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Research Article

Existence and Stability of Standing Waves for Nonlinear Fractional Schrödinger Equations with Hartree Type and Power Type Nonlinearities

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We consider the standing wave solutions for nonlinear fractional Schrödinger equations with focusing Hartree type and power type nonlinearities. We first establish the constrained minimization problem via applying variational method. Under certain conditions, we then show the existence of standing waves. Finally, we prove that the set of minimizers for the initial value problem of this minimization problem is stable.

1. Introduction

In the paper, we study the following nonlinear fractional Schödinger equations:

$$\begin{aligned} & i\psi_{1t} + (-\Delta)^{\alpha}\psi_1 - \left(|\cdot|^{-\gamma} * |\psi_2|^2\right)\psi_1 \\ & \quad = |\psi_1|^{p/2-2}|\psi_2|^{p/2}\psi_1, \\ & i\psi_{2t} + (-\Delta)^{\alpha}\psi_2 - \left(|\cdot|^{-\gamma} * |\psi_1|^2\right)\psi_2 \\ & \quad = |\psi_2|^{p/2-2}|\psi_1|^{p/2}\psi_2, \end{aligned} \tag{1}$$

where $0 < \alpha < 1$, $i = \sqrt{-1}$, $2 < p < 2 + 4\alpha/d$, $0 < \gamma < 2\alpha$, $\psi_i(x,t)$ is a complex-valued function on $\mathbb{R}^d \times \mathbb{R}$, $d \geqslant 1$, and $|\cdot|^{-\gamma} * |\psi_i|^2 := \int_{\mathbb{R}^d} |\psi_i(y)|^2/|x-y|^{\gamma}dy$, where $*$ denotes the convolution. The fractional Laplacian $(-\Delta)^{\alpha}$ is a nonlocal operator defined as

$$F\left[(-\Delta)^{\alpha}\psi_i\right](\xi) = |\xi|^{2\alpha}F\psi_i(\xi), \tag{2}$$

where the Fourier transform is defined by

$$F\psi_i(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\psi_i(x)e^{-i\xi\cdot x}dx. \tag{3}$$

Equations (1) appear in statistical physics; this problem stems from the Slater semirelativistic situation of Hartree-Fock model for particle interacting with each other via the Coulomb law; see [1]. The power type nonlinearity reflects the exchange effect produced by the Pauli principle, and the Hartree type nonlinearity describes the Coulomb effect of particles exclusion.

The fractional Schrödinger equations have widely applied to physical and other areas and have attracted much attention of researchers, especially in fractional quantum mechanics. Laskin spread the fractional operator to quantum mechanics and formulated Schrödinger equation (see [2–4]). In [5–7], equations with fractional Laplacians have been studied recently as the Lévy processes appear widely in physics, chemistry, and biology. Equation in [8] can describe ground state solutions for the L^2-critical boson star equation (when $\alpha = 1/2$). The authors in [9] considered the orbital stability of standing waves for classical nonlinear Schrödinger equations (when $\alpha = 1$). The Cauchy problems of systems of Schrödinger equations are important issues which have been studied by many researchers; see [10–13].

The existence and stability of standing waves is a very important topic of fractional Schrödinger equations. The standing waves have been raised in various fields of physics, for example, plasma physics, constructive field theory, nonlinear optics, and so on. Recently, the authors in [12–15] have been concerned with the fractional Schrödinger equations with Hartree type nonlinearity; they obtained a series of results about existence, continuity, and stability of standing waves. The results about this topic for power type nonlinearity have been studied by [16, 17]. In this paper, we are interested in considering the existence and stability of standing waves for nonlinear fractional Schrödinger equations with combined nonlinearities of Hartree type and power type.

It is well known that a standing wave for (1) is a solution of the form $(\psi_1(x,t),\psi_2(x,t)) = (e^{i\omega_1 t}u_1(x), e^{i\omega_2 t}u_2(x))$, where $\omega_i \in \mathbb{R}$ $(i=1,2)$. Therefore, it is easy to see that get a standing wave of (1) is equivalent to solving the following equations for $(u_1(x),u_2(x))$:

$$\begin{aligned}
&(-\Delta)^{\alpha} u_1 - \left(|\cdot|^{-\gamma} * |u_2|^2\right) u_1 - |u_1|^{p/2-2} |u_2|^{p/2} u_1 \\
&\quad = \omega_1 u_1, \quad x \in \mathbb{R}^d, \\
&(-\Delta)^{\alpha} u_2 - \left(|\cdot|^{-\gamma} * |u_1|^2\right) u_2 - |u_2|^{p/2-2} |u_1|^{p/2} u_2 \\
&\quad = \omega_2 u_2, \quad x \in \mathbb{R}^d,
\end{aligned} \tag{4}$$

where u_1, u_2 are complex-valued functions. In order to study the existence of solutions to (4), by the variational method, we consider the following constrained minimization problem:

$$\begin{aligned}
E_q \coloneqq \inf \{ & E(u_1,u_2);\ (u_1,u_2) \in H^{\alpha}(\mathbb{R}^d) \\
& \times H^{\alpha}(\mathbb{R}^d),\ M(u_1,u_2) = q \},
\end{aligned} \tag{5}$$

where $M(u_1,u_2)$ and $E(u_1,u_2)$ are defined by

$$\begin{aligned}
M(u_1,u_2) &= \int_{\mathbb{R}^d} \left(|u_1(x)|^2 + |u_2(x)|^2\right) dx, \\
E(u_1,u_2) &= \frac{1}{2} \int_{\mathbb{R}^d} \left(\left|(-\Delta)^{\alpha/2} u_1(x)\right|^2 \right. \\
&\quad \left. + \left|(-\Delta)^{\alpha/2} u_2(x)\right|^2 \right) dx - \frac{1}{4} \\
&\quad \cdot \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{|x-y|^{\gamma}} \left(|u_1(x)|^2 |u_2(y)|^2 \right. \\
&\quad \left. + |u_2(x)|^2 |u_1(y)|^2 \right) dx\,dy - \frac{2}{p} \\
&\quad \cdot \int_{\mathbb{R}^d} |u_1|^{p/2} |u_2|^{p/2} dx.
\end{aligned} \tag{6}$$

Denote the set of the minimizers of problem (5) by

$$\begin{aligned}
G_q \coloneqq \{ & (u_1,u_2) \in H^{\alpha}(\mathbb{R}^d) \times H^{\alpha}(\mathbb{R}^d);\ E(u_1,u_2) \\
& = E_q,\ M(u_1,u_2) = q \}.
\end{aligned} \tag{7}$$

We define the fractional order Sobolev space $H_p^{\alpha}(\mathbb{R}^d) \coloneqq \{u \in L^p : F^{-1}[(1+|\xi|^2)^{\alpha/2}Fu] \in L^p\}$; its norm is $\|u\|_{\alpha,p} = \|F^{-1}[(1+|\xi|^2)^{\alpha/2}Fu]\|_p$. We always write $H^{\alpha}(\mathbb{R}^d) = H_2^{\alpha}(\mathbb{R}^d)$ for brevity; we denote $\|\cdot\|_{H^{\alpha}(\mathbb{R}^d)} = \|\cdot\|$ and $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{R}^d)}$ in the following.

