &= \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \\ W_1 &= \begin{pmatrix} d & e \\ f & -d \end{pmatrix}, \\ W_2 &= \begin{pmatrix} g & h \\ k & -g \end{pmatrix}, \end{aligned} \tag{48}$$

and p, q, r_1, r_2, s_1, s_2 are dependent variables, if we substitute $\overline{U}, \overline{W}$ into the corresponding enlarged stationary zero curvature equation, we can get

$$\begin{aligned} 2a_x - bq\lambda - cq\lambda &= 0, \\ b_x + 2bp + aq\lambda &= 0, \\ c_x - 2cp + aq\lambda &= 0, \\ 2d_x - eq\lambda - fq\lambda - b\lambda r_2 - c\lambda r_2 &= 0, \\ e_x + 2ep + dq\lambda + 2br_1 + a\lambda r_2 &= 0, \\ f_x - 2fp + dq\lambda - 2cr_1 + a\lambda r_2 &= 0, \\ 2g_x - hq\lambda - kq\lambda - e\delta\lambda r_2 - f\delta\lambda r_2 - b\lambda s_2 - c\lambda s_2 &= 0, \\ h_x + 2hp + gq\lambda + 2e\delta r_1 + d\delta\lambda r_2 + 2bs_1 + a\lambda s_2 &= 0, \\ k_x - 2kp + gq\lambda - 2f\delta r_1 + d\delta\lambda r_2 - 2cs_1 + a\lambda s_2 &= 0. \end{aligned} \tag{49}$$

When $a, b, c, d, e, f, g, h, k$ are assumed to be

$$\begin{aligned} a &= \sum_{i\geq 0} a_i \lambda^{-2i}, \\ b &= \sum_{i\geq 0} b_i \lambda^{-2i-1}, \\ c &= \sum_{i\geq 0} c_i \lambda^{-2i-1}, \\ & \qquad i \geq 0, \\ d &= \sum_{i\geq 0} d_i \lambda^{-2i}, \\ e &= \sum_{i\geq 0} e_i \lambda^{-2i-1}, \\ f &= \sum_{i\geq 0} f_i \lambda^{-2i-1}, \\ & \qquad i \geq 0, \end{aligned}$$

$$g=\sum_{i\geq 0} g_i\lambda^{-2i},$$
$$h=\sum_{i\geq 0} h_i\lambda^{-2i-1},$$
$$k=\sum_{i\geq 0} k_i\lambda^{-2i-1},$$
$$i\geq 0, \tag{50}$$

and the initial values are taken as

$$\begin{aligned} a_0&=0,\\ b_0&=\alpha,\\ c_0&=-\alpha,\\ d_0&=0,\\ e_0&=\beta,\\ f_0&=-\beta,\\ g_0&=0,\\ h_0&=\iota,\\ k_0&=-\iota, \end{aligned} \tag{51}$$

we can write (49) in the following forms:

$$\begin{aligned} &-\frac{1}{2}b_iq-c_i\frac{q}{2}+a_{i,x}=0,\\ &2b_ip+a_{i+1}q+b_{i,x}=0,\\ &-2c_ip+a_{i+1}q\lambda+c_{i,x}=0,\\ &-\frac{1}{2}e_iq-f_i\frac{q}{2}+d_{i,x}-\frac{1}{2}b_ir_2-\frac{1}{2}c_ir_2=0,\\ &2e_ip+d_{i+1}q\lambda+e_{i,x}+2br_1+a_{i+1}r_2=0,\\ &-2f_ip+d_{i+1}q+f_{i,x}-2c_ir_1+a_{i+1}r_2=0,\\ &-\frac{1}{2}h_iq-\frac{k_iq}{2}+g_{i,x}-\frac{1}{2}e_i\delta r_2-\frac{1}{2}f_i\delta r_2-\frac{1}{2}b_is_2\\ &\quad-\frac{1}{2}c_is_2=0,\\ &2h_ip+g_{i+1}q+h_{i,x}+2e_i\delta r_1+d_{i+1}\delta r_2+2b_is_1\\ &\quad+a_{i+1}s_2=0,\\ &-2k_ip+g_{i+1}q+k_{i,x}-2f_i\delta r_1+d_{i+1}\delta\lambda r_2-2c_is_1\\ &\quad+a_{i+1}\lambda s_2=0. \end{aligned} \tag{52}$$

Furthermore, by imposing

$$\begin{aligned} a_i|_{u=0}&=b_i|_{u=0}=c_i|_{u=0}=0,\quad i\geq 1,\\ d_i|_{u=0}&=e_i|_{u=0}=f_i|_{u=0}=0,\quad i\geq 1,\\ g_i|_{u=0}&=h_i|_{u=0}=k_i|_{u=0}=0,\quad i\geq 1, \end{aligned} \tag{53}$$

the sequence of $\{a_i,b_i,c_i,d_i,e_i,f_i,g_i,h_i,k_i \mid i\geq 1\}$ has the unique expressions

$$\begin{aligned} a_0&=0,\\ b_0&=\alpha,\\ c_0&=-\alpha,\\ a_1&=-2\alpha\frac{p}{q},\\ b_1&=2\alpha\left(\frac{p}{q}\right)^2-2\alpha\frac{1}{q}\left(\frac{p}{q}\right)_x,\\ c_1&=-2\alpha\left(\frac{p}{q}\right)^2-2\alpha\frac{1}{q}\left(\frac{p}{q}\right)_x,\\ d_0&=0,\\ e_0&=\beta,\\ f_0&=-\beta,\\ d_1&=-2\alpha\frac{p}{q}-2\beta\frac{r_1}{q}+2\beta\frac{pr_2}{q^2},\\ e_1&=2\alpha\left(\frac{p}{q}\right)^2-2\alpha\frac{1}{q}\left(\frac{p}{q}\right)_x-2\beta\frac{1}{q}\left(\frac{r_1}{q}\right)_x\\ &\quad+4\beta\left(\frac{pr_1}{q^2}\right)_x+2\beta\frac{1}{q}\left(\frac{pr_2}{q^2}\right)_x\\ &\quad+2\beta\frac{r_2}{q^2}\left(\frac{p}{q}\right)_x-4\beta\left(\frac{p^2r_2}{q^3}\right)_x,\\ f_1&=-2\alpha\left(\frac{p}{q}\right)^2-2\alpha\frac{1}{q}\left(\frac{p}{q}\right)_x-2\beta\frac{1}{q}\left(\frac{r_1}{q}\right)_x\\ &\quad-4\beta\left(\frac{pr_1}{q^2}\right)_x+2\beta\frac{1}{q}\left(\frac{pr_2}{q^2}\right)_x\\ &\quad+2\beta\frac{r_2}{q^2}\left(\frac{p}{q}\right)_x+4\beta\left(\frac{p^2r_2}{q^3}\right)_x,\\ g_0&=0,\\ h_0&=\iota,\\ k_0&=-\iota,\\ g_1&=-2\iota\frac{p}{q}-2\beta\delta\frac{r_1}{q}-2\alpha\frac{s_1}{q}+2\alpha\delta\frac{pr_2}{q^2}+2\beta\delta\frac{r_1r_2}{q^2}\\ &\quad-2\beta\delta\frac{pr_2r_2}{q^3}+2\alpha\frac{ps_2}{q^2},\\ k_1&=-2\iota\frac{1}{q}\left(\frac{p}{q}\right)_x-2\beta\delta\frac{1}{q}\left(\frac{r_1}{q}\right)_x+2\beta\delta\frac{1}{q}\left(\frac{r_1r_2}{q^2}\right)_x\\ &\quad+2\beta\delta\frac{r_2}{q^2}\left(\frac{r_1}{q}\right)_x-2\beta\delta\frac{1}{q}\left(\frac{pr_2r_2}{q^3}\right)_x \end{aligned}$$

$$
\begin{aligned}
&-2\beta\delta\frac{r_2}{q^2}\left(\frac{pr_2}{q^2}\right)_x-2\beta\delta\frac{r_2r_2}{q^3}\left(\frac{p}{q}\right)_x\\
&-2\alpha\frac{1}{q}\left(\frac{s_1}{q}\right)_x+2\alpha\delta\frac{r_2}{q^2}\left(\frac{p}{q}\right)_x\\
&+2\alpha\delta\frac{1}{q}\left(\frac{pr_2}{q^2}\right)_x+2\alpha\frac{s_2}{q^2}\left(\frac{p}{q}\right)_x\\
&+2\alpha\frac{1}{q}\left(\frac{ps_2}{q^2}\right)_x-2\iota\left(\frac{p}{q}\right)^2-4\beta\delta\frac{pr_1}{q^2}\\
&+4\beta\delta\frac{pr_1r_2}{q^3}+4\beta\delta\frac{pr_1r_2}{q^3}-6\beta\delta\left(\frac{pr_2}{q^2}\right)^2\\
&-2\beta\delta\left(\frac{r_1}{q}\right)^2+4\alpha\delta\frac{p^2r_2}{q^3}-4\alpha\frac{ps_1}{q^2}\\
&+4\alpha\frac{p^2s_2}{q^3},\\
h_1=&-2\iota\frac{1}{q}\left(\frac{p}{q}\right)_x-2\beta\delta\frac{1}{q}\left(\frac{r_1}{q}\right)_x+2\beta\delta\frac{1}{q}\left(\frac{r_1r_2}{q^2}\right)_x\\
&+2\beta\delta\frac{r_2}{q^2}\left(\frac{r_1}{q}\right)_x-2\beta\delta\frac{1}{q}\left(\frac{pr_2r_2}{q^3}\right)_x\\
&-2\beta\delta\frac{r_2}{q^2}\left(\frac{pr_2}{q^2}\right)_x-2\beta\delta\frac{r_2r_2}{q^3}\left(\frac{p}{q}\right)_x\\
&-2\alpha\frac{1}{q}\left(\frac{s_1}{q}\right)_x+2\alpha\delta\frac{r_2}{q^2}\left(\frac{p}{q}\right)_x\\
&+2\alpha\delta\frac{1}{q}\left(\frac{pr_2}{q^2}\right)_x+2\alpha\frac{s_2}{q^2}\left(\frac{p}{q}\right)_x\\
&+2\alpha\frac{1}{q}\left(\frac{ps_2}{q^2}\right)_x+2\iota\left(\frac{p}{q}\right)^2+4\beta\delta\frac{pr_1}{q^2}\\
&-4\beta\delta\frac{pr_1r_2}{q^3}-4\beta\delta\frac{pr_1r_2}{q^3}+6\beta\delta\left(\frac{pr_2}{q^2}\right)^2\\
&+2\beta\delta\left(\frac{r_1}{q}\right)^2-4\alpha\delta\frac{p^2r_2}{q^3}+4\alpha\frac{ps_1}{q^2}\\
&-4\alpha\frac{p^2s_2}{q^3}.
\end{aligned}
\tag{54}
$$