The following is the main conclusion of the paper.

Theorem 1. *Let $d \geqslant 1, 0 < \alpha < 1, 2 = 1, 0 < \alpha < 1, 2$ and $0 < \gamma < \min\{2\alpha, d\}$. If $\{(u_{1n}, u_{2n})\}$ is a minimizing sequence of problem (5), then there exists a sequence $\{y_n\} \subset \mathbb{R}^d$ such that $\{(u_{1n}(\cdot - y_n), u_{2n}(\cdot - y_n))\}$ contents a convergent subsequence. In particular, there exists a minimizer for problem (5), which implies G_q is not an empty set, and we have*

$$\lim_{n\to+\infty} \inf_{(g_1,g_2)\in G_q} \left\|(u_{1n},u_{2n}) - (g_1,g_2)\right\| = 0. \tag{8}$$

Theorem 2. *Under the assumptions of Theorem 1, the set G_q is $H^{\alpha}(\mathbb{R}^d)\times H^{\alpha}(\mathbb{R}^d)$-stable with respect to (1); that is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if the initial condition (u_{10}, u_{20}) in (1) satisfies*

$$\inf_{(g_1,g_2)\in G_q} \left\|(u_{10},u_{20}) - (g_1,g_2)\right\| < \delta. \tag{9}$$

Then for any $t \in [0,T)$,

$$\inf_{(g_1,g_2)\in G_q} \left\|(u_1(\cdot,t),u_2(\cdot,t)) - (g_1,g_2)\right\| < \varepsilon, \tag{10}$$

where $(u_1,u_2) \in C([0,T), H^{\alpha}(\mathbb{R}^d)\times H^{\alpha}(\mathbb{R}^d))$ is a solution of (1) corresponding to the initial condition (u_{10},u_{20}).

Remark 3. By the Hermiticity of the fractional Schrödinger operator in [4], we obtain that the solution $(\psi_1(x,t),\psi_2(x,t))$ of (1) with initial value $(\psi_{10},\psi_{20}) = (\psi_1(x,0),\psi_2(x,0))$ satisfies the following conservation laws:

(1) Conservation of mass:

$$\begin{aligned}
M(\psi_1(t),\psi_2(t)) &= \int_{\mathbb{R}^d} \left(|\psi_1(t,x)|^2 + |\psi_2(t,x)|^2\right) dx \\
&= M(\psi_{10},\psi_{20})
\end{aligned} \tag{11}$$

(2) Conservation of energy:

$$\begin{aligned}
E(\psi_1(t),\psi_2(t)) &= \frac{1}{2} \int_{\mathbb{R}^d} \left(\left|(-\Delta)^{\alpha/2} \psi_1(t,x)\right|^2 \right. \\
&\quad \left. + \left|(-\Delta)^{\alpha/2} \psi_2(t,x)\right|^2 \right) dx - \frac{1}{4} \\
&\quad \cdot \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{|x-y|^{\gamma}} \left(|\psi_1(t,x)|^2 |\psi_2(t,y)|^2 \right. \\
&\quad \left. + |\psi_2(t,x)|^2 |\psi_1(t,y)|^2 \right) dx\,dy - \frac{2}{p} \\
&\quad \cdot \int_{\mathbb{R}^d} |\psi_1(t,x)|^{p/2} |\psi_2(t,x)|^{p/2} dx = E(\psi_{10},\psi_{20})
\end{aligned} \tag{12}$$

The conservations are very important to the proof of the $H^{\alpha}(\mathbb{R}^d)\times H^{\alpha}(\mathbb{R}^d)$-stability.

2. Preliminaries

In the section, we will list some lemmas, which will have great effects for the following proofs.

Lemma 4. *Let $0 < \alpha < 1$; there are two properties with $H^{\alpha}(\mathbb{R}^d)$.*

(i) The norm $\|\cdot\|_{\alpha,2}$ of $H^{\alpha}(\mathbb{R}^d)$ is equivalent to

$$\|\cdot\| = \left\| F^{-1}\left[\left(1+|\xi|^{\alpha}\right)F\cdot\right]\right\|_2 = \|\cdot\|_2 + \|\cdot\|_{\dot{H}^{\alpha}(\mathbb{R}^d)}. \tag{13}$$

This result follows easily from the fundamental inequality

$$1+|\xi|^{\alpha} \leqslant \left(1+|\xi|^2\right)^{\alpha/2} \leqslant C\left(1+|\xi|^{\alpha}\right) \tag{14}$$

and the definitions of $\|\cdot\|_{\alpha,2}$ and $\|\cdot\|$.

(ii) $\forall u \in H^{\alpha}(\mathbb{R}^d)$,

$$c\|u\|_{W^{\alpha,2}(\mathbb{R}^d)} \leqslant \|u\| \leqslant C\|u\|_{W^{\alpha,2}(\mathbb{R}^d)}, \tag{15}$$

which implies $H^{\alpha}(\mathbb{R}^d) = W^{\alpha,2}(\mathbb{R}^d)$, where $W^{\alpha,2}(\mathbb{R}^d)$ is defined by the trace interpolation (see [18]), and

$$\|u\|_{W^{\alpha,2}(\mathbb{R}^d)} = \left(|u|_2^2 + \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\alpha}}\,dy\,dx\right)^{1/2}. \tag{16}$$

Lemma 5. *If $0 < \alpha < 1$, $x \in \mathbb{R}^d$ and $u \in S$, S represent the Schwartz class; then the fractional Laplacian $(-\Delta)^{\alpha}$ of u is also expressed by the formula*

$$(-\Delta)^{\alpha} u(x) = C_{d,\alpha} P.V. \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2\alpha}}\,dy, \tag{17}$$

where P.V. means the Cauchy principal value on the integral and $C_{d,\alpha}$ is some positive normalization constant.

In Lemma 5, the other definition of the fractional Laplacian is given. The proof for the equivalence of two definitions of the fractional Laplacian can be found in [7], so we omit the details.

Lemma 6 ((Hardy's inequality)(see [19])). *For $0 < \gamma < d$, we have*

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x-y|^{\gamma}}\,dx \leqslant C\|u\|^2_{\dot{H}^{\gamma/2}}, \tag{18}$$

where the constant C depends on d and γ.

The following commutator estimates were developed in [17] by using Katö and Ponce's result in [20].

Lemma 7 (commutator estimates). *If $0 < \alpha < 1$, $f, g \in S$, the Schwartz class, then*

$$\begin{aligned}\left\|(-\Delta)^{\alpha/2}(fg) - f(-\Delta)^{\alpha/2} g\right\|_2 &\leqslant C\left(\|\nabla f\|_{p_1}\left\|(-\Delta)^{(\alpha-1)/2} g\right\|_{q_1}\right.\\ &\quad \left.+ \left\|(-\Delta)^{\alpha/2} f\right\|_{p_2}\|g\|_{q_2}\right),\end{aligned} \tag{19}$$

where $q_1, p_2 \in [2,+\infty)$ satisfying $1/p_1 + 1/q_1 = 1/p_2 + 1/q_2 = 1/2$.

Lemma 8 ((fractional Rellich compactness theorem) (see [21])). *Let $0 < \alpha < 1$ and $2 \leqslant q < 2d/(d-2\alpha)$. If Ω is a bounded domain, then every bounded sequence $\{u_n\} \in H^{\alpha}(\mathbb{R}^d)$ has a convergent subsequence in $L^q(\Omega)$.*

Lemma 9 (see [18]). *For $0 < \alpha < 1$, $2 \leqslant p_1 \leqslant 2d/(d-2\alpha)$, $H^{\alpha}(\mathbb{R}^d) \hookrightarrow L^{p_1}(\mathbb{R}^d)$ and $W^{\alpha,2}(B(y,r)) \hookrightarrow L^{p_1}(B(y,r))$ for $y \in \mathbb{R}^d$ and $r > 0$, where $B(y,r) = \{x; x \in \mathbb{R}^d, |x-y| < r\}$. Moreover, for $2 \leqslant p < 2d/(d-2\alpha)$, if $u_n \to u$ in $H^{\alpha}(\mathbb{R}^d)$, then $u_n \rightharpoonup u$ in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$, and $u_n \to u$ a.e. in $\mathbb{R}^d$.*

Lemma 10. *Suppose $\{(u_{1n}, u_{2n})\}$ is a minimizing sequence for the problem (5) satisfying*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^d} \int_{B(y,r)} \left(|u_{1n}|^2 + |u_{2n}|^2\right) dx = 0 \quad \text{for some } r > 0. \tag{20}$$

Then we have

$$\begin{aligned}&\lim_{n\to\infty}(u_{1n}, u_{2n}) = (0,0),\\ &\lim_{n\to\infty} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{|x-y|^{\gamma}}\left(|u_{1n}(x)|^2|u_{2n}(y)|^2\right.\\ &\quad \left.+ |u_{2n}(x)|^2|u_{1n}(y)|^2\right) dx\,dy = 0,\end{aligned} \tag{21}$$

in $L^p(\mathbb{R}^d)\times L^p(\mathbb{R}^d)$, where $2 < p < 2d/(d-2\alpha)$.

Proof. For $(u_1, u_2) \in H^{\alpha}(\mathbb{R}^d)\times H^{\alpha}(\mathbb{R}^d)$, let $2 < p < 2d/(d-2\alpha)$; by the Hölder inequality and Lemma 9, we deduce that

$$\begin{aligned}\|u_{in}\|_{L^p(B(y,r))} &\leqslant \|u_{in}\|^{1-\lambda}_{L^2(B(y,r))}\|u_{in}\|^{\lambda}_{L^{2d/(d-2\alpha)}(B(y,r))}\\ &\leqslant c\|u_{in}\|^{1-\lambda}_{L^2(B(y,r))}\|u_{in}\|^{\lambda}_{W^{\alpha,2}(B(y,r))} \leqslant c\|u_{in}\|^{1-\lambda}_{L^2(B(y,r))}\\ &\quad\cdot\left(\int_{B(y,r)}|u_{in}(x)|^2\,dx\right.\\ &\quad\left.+\iint_{B(y,r)\times B(y,r)}\frac{|u_{in}(x)-u_{in}(z)|^2}{|x-z|^{d+2\alpha}}\,dz\,dx\right)^{\lambda/2},\\ &\qquad i = 1,2,\end{aligned} \tag{22}$$

where $(1-\lambda)/2 + \lambda/(2d/(d-2\alpha)) = 1/p$. Let $\lambda = 2/p$; from (22), we have

$$\|u_{in}\|^p_{L^p(B(y,r))} \leqslant c\|u_{in}\|^{(1-1/p)p}_{L^2(B(y,r))}\|u_{in}\|^2_{W^{\alpha,2}(B(y,r))}, \quad i = 1,2. \tag{23}$$

Next we divide the domain. Covering $\mathbb{R}^d$ by countable balls $\{B(y_i,r)\}$ in such a way that every point of $\mathbb{R}^d$ belongs to at most $d+1$ balls, by (23) and Lemma 4, we get

$$\begin{aligned}
&\|u_{in}\|_p^p \\
&\leqslant c(d+1)\left(\sup_{y\in\mathbb{R}^d}\|u_{in}\|_{L^2(B(y,r))}\right)^{(1-2/p)p}\|u_{in}\|^2_{W^{\alpha,2}(\mathbb{R}^d)} \\
&\leqslant c(d+1)\left(\sup_{y\in\mathbb{R}^d}\|u_{in}\|_{L^2(B(y,r))}\right)^{(1-2/p)p}\|u_{in}\|^2, \\
&\qquad i=1,2.
\end{aligned} \tag{24}$$

Since $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^d}\int_{B(y,r)}(|u_{1n}(x)|^2+|u_{2n}(x)|^2)dx=0$ for some $r>0$, applying (24), then we obtain $\lim_{n\to\infty}(u_{1n},u_{2n})=(0,0)$ in $L^p(\mathbb{R}^d)\times L^p(\mathbb{R}^d)$. For every $\varepsilon>0$, we have $r_\varepsilon>0$ such that

$$\begin{aligned}
&\iint_{|x-y|\geqslant r_\varepsilon}\frac{1}{|x-y|^\gamma}|u_{in}(x)|^2|u_{jn}(y)|^2\,dx\,dy\leqslant\frac{\varepsilon}{2}, \\
&\qquad i,j=1,2,\ i\neq j.
\end{aligned} \tag{25}$$

Since we have divided the domain, we know that

$$\sum_{i=1}^{\infty}\int_{B(y_i,r)}|u_{in}(x)|^2\,dx\leqslant(d+1)\|u_{in}\|_2^2. \tag{26}$$