In order to get the integrable couplings systems from the enlarged zero curvature equations

$$\overline{U}_{t_m}-V_x^{[m]}+\left[U,V^{[m]}\right]=0, \tag{55}$$

we define the Lax matrices and modification terms as

$$V^{[m]}=\lambda\left(\lambda^{2m+1}W\right)_++\Delta_m,\quad m\geq 0, \tag{56}$$

where

$$
\begin{aligned}
\Delta_m&=\begin{pmatrix}\Delta_{1m} & \Delta_{2m} & \Delta_{3m}\\ 0 & \Delta_{1m} & \delta\Delta_{2m}\\ 0 & 0 & \Delta_{1m}\end{pmatrix},\\
\Delta_{1m}&=\begin{pmatrix}-a_{m+1} & \lambda\frac{q}{p}a_{m+1}\\ -\lambda\frac{q}{p}a_{m+1} & a_{m+1}\end{pmatrix},\\
\Delta_{2m}&=\begin{pmatrix}-d_{m+1}-\frac{2a_{1+m}r_1}{p} & \lambda\frac{q}{p}d_{m+1}+\lambda\frac{r_2}{p}a_{1+m}\\ -\lambda\frac{q}{p}d_{m+1}-\lambda\frac{r_2}{p}a_{1+m} & d_{m+1}+\frac{2a_{1+m}r_1}{p}\end{pmatrix},\\
\Delta_{1m}&=\begin{pmatrix}-g_{m+1}-2\delta\frac{r_1}{p}d_{m+1}-\frac{2s_1}{p}a_{1+m} & \lambda\frac{q}{p}g_{m+1}+\delta\lambda\frac{r_2}{p}d_{m+1}+\lambda\frac{s_2}{p}a_{1+m}\\ -\lambda\frac{q}{p}g_{m+1}-\delta\lambda\frac{r_2}{p}d_{m+1}-\lambda\frac{s_2}{p}a_{1+m} & g_{m+1}+2\delta\frac{r_1}{p}d_{m+1}+\frac{2s_1}{p}a_{1+m}\end{pmatrix},
\end{aligned}
\tag{57}
$$

and P_+ denotes the polynomial part of P.

In this condition, the integrable couplings system can be obtained as

$$u_{t_m} = \begin{pmatrix} p_{t_m} \\ q_{t_m} \\ r_{1t_m} \\ r_{2t_m} \\ s_{1t_m} \\ s_{2t_m} \end{pmatrix} = \begin{pmatrix} \partial a_{m+1} \\ 2\partial \frac{q}{p} a_{m+1} \\ \partial d_{m+1} + 2\partial \frac{a_{1+m} r_1}{p} \\ 2\partial \frac{q}{p} d_{m+1} + 2\partial \frac{r_2}{p} a_{1+m} \\ \partial g_{m+1} + 2\delta\partial \frac{r_1}{p} d_{m+1} + 2\partial \frac{s_1}{p} a_{1+m} \\ 2\partial \frac{q}{p} g_{m+1} + 2\delta\partial \frac{r_2}{p} d_{m+1} + 2\partial \frac{s_2}{p} a_{1+m} \end{pmatrix}, \tag{58}$$

which is equal to

$$u_{t_m} = \overline{J} \begin{pmatrix} -4a_{m+1}\eta_1 - 4d_{m+1}\eta_2 - 4g_{m+1}\eta_3 \\ -b_{m+1}\eta_1 + c_{m+1}\eta_1 - e_{m+1}\eta_2 + f_{m+1}\eta_2 - h_{m+1}\eta_3 + k_{m+1}\eta_3 \\ -4a_{m+1}\eta_2 - 4d_{m+1}\delta\eta_3 \\ -b_{m+1}\eta_2 + c_{m+1}\eta_2 - e_{m+1}\delta\eta_3 + f_{m+1}\delta\eta_3 \\ -4a_{m+1}\eta_3 \\ -b_{m+1}\eta_3 + c_{m+1}\eta_3 \end{pmatrix}, \tag{59}$$