If x in some $B(y_i,r)$, $|x-y|\leqslant r_\varepsilon$, then there exists at most N_ε balls such that

$$\left\{y\in\mathbb{R}^d;\ |x-y|\leqslant r_\varepsilon,\ x\in B(y_i,r)\right\}\subset\sum_{k=1}^{N_\varepsilon}B\left(y_{i_k},r\right), \tag{27}$$

where N_ε only depends on ε. Then by Hölder and Hardy's inequality, we can get

$$\begin{aligned}
&\iint_{|x-y|\leqslant r_\varepsilon}\frac{1}{|x-y|^\gamma}|u_{in}(x)|^2|u_{jn}(y)|^2\,dx\,dy\leqslant\sum_{i=1}^{\infty}\int_{B_x(y_i,r)}\left[\sum_{k=1}^{N_\varepsilon}\int_{B_y(y_{i_k},r)}\frac{1}{|x-y|^\gamma}|u_{in}(x)|^2|u_{jn}(y)|^2\,dy\right]dx \\
&\leqslant\sum_{i=1}^{\infty}\|u_{in}(x)\|^2_{L^2(B_x(y_i,r))}\left[\sum_{k=1}^{N_\varepsilon}\sup_{x\in B_x(y_i,r)}\int_{B_y(y_{i_k},r)}\frac{1}{|x-y|^\gamma}|u_{jn}(y)|^2\,dy\right] \\
&\leqslant\sum_{i=1}^{\infty}\|u_{in}(x)\|^2_{L^2(B_x(y_i,r))}\sum_{k=1}^{N_\varepsilon}\sup_{x\in B_x(y_i,r)}\|u_{in}(x)\|^{2-\gamma/\alpha}_{L^2(B_x(y_{i_k},r))}\left(\int_{B_y(y_{i_k},r)}\frac{1}{|x-y|^\gamma}|u_{jn(y)}|^2\,dy\right)^{\gamma/2\alpha} \\
&\leqslant cN_\varepsilon\|u_{in}(x)\|^{\gamma/2\alpha}\left(\sum_{i=1}^{\infty}\|u_{in}(x)\|^2_{L^2(B_x(y_i,r))}\right)\left(\sup_{y\in\mathbb{R}^d}\int_{B(y,r)}|u_{in}(x)|^2\,dx\right)^{1-\gamma/2\alpha} \\
&\leqslant c(d+1)N_\varepsilon\|u_{in}\|_2^2\|u_{in}(x)\|^{\gamma/2\alpha}\left(\sup_{y\in\mathbb{R}^d}\int_{B(y,r)}|u_{in}(x)|^2\,dx\right)^{1-\gamma/2\alpha},\quad i,j=1,2,\ i\neq j.
\end{aligned} \tag{28}$$

If we take n to infinity, we find the second part can also be bounded by $\varepsilon/2$, and we finish the proof of the lemma. □

Theorem 11. *Let $d\geqslant 1$, $0<\alpha<1$, $0<\gamma<\min\{2\alpha,d\}$. If $(\psi_{10},\psi_{20})\in H^\alpha(\mathbb{R}^d)\times H^\alpha(\mathbb{R}^d)$, then there exists a global solution $(\psi_1(x,t),\psi_2(x,t))\in C(0,+\infty;H^\alpha(\mathbb{R}^d)\times H^\alpha(\mathbb{R}^d))$ to the Cauchy problem of nonlinear fractional Schrödinger equations (1) with the initial date $(\psi_1(x,0),\psi_2(x,0))=(\psi_{10}(x),\psi_{20}(x))$.*

Remark 12. The authors in [22] concerned with a class of systems of fractional nonlinear Schrödinger equations. By Faedo-Galërkin method, they have obtained the existence and uniqueness of the global solution to the periodic boundary value problem. In the proof of Theorem 4.1 in [22], let the period $2\pi e_i\to+\infty$; we can get Theorem 11. In [10–13], many researchers have also studied the Cauchy problem of Schrödinger equations. Here we will prove Theorem 11 by the semigroup method.

Proof of Theorem 11. Let $A\psi_i=i(-\Delta)^\alpha\psi_i$, $i=1,2$. By Stone's Theorem [23], $-A$ is the infinitesimal generator of a C_0 group of unitary operator $S(t)$, $-\infty<t<+\infty$, on $L^2(\mathbb{R}^d)$. Therefore, (1) can be rewritten in the form of the integral equation

$$\begin{aligned}
\psi_i(t)&=S(t)\psi_{i0}+i\int_0^t S(t-s) \\
&\quad\cdot\left(|\psi_i|^{p/2-2}|\psi_j|^{p/2}+\left(|\cdot|^{-\gamma}*|\psi_j|^2\right)\right)\psi_i\,ds, \\
&\qquad i,j=1,2,\ i\neq j.
\end{aligned} \tag{29}$$

By Banach Fixed Point Theorem, for $T > 0$, we know that there exist local weak solutions $(\psi_1(t), \psi_2(t)) \in C(0, T; H^{\alpha}(\mathbb{R}^d) \times H^{\alpha}(\mathbb{R}^d))$ to (1) with initial data $(\psi_1(0), \psi_2(0))$. We can use the standard contraction mapping argument to prove the local existence of solutions; since the argument has been considered by many authors (see [10, 12, 13, 23]), we omit it. Next we will give the uniform estimates of local weak solutions. By the conservation laws, we have

$$\begin{aligned} &\left|\psi_i(t)\right|_2^2 = \left|\psi_{i0}\right|_2^2, \\ &E\left(\psi_1(t), \psi_2(t)\right) = E\left(\psi_{10}, \psi_{20}\right), \\ &\qquad i = 1, 2. \end{aligned} \tag{30}$$

By the Gagliardo-Nirenberg inequality, we obtain

$$\left|\psi_i\right|_p^p \leqslant C\left|(-\Delta)^{\alpha/2}\psi_i\right|_2^{\lambda p}\left|\psi_i\right|_2^{(1-\lambda)p}, \tag{31}$$

where λ satisfies $1/p = \lambda(1/2 - \alpha/d) + (1-\lambda)(1/2)$; that is, $\lambda = d(p-2)/2\alpha p$. Applying the Hardy's inequality and the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} &\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{|x-y|^{\gamma}}\left|\psi_i(t,x)\right|^2\left|\psi_j(t,y)\right|^2 dx\,dy \\ &\quad \leqslant C\left|(-\Delta)^{\alpha/2}\psi_i\right|_2^{\gamma}\left|\psi_j\right|_2^{4-\gamma}. \end{aligned} \tag{32}$$

Since $2 < p < 2 + 4\alpha/d$ implies that $0 < \lambda p < 2$, $0 < \gamma < \min\{2\alpha, d\}$, using Young inequality and combining (31) and (32), we have

$$\begin{aligned} &\int_{\mathbb{R}^d}\left(\left|(-\Delta)^{\alpha/2}\psi_1(t,x)\right|^2 + \left|(-\Delta)^{\alpha/2}\psi_2(t,x)\right|^2\right)dx \\ &\quad \leqslant CE\left(\psi_{10}, \psi_{20}\right) + C. \end{aligned} \tag{33}$$

From the above inequality and (30), for $0 \leqslant t < T$,

$$\left\|\psi_i\right\| \leqslant C. \tag{34}$$

This completes the proof of Theorem 11. □

3. Main Results

In the section, we will give proofs of our main results which are listed in the first section. Before going to the proofs of Theorems 1 and 2, we give some important lemmas.

Lemma 13. *For every $q > 0$, we have $-\infty < E_q < 0$.*

Proof. For given $(u_1, u_2) \in H^{\alpha}(\mathbb{R}^d) \times H^{\alpha}(\mathbb{R}^d)$, $\|u_1\|_2^2 + \|u_2\|_2^2 = q$, setting $u_{i\lambda}(x) = \lambda^{1/2}u_i(\lambda^{1/d}x)$ $(i = 1, 2)$, $\|u_{1\lambda}\|_2^2 + \|u_{2\lambda}\|_2^2 = q$. By Lemma 5, we obtain

$$\begin{aligned} E\left(u_{1\lambda}, u_{2\lambda}\right) &= \frac{1}{2}\lambda^{2\alpha/d}\int_{\mathbb{R}^d}\left(\left|(-\Delta)^{\alpha/2}u_1(x)\right|^2\right. \\ &\quad \left.+ \left|(-\Delta)^{\alpha/2}u_2(x)\right|^2\right)dx - \frac{1}{4} \\ &\quad \cdot \lambda^{\gamma/d}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{|x-y|^{\gamma}}\left(\left|u_1(x)\right|^2\left|u_2(y)\right|^2\right. \\ &\quad \left.+ \left|u_2(x)\right|^2\left|u_1(y)\right|^2\right)dx\,dy - \frac{2}{p} \\ &\quad \cdot \lambda^{p/2-1}\int_{\mathbb{R}^d}\left(\left|u_1(x)\right|^{p/2}\left|u_2(x)\right|^{p/2}\right)dx. \end{aligned} \tag{35}$$

Since $0 < \gamma < 2\alpha$, $0 < p/2 - 1 < 2\alpha/d$, we have $\lambda > 0$ sufficient small such that $E(u_{1\lambda}, u_{2\lambda}) < 0$. Therefore, $E_q < E(u_{1\lambda}, u_{2\lambda}) < 0$.

By Hardy's inequality, we can deduce that

$$\begin{aligned} &\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{|x-y|^{\gamma}}\left|u_i(x)\right|^2\left|u_j(y)\right|^2 dx\,dy \\ &\quad \leqslant \sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\left|u_i(x)\right|^2}{|x-y|^{\gamma}}dx\left\|u_j\right\|_2^2 \leqslant C\left\|u_i\right\|_{\dot{H}^{\gamma/2}}^2\left\|u_j\right\|_2^2, \\ &\qquad i, j = 1, 2,\ i \neq j. \end{aligned} \tag{36}$$

Sobolev's inequality and Young's inequality imply that

$$\begin{aligned} &\frac{1}{4}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{|x-y|^{\gamma}}\left|u_i(x)\right|^2\left|u_j(y)\right|^2 dx\,dy \\ &\quad \leqslant C\left\|u_i\right\|_{\dot{H}^{\alpha}}^{\gamma/\alpha}\left\|u_j\right\|_2^{4-\gamma/\alpha} \\ &\quad \leqslant \varepsilon\left\|u_i\right\|_{\dot{H}^{\alpha}}^2 + C(\varepsilon)\left\|u_j\right\|^{(8\alpha-2\gamma)/(2\alpha-\gamma)}, \\ &\qquad i, j = 1, 2,\ i \neq j, \end{aligned} \tag{37}$$

where ε is a sufficiently small positive constant. Using Hölder inequality and the Sobolev inequality, we have

$$\begin{aligned} &\left|u_i\right|_p \leqslant \left|u_i\right|_2^{(1-\theta)}\left|u_i\right|_{2d/(d-2\alpha)}^{\theta} \leqslant c\left|u_i\right|_2^{1-\theta}\left\|u_i\right\|^{\theta}, \\ &\qquad i = 1, 2, \end{aligned} \tag{38}$$

where $(1-\theta)p/2 + \theta p/(2d/(d-2\alpha)) = 1$, $\theta = d(p-2)/2\alpha p$. From Young inequality and $0 < \theta p < 2$, for $\eta > 0$, from (38), we can get

$$\left|u_i\right|_p^p \leqslant \eta\left\|u_i\right\|^2 + c(\eta)\left|u_i\right|_2^{2p(1-\theta)/(2-\theta p)}, \quad i = 1, 2. \tag{39}$$

Hence, for $(u_1, u_2) \in H^\alpha(\mathbb{R}^d) \times H^\alpha(\mathbb{R}^d)$, $\|u_1\|_2^2 + \|u_2\|_2^2 = q$, and sufficiently small $\varepsilon, \eta > 0$, we have

$$\begin{aligned} E(u_1, u_2) &\geqslant \frac{1}{2}\left(\|u_1\|^2 + \|u_2\|^2\right) - \frac{1}{2}q \\ &\quad - \left(\varepsilon + \frac{\eta}{p}\right)\left(\|u_1\|^2 + \|u_2\|^2\right) \\ &\quad - C(\varepsilon)\left(q_1^{(4\alpha-\gamma)/(2\alpha-\gamma)} + q_2^{(4\alpha-\gamma)/(2\alpha-\gamma)}\right) \\ &\quad - \frac{c(\eta)}{p}\left(q_1^{p(1-\theta)/(2-\theta p)} + q_2^{p(1-\theta)/(2-\theta p)}\right) \\ &\geqslant -\frac{1}{2}q \\ &\quad - C(\varepsilon)\left(q_1^{(4\alpha-\gamma)/(2\alpha-\gamma)} + q_2^{(4\alpha-\gamma)/(2\alpha-\gamma)}\right) \\ &\quad - \frac{c(\eta)}{p}\left(q_1^{p(1-\theta)/(2-\theta p)} + q_2^{p(1-\theta)/(2-\theta p)}\right) \\ &> -\infty. \end{aligned} \tag{40}$$

Hence, $-\infty < E_q < 0$. □

To solve the constrained minimization problem (5), the most difficult problem is the lack of compactness of the minimizing sequences (u_{1n}, u_{2n}). However, there are two scenarios and they are impossible for the problem: (1) vanishing $(u_{1n}, u_{2n}) \rightharpoonup 0$; (2) dichotomy $(u_{1n}, u_{2n}) \rightharpoonup (u_1, u_2)$ and $\|u_1\|_2^2 + \|u_2\|_2^2 \neq q$.

In order to rule out the above two cases and to show that the infimum is achieved, we apply the concentration compactness principle in [24, 25]. At first, we introduce the *Lévy* concentration function

$$Q_n(r) := \sup_{y\in\mathbb{R}^d} \int_{B(y,r)} \left(|u_{1n}(x)|^2 + |u_{2n}(x)|^2\right) dx. \tag{41}$$

$\{Q_n\}$ is nondecreasing on $(0, +\infty)$. By Helly's selection theorem, we find a convergent subsequence of $\{Q_n\}$ denoted again by $\{Q_n\}$ such that

$$\lim_{n\to+\infty} Q_n(r) = Q(r), \quad \forall r > 0, \tag{42}$$

where $Q(r)$ is a nondecreasing function. We know that $0 \leqslant Q_n(r) \leqslant q$, so there exists β, $\beta \in [0, q]$, such that

$$\lim_{n\to+\infty} Q(r) = \beta. \tag{43}$$

Lemma 14. *Every minimizing sequence $\{(u_{1n}, u_{2n})\}$ for problems E_q is bounded in $H^\alpha(\mathbb{R}^d) \times H^\alpha(\mathbb{R}^d)$, and for sufficiently large n, there exists a constant δ such that*

$$\begin{aligned} &\frac{1}{4}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{|x-y|^\gamma}\left(|u_{1n}(x)|^2 |u_{2n}(y)|^2 \right. \\ &\quad \left. + |u_{2n}(x)|^2 |u_{1n}(y)|^2\right) dx\,dy + \frac{2}{p} \\ &\quad \cdot \int_{\mathbb{R}^d} \left(|u_{1n}(x)|^{p/2} |u_{2n}(x)|^{p/2}\right) dx \geqslant \delta > 0. \end{aligned} \tag{44}$$