where

$$\begin{aligned}
\overline{J} &= \begin{pmatrix} 0 & 0 & J_1 \\ 0 & \frac{1}{\delta} J_1 & J_2 \\ J_1 & J_2 & J_3 \end{pmatrix}, \\
J_1 &= \begin{pmatrix} j_1^1 & 0 \\ 0 & j_1^4 \end{pmatrix}, \\
J_2 &= \begin{pmatrix} j_2^1 & j_2^2 \\ j_2^3 & j_2^4 \end{pmatrix}, \\
J_3 &= \begin{pmatrix} j_3^1 & j_3^2 \\ j_3^3 & j_3^4 \end{pmatrix}, \\
j_1^1 &= -\frac{1}{4\eta_3}\partial, \\
j_1^4 &= \frac{1}{2\eta_3}\partial \frac{q}{p}\partial^{-1}\frac{q}{p}\partial, \\
j_2^1 &= \frac{\eta_2}{4\delta\eta_3\eta_3}\partial, \\
j_2^2 &= \frac{1}{2\eta_3}\partial \frac{r_1}{p}\partial^{-1}\frac{q}{p}\partial, \\
j_2^3 &= \frac{1}{2\eta_3}\partial \frac{q}{p}\partial^{-1}\frac{r_1}{p}\partial, \\
j_2^4 &= \frac{1}{2\eta_3}\partial \frac{q}{p}\partial^{-1}\frac{r_2}{p}\partial + \frac{1}{2\eta_3}\partial \frac{r_2}{p}\partial^{-1}\frac{q}{p}\partial \\
&\quad - \frac{\eta_2}{2\delta\eta_3\eta_3}\partial \frac{q}{p}\partial^{-1}\frac{q}{p}\partial, \\
j_3^1 &= \frac{1}{2\eta_3}\delta\partial \frac{r_1}{p}\partial^{-1}\frac{r_1}{p}\partial + \frac{\eta_1}{4\eta_3\eta_3}\partial - \frac{\eta_2\eta_2}{4\delta\eta_3\eta_3\eta_3}\partial, \\
j_3^2 &= \frac{1}{2\eta_3}\delta\partial \frac{r_1}{p}\partial^{-1}\frac{r_2}{p}\partial + \frac{1}{2\eta_3}\partial \frac{s_1}{p}\partial^{-1}\frac{q}{p}\partial \\
&\quad - \frac{\eta_2}{2\eta_3\eta_3}\partial \frac{r_1}{p}\partial^{-1}\frac{q}{p}\partial, \\
j_3^3 &= \frac{1}{2\eta_3}\delta\partial \frac{r_2}{p}\partial^{-1}\frac{r_1}{p}\partial + \frac{1}{2\eta_3}\partial \frac{q}{p}\partial^{-1}\frac{s_1}{p}\partial \\
&\quad - \frac{\eta_2}{2\eta_3\eta_3}\partial \frac{q}{p}\partial^{-1}\frac{r_1}{p}\partial, \\
j_3^4 &= \frac{1}{2\eta_3}\partial \frac{q}{p}\partial^{-1}\frac{s_2}{p}\partial + \frac{1}{2\eta_3}\partial \frac{s_2}{p}\partial^{-1}\frac{q}{p}\partial \\
&\quad + \frac{1}{2\eta_3}\delta\partial \frac{r_2}{p}\partial^{-1}\frac{r_2}{p}\partial - \frac{\eta_1}{2\eta_3\eta_3}\partial \frac{q}{p}\partial^{-1}\frac{q}{p}\partial \\
&\quad - \frac{\eta_2}{2\eta_3\eta_3}\partial \frac{q}{p}\partial^{-1}\frac{r_2}{p}\partial - \frac{\eta_2}{2\eta_3\eta_3}\partial \frac{r_2}{p}\partial^{-1}\frac{q}{p}\partial \\
&\quad + \frac{\eta_2\eta_2}{2\delta\eta_3\eta_3\eta_3}\partial \frac{q}{p}\partial^{-1}\frac{q}{p}\partial.
\end{aligned} \tag{60}$$

Remark 4. $\overline{U}$ and $V^{[m]}$ are the Lax pairs of the integrable couplings of the 2-CQLTH; this implies that the integrable couplings are integrable in Lax sense.

5. Bi-Hamiltonian Structures and Liouville Integrability of the Bi-Integrable Couplings

Because the associated matrix Lie algebras are nonsemisimple, we should use the variational identity (11) to furnish the Hamiltonian structures of the integrable couplings system (58) (or (59)). Before doing this, we should find out the nondegenerate, symmetric, and adinvariant bilinear form corresponding to the nonsemisimple matrix loop algebras first [22].

By defining a mapping from $\overline{g}$ to R^9 as

$$\begin{aligned} \sigma : \overline{g} &\longrightarrow R^9, \\ A &\longmapsto (a_1, \ldots, a_9)^T, \end{aligned} \tag{61}$$

where

$$\begin{aligned} A &= M(A_1, A_2, A_3) \in \overline{g}, \\ A_i &= \begin{pmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2} \end{pmatrix}, \\ & i = 1, 2, 3, \end{aligned} \tag{62}$$

we have the following results according to [22, 28, 30].

The mapping (61) is a Lie algebra isomorphism from $\overline{g}$ to R^9 and the Lie bracket $[\cdot, \cdot]$ can be calculated as

$$\begin{aligned} [a, b] &= a^T R(b), \\ a &= (a_1, a_2, \ldots, a_9), \\ b &= (b_1, b_2, \ldots, b_9) \in R^9, \end{aligned} \tag{63}$$

where

$$\begin{aligned} R(b) &= M(R_1, R_2, R_3), \\ R &= \begin{pmatrix} 0 & 2b_{3i-1} & -2b_{3i} \\ b_{3i} & -2b_{3i-2} & 0 \\ -b_{3i-1} & 0 & 2b_{3i-2} \end{pmatrix}, \quad i = 1, 2, 3. \end{aligned} \tag{64}$$

Assuming that the bilinear form on R^9 possesses the following form:

$$\langle a, b \rangle = a^T F b, \tag{65}$$

where F is a constant matrix, the symmetric property and adinvariance property of F

$$\begin{aligned} \langle a, b \rangle &= \langle b, a \rangle, \\ \langle a, [b, c] \rangle &= \langle [a, b], c \rangle \end{aligned} \tag{66}$$

give rise to

$$\begin{aligned} F &= F^T, \\ F(R(b))^T &= -R(b) F, \quad b \in R^9. \end{aligned} \tag{67}$$

So the form of the constant matrix F can be written as

$$F = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \delta\eta_3 & 0 \\ \eta_3 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{68}$$

where η_1, η_2, and η_3 are arbitrary constants. Thus, the bilinear form on the semidirect sums $\overline{g}$ of g and g_c can be defined as

$$\begin{aligned} \langle A, B \rangle &= \langle \sigma(A), \sigma(B) \rangle_{R^9} \\ &= (a_1, a_2, \ldots, a_9) F (b_1, b_2, \ldots, b_9)^T, \end{aligned} \tag{69}$$

where $A = \sigma^{-1}((a_1, a_2, \ldots, a_9)^T)$, $B = \sigma^{-1}((b_1, b_2, \ldots, b_9)^T) \in \overline{g}$.

In order to guarantee that the bilinear form is nondegenerate, we should require

$$\det(F) = 8\delta^3 \eta_3^9 \neq 0; \tag{70}$$

that is to say, $\delta \neq 0$ and $\eta_3 \neq 0$.

By calculation, we have

$$\frac{\partial \overline{U}}{\partial \lambda} = \begin{pmatrix} 0 & \frac{q}{2} & 0 & \frac{r_2}{2} & 0 & \frac{s_2}{2} \\ -\frac{q}{2} & 0 & -\frac{r_2}{2} & 0 & -\frac{s_2}{2} & 0 \\ 0 & 0 & 0 & \frac{q}{2} & 0 & \delta\frac{r_2}{2} \\ 0 & 0 & -\frac{q}{2} & 0 & -\delta\frac{r_2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{q}{2} \\ 0 & 0 & 0 & 0 & -\frac{q}{2} & 0 \end{pmatrix},$$

$$\frac{\partial \overline{U}}{\partial p} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\frac{\partial \overline{U}}{\partial q} = \begin{pmatrix} 0 & \frac{\lambda}{2} & 0 & 0 & 0 & 0 \\ -\frac{\lambda}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda}{2} & 0 & 0 \\ 0 & 0 & -\frac{\lambda}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\lambda}{2} \\ 0 & 0 & 0 & 0 & -\frac{\lambda}{2} & 0 \end{pmatrix},$$

$$\frac{\partial \overline{U}}{\partial r_1} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \frac{\partial \overline{U}}{\partial s_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\frac{\partial \overline{U}}{\partial r_2} = \begin{pmatrix} 0 & 0 & 0 & \frac{\lambda}{2} & 0 & 0 \\ 0 & 0 & -\frac{\lambda}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta\frac{\lambda}{2} \\ 0 & 0 & 0 & 0 & -\delta\frac{\lambda}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \frac{\partial \overline{U}}{\partial s_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{\lambda}{2} \\ 0 & 0 & 0 & 0 & -\frac{\lambda}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{71}$$

Further, we have

$$\left\langle W \ \frac{\partial U}{\partial \lambda} \right\rangle = -\frac{1}{2}bq\eta_1 + \frac{1}{2}cq\eta_1 - \frac{1}{2}eq\eta_2 + \frac{1}{2}fq\eta_2 - \frac{1}{2}br_2\eta_2 + \frac{1}{2}cr_2\eta_2 - \frac{1}{2}hq\eta_3 + \frac{1}{2}kq\eta_3 - \frac{1}{2}e\delta r_2\eta_3 + \frac{1}{2}f\delta r_2\eta_3 - \frac{1}{2}bs_2\eta_3 + \frac{1}{2}cs_2\eta_3,$$

$$\left\langle W \ \frac{\partial U}{\partial u} \right\rangle = \begin{pmatrix} -2a\eta_1 - 2d\eta_2 - 2g\eta_3 \\ -\frac{1}{2}b\lambda\eta_1 + \frac{1}{2}c\lambda\eta_1 - \frac{1}{2}e\lambda\eta_2 + \frac{1}{2}f\lambda\eta_2 - \frac{1}{2}h\lambda\eta_3 + \frac{1}{2}k\lambda\eta_3 \\ -2a\eta_2 - 2d\delta\eta_3 \\ -\frac{1}{2}b\lambda\eta_2 + \frac{1}{2}c\lambda\eta_2 - \frac{1}{2}e\delta\lambda\eta_3 + \frac{1}{2}f\delta\lambda\eta_3 \\ -2a\eta_3 \\ -\frac{1}{2}b\lambda\eta_3 + \frac{1}{2}c\lambda\eta_3 \end{pmatrix}. \tag{72}$$