Proof. At first, from (37) and (39), we obtain

$$\begin{aligned} &\frac{1}{2}\left(\|u_{1n}\|^2 + \|u_{2n}\|^2\right) \leqslant E(u_{1n}, u_{2n}) + \frac{1}{2}\left(\|u_{1n}\|_2^2 \right. \\ &\quad \left. + \|u_{2n}\|_2^2\right) + \frac{1}{4} \\ &\quad \cdot \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{|x-y|^\gamma}\left(|u_{1n}(x)|^2 |u_{2n}(y)|^2 \right. \\ &\quad \left. + |u_{2n}(x)|^2 |u_{1n}(y)|^2\right) dx\,dy + \frac{2}{p} \\ &\quad \cdot \int_{\mathbb{R}^d} \left(|u_{1n}(x)|^{p/2} |u_{2n}(x)|^{p/2}\right) dx \leqslant E(u_{1n}, \\ &\quad u_{2n}) + \frac{1}{2}q + \left(\varepsilon + \frac{\eta}{p}\right)\left(\|u_{1n}\|^2 + \|u_{2n}\|^2\right) + C(\varepsilon) \\ &\quad \cdot \left(q_1^{(4\alpha-\gamma)/(2\alpha-\gamma)} + q_2^{(4\alpha-\gamma)/(2\alpha-\gamma)}\right) \\ &\quad + \frac{c(\eta)}{p}\left(q_1^{p(1-\theta)/(2-\theta p)} + q_2^{p(1-\theta)/(2-\theta p)}\right). \end{aligned} \tag{45}$$

Since $\{(u_{1n}, u_{2n})\}$ is a minimizing sequence, by $(\varepsilon + \eta/p) < 1/2$, we get the result.

We prove the second part of Lemma 14 with an argument by contradiction. Suppose δ does not exist; then there exists a subsequence $\{(u_{1n_k}, u_{2n_k})\}$ of $\{(u_{1n}, u_{2n})\}$ such that

$$\begin{aligned} &\frac{1}{4}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{|x-y|^\gamma}\left(|u_{1n_k}(x)|^2 |u_{2n_k}(y)|^2 \right. \\ &\quad \left. + |u_{2n_k}(x)|^2 |u_{1n_k}(y)|^2\right) dx\,dy + \frac{2}{p} \\ &\quad \cdot \int_{\mathbb{R}^d} \left(|u_{1n_k}(x)|^{p/2} |u_{2n_k}(x)|^{p/2}\right) dx \longrightarrow 0, \\ &\qquad \text{as } k \longrightarrow +\infty. \end{aligned} \tag{46}$$

Using the definition of energy, we deduce that

$$E\left(u_{1n_k}, u_{2n_k}\right) \longrightarrow E_q \geqslant 0, \quad \text{as } k \longrightarrow +\infty, \tag{47}$$

which contradicts $-\infty < E_q < 0$. The proof of Lemma 14 is completed. □

Lemma 15. *Vanishing does not occur, that is, $\beta > 0$, for any $q > 0$.*

Proof. We prove the lemma with an argument by contradiction. If $\beta = 0$, then there exist a positive r_0 and a subsequence $\{(u_{1n_k}, u_{2n_k})\}$ of a minimizing sequence $\{(u_{1n}, u_{2n})\}$ such that

$$\begin{aligned} &\sup_{y\in\mathbb{R}^d} \int_{B(y,r_0)} \left(|u_{1n_k}(x)|^2 + |u_{2n_k}(x)|^2\right) dx \longrightarrow 0, \\ &\qquad \text{as } k \longrightarrow \infty. \end{aligned} \tag{48}$$

As $\{(u_{1n_k}, u_{2n_k})\}$ is also a minimizing sequence, according to Lemma 10, we can find

$$\frac{1}{4}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{|x-y|^\gamma}\left(\left|u_{1n_k}(x)\right|^2\left|u_{2n_k}(y)\right|^2 + \left|u_{2n_k}(x)\right|^2\left|u_{1n_k}(y)\right|^2\right)dx\,dy + \frac{2}{p} \cdot \int_{\mathbb{R}^d}\left(\left|u_{1n_k}(x)\right|^{p/2}\left|u_{2n_k}(x)\right|^{p/2}\right)dx \longrightarrow 0, \quad \text{as } k\longrightarrow\infty, \tag{49}$$

which contradicts Lemma 14. □

By Lemma 15, we exclude the possibility of vanishing. We prove dichotomy will not occur yet by the next two Lemmas.

Lemma 16. *Let $q_1, q_2 > 0$; then $E_{q_1+q_2} < E_{q_1} + E_{q_2}$.*

Since the proof of Lemma 16 is similar to that of Lemma 3.3 in [17] and Lemma 13 in [14], we omit the details.

Lemma 17. *Suppose $0 < \beta < q$, then $E_\beta + E_{q-\beta} \leqslant E_q$.*

Proof. According to the boundedness of $\{(u_{1n}, u_{2n})\}$ in $H^\alpha(\mathbb{R}^d)\times H^\alpha(\mathbb{R}^d)$, the definition of β, we can get for $\forall\varepsilon > 0$, $\exists r_\varepsilon > 0$ such that

$$\frac{1}{4}\iint_{|x-y|\geqslant r_\varepsilon}\frac{1}{|x-y|^\gamma}\left(\left|u_{1n}(x)\right|^2\left|u_{2n}(y)\right|^2 + \left|u_{2n}(x)\right|^2\left|u_{1n}(y)\right|^2\right)dx\,dy \leqslant \frac{\varepsilon}{2}, \tag{50}$$

and for all $R > r_\varepsilon$, we have

$$\beta - \frac{\varepsilon}{4} < Q(r_\varepsilon) \leqslant Q(R) \leqslant \beta. \tag{51}$$

Then there exists $N_\varepsilon \in \mathbb{N}^+$ such that, for $\forall n \geqslant N_\varepsilon$, we obtain

$$\beta - \frac{\varepsilon}{2} < Q_n(r_\varepsilon) \leqslant Q_n(R) < \beta + \frac{\varepsilon}{2}. \tag{52}$$

Next we let $\{y_n\}\subset\mathbb{R}^d$ such that

$$\beta - \varepsilon < \int_{B(y_n,r_\varepsilon)}\left(\left|u_{1n}\right|^2 + \left|u_{2n}\right|^2\right)dx \leqslant \int_{B(y_n,R)}\left(\left|u_{1n}\right|^2 + \left|u_{2n}\right|^2\right)dx < \beta + \varepsilon. \tag{53}$$