Substituting the above expressions into the variational identity (11), we can obtain

$$\frac{\delta}{\delta \overline{u}} \int \left(-\frac{1}{2}bq\eta_1 + \frac{1}{2}cq\eta_1 - \frac{1}{2}eq\eta_2 + \frac{1}{2}fq\eta_2 - \frac{1}{2}br_2\eta_2 + \frac{1}{2}cr_2\eta_2 - \frac{1}{2}hq\eta_3 + \frac{1}{2}kq\eta_3 - \frac{1}{2}e\delta r_2\eta_3 + \frac{1}{2}f\delta r_2\eta_3 - \frac{1}{2}bs_2\eta_3 + \frac{1}{2}cs_2\eta_3 \right) dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma} \begin{pmatrix} -2a\eta_1 - 2d\eta_2 - 2g\eta_3 \\ -\frac{1}{2}b\lambda\eta_1 + \frac{1}{2}c\lambda\eta_1 - \frac{1}{2}e\lambda\eta_2 + \frac{1}{2}f\lambda\eta_2 - \frac{1}{2}h\lambda\eta_3 + \frac{1}{2}k\lambda\eta_3 \\ -2a\eta_2 - 2d\delta\eta_3 \\ -\frac{1}{2}b\lambda\eta_2 + \frac{1}{2}c\lambda\eta_2 - \frac{1}{2}e\delta\lambda\eta_3 + \frac{1}{2}f\delta\lambda\eta_3 \\ -2a\eta_3 \\ -\frac{1}{2}b\lambda\eta_3 + \frac{1}{2}c\lambda\eta_3 \end{pmatrix}. \tag{73}$$

Comparing the coefficients of all powers of λ in (73), we know that

$$\frac{\delta}{\delta \overline{u}} \int \frac{z_m}{1-2m} dx = \begin{pmatrix} -4a_m\eta_1 - 4d_m\eta_2 - 4g_m\eta_3 \\ -b_m\eta_1 + c_m\eta_1 - e_m\eta_2 + f_m\eta_2 - h_m\eta_3 + k_m\eta_3 \\ -4a_m\eta_2 - 4d_m\delta\eta_3 \\ -b_m\eta_2 + c_m\eta_2 - e_m\delta\eta_3 + f_m\delta\eta_3 \\ -4a_m\eta_3 \\ -b_m\eta_3 + c_m\eta_3 \end{pmatrix}, \quad m \geq 0, \tag{74}$$

where

$$\begin{aligned} z_m = {} & -b_m q\eta_1 + c_m q\eta_1 - e_m q\eta_2 + f_m q\eta_2 - b_m r_2\eta_2 \\ & + c_m r_2\eta_2 - h_m q\eta_3 + k_m q\eta_3 - e_m\delta r_2\eta_3 \\ & + f_m\delta r_2\eta_3 - b_m s_2\eta_3 + c_m s_2\eta_3. \end{aligned} \tag{75}$$

Setting $m = 1$ in (74), we know that the constant $\gamma = 1$, so we have

$$\frac{\delta \mathscr{H}_m}{\delta u} = \begin{pmatrix} -4a_m\eta_1 - 4d_m\eta_2 - 4g_m\eta_3 \\ -b_m\eta_1 + c_m\eta_1 - e_m\eta_2 + f_m\eta_2 - h_m\eta_3 + k_m\eta_3 \\ -4a_m\eta_2 - 4d_m\delta\eta_3 \\ -b_m\eta_2 + c_m\eta_2 - e_m\delta\eta_3 + f_m\delta\eta_3 \\ -4a_m\eta_3 \\ -b_m\eta_3 + c_m\eta_3 \end{pmatrix}. \tag{76}$$

Thus, the Hamiltonian functions can be expressed as

$$\mathscr{H}_m = \int \frac{z_m}{1-2m} dx \tag{77}$$

and the Hamiltonian structures of (58) as

$$u_{t_m} = J\frac{\delta \mathscr{H}_{m+1}}{\delta u}, \quad m \geq 0. \tag{78}$$

Furthermore, by using (52), we can get

$$u_{t_m} = J\frac{\delta \mathscr{H}_{m+1}}{\delta u} = M\frac{\delta \mathscr{H}_m}{\delta u}, \tag{79}$$

where

$$\overline{M} = \begin{pmatrix} 0 & 0 & \overline{J}_1 \\ 0 & \frac{1}{\delta}\overline{J}_1 & \overline{J}_2 \\ \overline{J}_1 & \overline{J}_2 & \overline{J}_3 \end{pmatrix},$$

$$\overline{J}_1 = \begin{pmatrix} \overline{j}_1^1 & 0 \\ 0 & \overline{j}_1^4 \end{pmatrix},$$

$$\overline{J}_2 = \begin{pmatrix} \overline{j}_2^1 & \overline{j}_2^2 \\ \overline{j}_2^3 & \overline{j}_2^4 \end{pmatrix},$$

$$\overline{J}_3 = \begin{pmatrix} \overline{j}_3^1 & \overline{j}_3^2 \\ \overline{j}_3^3 & \overline{j}_3^4 \end{pmatrix},$$

$$\overline{j}_1^1 = \frac{1}{4\eta_3}\partial\frac{1}{q}\partial\frac{1}{q}\partial - \frac{1}{\eta_3}\partial\frac{p}{q}\partial^{-1}\frac{p}{q}\partial,$$

$$\overline{j}_1^4 = 2\frac{1}{\eta_3}\partial - \frac{1}{2\eta_3}\partial\frac{1}{p}\partial\frac{1}{p}\partial,$$

$$\begin{aligned} \overline{j}_2^1 = {} & -\frac{1}{4\eta_3}\partial\frac{r_2}{q^2}\partial\frac{1}{q}\partial - \frac{1}{4\eta_3}\partial\frac{1}{q}\partial\frac{r_2}{q^2}\partial + \frac{1}{\eta_3}\partial\frac{p}{q}\partial^{-1}\frac{pr_2}{q^2}\partial \\ & + \frac{1}{\eta_3}\partial\frac{pr_2}{q^2}\partial^{-1}\frac{p}{q}\partial - \frac{1}{\eta_3}\partial\frac{p}{q}\partial^{-1}\frac{r_1}{q}\partial - \frac{1}{\eta_3}\partial\frac{r_1}{q}\partial^{-1}\frac{p}{q}\partial \\ & + \frac{\eta_2}{\delta\eta_3\eta_3}\partial\frac{p}{q}\partial^{-1}\frac{p}{q}\partial - \frac{\eta_2}{4\delta\eta_3\eta_3}\partial\frac{1}{q}\partial\frac{1}{q}\partial, \end{aligned}$$

$$\overline{j}_2^2 = -\frac{1}{2\eta_3}\partial\frac{r_1}{pq}\partial\frac{1}{p}\partial + 2\frac{1}{\eta_3}\partial\frac{r_1}{q},$$

$$\overline{j}_2^3 = -\frac{1}{2\eta_3}\partial\frac{1}{p}\partial\frac{r_1}{pq}\partial + 2\frac{1}{\eta_3}\frac{r_1}{q}\partial,$$

$$\begin{aligned} \overline{j}_2^4 = {} & 2\frac{1}{\eta_3}\partial\frac{r_1}{p} + 2\frac{1}{\eta_3}\frac{r_1}{p}\partial - 2\frac{\eta_2}{\delta\eta_3\eta_3}\partial \\ & + \frac{\eta_2}{2\delta\eta_3\eta_3}\partial\frac{1}{p}\partial\frac{1}{p}\partial, \end{aligned}$$

$$\begin{aligned} \overline{j}_3^1 = {} & -\frac{1}{4\eta_3}\Big(4\partial\frac{p}{q}\partial^{-1}\frac{s_1}{q}\partial + 4\partial\frac{s_1}{q}\partial^{-1}\frac{p}{q}\partial - \delta\partial\frac{1}{q}\partial\frac{r_2r_2}{q^3}\partial \\ & - \delta\partial\frac{r_2r_2}{q^3}\partial\frac{1}{q}\partial + \partial\frac{1}{q}\partial\frac{s_2}{q^2}\partial + \partial\frac{s_2}{q^2}\partial\frac{1}{q}\partial \\ & - 4\delta\partial\frac{r_1}{q}\partial^{-1}\frac{pr_2}{q^2}\partial - 4\delta\partial\frac{pr_2}{q^2}\partial^{-1}\frac{r_1}{q}\partial \\ & + 4\delta\partial\frac{r_1}{q}\partial^{-1}\frac{r_1}{q}\partial - \delta\partial\frac{r_2}{q^2}\partial\frac{r_2}{q^2}\partial + 4\delta\partial\frac{pr_2}{q^2}\partial^{-1}\frac{pr_2}{q^2}\partial \\ & + 4\delta\partial\frac{pr_2r_2}{q^3}\partial^{-1}\frac{p}{q}\partial + 4\delta\partial\frac{p}{q}\partial^{-1}\frac{pr_2r_2}{q^3}\partial \\ & - 4\partial\frac{ps_2}{q^2}\partial^{-1}\frac{p}{q}\partial - 4\partial\frac{p}{q}\partial^{-1}\frac{ps_2}{q^2}\partial - 4\delta\partial\frac{p}{q}\partial^{-1}\frac{r_1r_2}{q^2}\partial \end{aligned}$$