We define $\phi_{ir}(x) = \phi_i(x/r)$ and $\widetilde{\phi}_{ir}(x) = \widetilde{\phi}_i(x/r)$ for $r > r_\varepsilon$, $i = 1, 2$, where $\phi_i \in C_0^\infty(B(0,2))$ is a smooth cutoff function satisfying $0 \leqslant \phi_i(x) \leqslant 1$, $\phi_i(x) = 1$ for $|x| \leqslant 1$, $\phi_i(x) = 0$ for $|x| \geqslant 2$ and $\widetilde{\phi}_i^2(x) = 1 - \phi_i^2(x)$. With this notation, we write

$$v_{in}(x) = \phi_{ir}(x-y_n)\,u_{in}(x), \quad w_{in}(x) = \widetilde{\phi}_{ir}(x-y_n)\,u_{in}(x), \quad i = 1, 2. \tag{54}$$

It follows easily from (53) that

$$\beta - \varepsilon < \int_{\mathbb{R}^d}\left(\left|v_{1n}(x)\right|^2 + \left|v_{2n}(x)\right|^2\right)dx < \beta + \varepsilon, \quad q - \beta - \varepsilon < \int_{\mathbb{R}^d}\left(\left|w_{1n}(x)\right|^2 + \left|w_{2n}(x)\right|^2\right)dx < q - \beta + \varepsilon. \tag{55}$$

We can get the conclusion if

$$E(v_{1n}, v_{2n}) + E(w_{1n}, w_{2n}) \leqslant E(u_{1n}, u_{2n}) + c\varepsilon, \tag{56}$$

for some positive constant c. Indeed, (55) implies that there exist $\mu_{in}, \nu_{in} \in [1-\varepsilon, 1+\varepsilon]$ $(i = 1, 2)$ such that

$$\left\|\mu_{1n}v_{1n}\right\|_2^2 + \left\|\mu_{2n}v_{2n}\right\|_2^2 = \beta, \quad \left\|\nu_{1n}w_{1n}\right\|_2^2 + \left\|\nu_{2n}w_{2n}\right\|_2^2 = q - \beta. \tag{57}$$

Therefore, we deduce that

$$E_\beta \leqslant E(\mu_{1n}v_{1n}, \mu_{2n}v_{2n}) \leqslant E(v_{1n}, v_{2n}) + c\varepsilon, \quad E_{q-\beta} \leqslant E(\nu_{1n}w_{1n}, \nu_{2n}w_{2n}) \leqslant E(w_{1n}, w_{2n}) + c\varepsilon. \tag{58}$$

Using the above two inequalities and (56), we have

$$E_\beta + E_{q-\beta} \leqslant E(v_{1n}, v_{2n}) + E(w_{1n}, w_{2n}) + c\varepsilon \leqslant E(u_{1n}, u_{2n}) + c\varepsilon. \tag{59}$$

Therefore, we obtain $E_\beta + E_{q-\beta} \leqslant E_q$.

Now we prove (56). By the definitions of v_{in} and ω_{in}, we have

$$\begin{aligned} E(v_{1n}, v_{2n}) + E(w_{1n}, w_{2n}) &= \frac{1}{2}\int_{\mathbb{R}^d}\left(\left|(-\Delta)^{\alpha/2}v_{1n}(x)\right|^2 + \left|(-\Delta)^{\alpha/2}v_{2n}(x)\right|^2\right)dx + \frac{1}{2} \\ &\cdot \int_{\mathbb{R}^d}\left(\left|(-\Delta)^{\alpha/2}w_{1n}(x)\right|^2 + \left|(-\Delta)^{\alpha/2}w_{2n}(x)\right|^2\right)dx - \frac{1}{4} \\ &\cdot \iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{|x-y|^\gamma}\left[\left|v_{1n}(x)\right|^2\left|v_{2n}(y)\right|^2 + \left|v_{2n}(x)\right|^2\left|v_{1n}(y)\right|^2\right]dx\,dy - \frac{1}{4} \\ &\cdot \iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{|x-y|^\gamma}\left[\left|w_{1n}(x)\right|^2\left|w_{2n}(y)\right|^2 + \left|w_{2n}(x)\right|^2\left|w_{1n}(y)\right|^2\right]dx\,dy - \frac{2}{p} \\ &\cdot \int_{\mathbb{R}^d}\left(\left|v_{1n}(x)\right|^{p/2}\left|v_{2n}(x)\right|^{p/2}\right)dx - \frac{2}{p} \\ &\cdot \int_{\mathbb{R}^d}\left(\left|w_{1n}(x)\right|^{p/2}\left|w_{2n}(x)\right|^{p/2}\right)dx. \end{aligned} \tag{60}$$

Applying Lemma 7 and Sobolev's inequalities, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|(-\Delta)^{\alpha/2} \left[\phi_{ir}(x-y_n) u_{in}(x)\right]\right|^2 dx \\
&\quad - \int_{\mathbb{R}^d} \phi_{ir}^2 (x-y_n) \left|(-\Delta)^{\alpha/2} u_{in}(x)\right|^2 dx \\
&\quad \leqslant C \Big(\left\|\nabla \phi_{ir}\right\|_{d/(1-\alpha)} \left\|(-\Delta)^{(\alpha-1)/2} u_{in}\right\|_{2d/(d-2(1-\alpha))} \\
&\quad + \left\|(-\Delta)^{\alpha/2} \phi_{ir}\right\|_{2+d/\alpha} \left\|u_{in}\right\|_{2+4\alpha/d} \Big) \\
&\quad \leqslant C \Big(\frac{1}{r^d} \left\|\nabla \phi_i\right\|_{d/(1-\alpha)} \left\|u_{in}\right\|_2 \\
&\quad + \frac{1}{r^{2\alpha^2/(2\alpha+d)}} \left\|(-\Delta)^{\alpha/2} \phi_i\right\|_{2+d/\alpha} \left\|u_{in}\right\| \Big).
\end{aligned} \tag{61}$$

After taking $r = R_\varepsilon$ large enough and $R_\varepsilon > r_\varepsilon$, from the above inequality, we derive that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|(-\Delta)^{\alpha/2} \left[\phi_{iR_\varepsilon}(x-y_n) u_{in}(x)\right]\right|^2 dx \\
&\quad \leqslant \int_{\mathbb{R}^d} \phi_{iR_\varepsilon}^2 (x-y_n) \left|(-\Delta)^{\alpha/2} u_{in}(x)\right|^2 dx + c\varepsilon, \\
&\qquad i = 1, 2.
\end{aligned} \tag{62}$$