$$
\begin{aligned}
&- 4\delta\partial\frac{r_1r_2}{q^2}\partial^{-1}\frac{p}{q}\partial + \frac{\eta_1}{\eta_3}\partial\frac{1}{q}\partial\frac{1}{q}\partial - \frac{\eta_2\eta_2}{\delta\eta_3\eta_3}\partial\frac{1}{q}\partial\frac{1}{q}\partial \\
&- 4\frac{\eta_1}{\eta_3}\partial\frac{p}{q}\partial^{-1}\frac{p}{q}\partial + 4\frac{\eta_2\eta_2}{\delta\eta_3\eta_3}\partial\frac{p}{q}\partial^{-1}\frac{p}{q}\partial \\
&- \frac{\eta_2}{\eta_3}\partial\frac{r_2}{q^2}\partial\frac{1}{q}\partial - \frac{\eta_2}{\eta_3}\partial\frac{1}{q}\partial\frac{r_2}{q^2}\partial - 4\frac{\eta_2}{\eta_3}\partial\frac{p}{q}\partial^{-1}\frac{r_1}{q}\partial \\
&- 4\frac{\eta_2}{\eta_3}\partial\frac{r_1}{q}\partial^{-1}\frac{p}{q}\partial + 2\delta\partial\frac{r_1}{pq}\partial\frac{r_1}{pq}\partial \\
&+ 4\frac{\eta_2}{\eta_3}\partial\frac{p}{q}\partial^{-1}\frac{pr_2}{q^2}\partial + 4\frac{\eta_2}{\eta_3}\partial\frac{pr_2}{q^2}\partial^{-1}\frac{p}{q}\partial\Big), \\
\bar{j}_3^2 = {} & \frac{1}{2\eta_3}\delta\partial\frac{r_1r_2}{pq^2}\partial\frac{1}{p}\partial - 2\frac{1}{\eta_3}\delta\partial\frac{r_1r_2}{q^2} + 2\frac{1}{\eta_3}\delta\partial\frac{r_1r_1}{pq} \\
&- \frac{1}{2\eta_3}\partial\frac{s_1}{pq}\partial\frac{1}{p}\partial + 2\frac{1}{\eta_3}\partial\frac{s_1}{q} + \frac{1}{2}\frac{\eta_2}{\eta_3\eta_3}\partial\frac{r_1}{pq}\partial\frac{1}{p}\partial - 2 \\
&\cdot \frac{\eta_2}{\eta_3\eta_3}\partial\frac{r_1}{q}, \\
\bar{j}_3^3 = {} & \frac{1}{2\eta_3}\delta\partial\frac{1}{p}\partial\frac{r_1r_2}{pq^2}\partial - 2\frac{1}{\eta_3}\delta\frac{r_1r_2}{q^2}\partial + 2\frac{1}{\eta_3}\delta\frac{r_1r_1}{pq}\partial \\
&- \frac{1}{2\eta_3}\partial\frac{1}{p}\partial\frac{s_1}{pq}\partial + 2\frac{1}{\eta_3}\frac{s_1}{q}\partial + \frac{1}{2}\frac{\eta_2}{\eta_3\eta_3}\partial\frac{1}{p}\partial\frac{r_1}{pq}\partial - 2 \\
&\cdot \frac{\eta_2}{\eta_3\eta_3}\frac{r_1}{q}\partial, \\
\bar{j}_3^4 = {} & -2\frac{1}{\eta_3}\partial\frac{s_1}{p} - 2\frac{1}{\eta_3}\frac{s_1}{p}\partial + \frac{\eta_1}{2\eta_3\eta_3}\partial\frac{1}{p}\partial\frac{1}{p}\partial + 2 \\
&\cdot \frac{\eta_2}{\eta_3\eta_3}\partial\frac{r_1}{p} + 2\frac{\eta_2}{\eta_3\eta_3}\frac{r_1}{p}\partial - \frac{\eta_2\eta_2}{2\delta\eta_3\eta_3\eta_3}\partial\frac{1}{p}\partial\frac{1}{p}\partial - 2 \\
&\cdot \frac{\eta_2\eta_2}{\delta\eta_3\eta_3\eta_3}\partial + 2\frac{\eta_1}{\eta_3\eta_3}\partial.
\end{aligned}
\tag{80}
$$

By a complicated computation, we know that

$$
\begin{aligned}
\bar{J} &= -\bar{J}^* \\
\overline{M} &= -\overline{M}^*;
\end{aligned}
\tag{81}
$$

namely, the hierarchy (58) possess bi-Hamiltonian structures

$$
u_{t_m} = J\frac{\delta\mathscr{H}_{m+1}}{\delta u} = M\frac{\delta\mathscr{H}_m}{\delta u}, \quad m \geq 0, \tag{82}
$$

where $\bar{J}$ and $\overline{M}$ are the Hamiltonian pairs of the integrable couplings of the 2-CQLTH. So the integrable couplings (58) are Liouville integrable.

6. Conclusion

In this paper, we construct a new eigenvalue problem and give the two-component Casimir-Qiao-Liu type hierarchy (2-CQLTH). The 2-CQLTH has Lax representation and possesses bi-Hamiltonian structures, which implies that it is integrable in both Lax and Liouville sense. Furthermore, we obtain the integrable couplings of the integrable hierarchy as well as its Lax pairs and bi-Hamiltonian structures and prove that the integrable couplings are also integrable in both Lax and Liouville senses.

Competing Interests

The authors declare that they have no competing interests.

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