Similarly, we get that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|(-\Delta)^{\alpha/2} \left[\widetilde{\phi}_{iR_\varepsilon}(x-y_n) u_{in}(x)\right]\right|^2 dx \\
&\quad \leqslant \int_{\mathbb{R}^d} \widetilde{\phi}_{iR_\varepsilon}^2 (x-y_n) \left|(-\Delta)^{\alpha/2} u_{in}(x)\right|^2 dx + c\varepsilon, \\
&\qquad i = 1, 2.
\end{aligned} \tag{63}$$

For $0 \leqslant \phi_i, \widetilde{\phi}_i \leqslant 1$, from the above two inequalities, we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left(\left|(-\Delta)^{\alpha/2} v_{1n}(x)\right|^2 + \left|(-\Delta)^{\alpha/2} v_{2n}(x)\right|^2 \right) dx \\
&\quad + \int_{\mathbb{R}^d} \left(\left|(-\Delta)^{\alpha/2} w_{1n}(x)\right|^2 + \left|(-\Delta)^{\alpha/2} w_{2n}(x)\right|^2 \right) dx \\
&\quad \leqslant \int_{\mathbb{R}^d} \left(\left|(-\Delta)^{\alpha/2} u_{1n}(x)\right|^2 + \left|(-\Delta)^{\alpha/2} u_{2n}(x)\right|^2 \right) dx \\
&\quad + c\varepsilon.
\end{aligned} \tag{64}$$

Now let us prove

$$\begin{aligned}
&\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\gamma} \Big[\left|u_{in}(x)\right|^2 \left|u_{jn}(y)\right|^2 \\
&\quad - \left|v_{in}(x)\right|^2 \left|v_{jn}(y)\right|^2 \\
&\quad - \left|w_{in}(x)\right|^2 \left|w_{jn}(y)\right|^2 \Big] dx\, dy \leqslant c\varepsilon, \\
&\qquad i, j = 1, 2,\ i \neq j.
\end{aligned} \tag{65}$$

Expanding the left-hand side of (65) and combining the equivalent terms, we have

$$\begin{aligned}
&2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\gamma} \Big(\left|v_{in}(x)\right|^2 \left|w_{in}(x)\right|^2 + 2 \left|v_{in}(x)\right| \\
&\quad \cdot \left|w_{in}(x)\right| \left|v_{jn}(y)\right|^2 + 2 \left|v_{in}(x)\right| \left|w_{in}(x)\right| \left|w_{jn}(y)\right|^2 \\
&\quad + 2 \left|v_{in}(x)\right| \left|v_{jn}(y)\right| \left|w_{in}(x)\right| \\
&\quad \cdot \left|w_{jn}(y)\right| \Big) dx\, dy, \quad i, j = 1, 2,\ i \neq j.
\end{aligned} \tag{66}$$

Indeed, except for the first term $|v_{in}(x)|^2 |\omega_{in}(x)|^2$, the remainder are integral on the ring $B(y_n, 2R_\varepsilon) \setminus B(y_n, R_\varepsilon)$ in $\mathbb{R}_x^d$ or $\mathbb{R}_y^d$ (or both). Therefore, for (53), we deduce that

$$\begin{aligned}
&\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\gamma} \left|v_{in}(x)\right| \left|w_{in}(x)\right| \left|v_{jn}(y)\right|^2 dx\, dy \\
&\quad \leqslant \sup_{x \in \mathbb{R}^d} \int_{B_y(y_n, 2R_\varepsilon)} \frac{1}{|x-y|^\gamma} \left|v_{jn}(y)\right|^2 dy \\
&\quad \cdot \int_{B_x(y_n, 2R_\varepsilon) \setminus B_x(y_n, R_\varepsilon)} \left|v_{in}(x)\right| \left|w_{in}(x)\right| dx \leqslant \left\|u_{in}\right\| \\
&\quad \cdot \int_{B(y_n, 2R_\varepsilon) \setminus B(y_n, R_\varepsilon)} \left|u_{in}\right|^2 dx \leqslant c, \quad i, j = 1, 2,\ i \neq j.
\end{aligned} \tag{67}$$

Similarly,

$$\begin{aligned}
&\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^\gamma} \left|v_{in}(x)\right| \left|w_{in}(x)\right| \left|w_{jn}(y)\right|^2 \\
&\quad + \left|v_{in}(x)\right| \left|v_{jn}(y)\right| \left|w_{in}(x)\right| \left|w_{jn}(y)\right| dx\, dy \\
&\quad \leqslant c\varepsilon, \quad i, j = 1, 2,\ i \neq j.
\end{aligned} \tag{68}$$

To estimate the first term, by (50), we should deal with the integral on the set $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d;\ |x-y| \leqslant r_\varepsilon\}$. Therefore, we have

$$\begin{aligned}
&\iint_{|x-y| \leqslant r_\varepsilon} \frac{1}{|x-y|^\gamma} \left|v_{in}(x)\right|^2 \left|w_{jn}(y)\right|^2 dx\, dy \\
&\quad \leqslant \sup_{y \in \mathbb{R}^d} \int_{B_y(y_n, 2R_\varepsilon)} \frac{1}{|x-y|^\gamma} \left|v_{jn}(y)\right|^2 \\
&\quad \cdot \int_{B_x(y_n, 3r_\varepsilon) \setminus B_x(y_n, r_\varepsilon)} \left|w_{in}(x)\right|^2 dx \leqslant \left\|u_{in}\right\| \\
&\quad \cdot \int_{B(y_n, 2R_\varepsilon + r_\varepsilon) \setminus B(y_n, R_\varepsilon)} \left|u_{in}(x)\right|^2 dx \leqslant c\varepsilon, \\
&\qquad i, j = 1, 2,\ i \neq j.
\end{aligned} \tag{69}$$

From (66)–(69), (65) is proved. At last, using (38), (53), the definitions of ϕ_{ir}, $\widetilde{\phi}_{ir}$, and Lemma 13, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} |u_{1n}|^{p/2} |u_{2n}|^{p/2} dx - \int_{\mathbb{R}^d} \left|\phi_{1R_\varepsilon}(x-y_n) u_{1n}\right|^{p/2} \\
&\cdot \left|\phi_{2R_\varepsilon}(x-y_n) u_{2n}\right|^{p/2} dx \\
&- \int_{\mathbb{R}^d} \left|\widetilde{\phi}_{1R_\varepsilon}(x-y_n) u_{1n}\right|^{p/2} \\
&\cdot \left|\widetilde{\phi}_{2R_\varepsilon}(x-y_n) u_{2n}\right|^{p/2} dx = \int_{\mathbb{R}^d} \Big(1 \\
&- \left|\phi_{1R_\varepsilon}(x-y_n)\right|^{p/2} \left|\phi_{2R_\varepsilon}(x-y_n)\right|^{p/2} \\
&- \left|\widetilde{\phi}_{1R_\varepsilon}(x-y_n)\right|^{p/2} \left|\widetilde{\phi}_{2R_\varepsilon}(x-y_n)\right|^{p/2}\Big) |u_{1n}|^{p/2} \\
&\cdot |u_{2n}|^{p/2} dx \leqslant \frac{3}{p} \int_\Omega |u_{1n}|^{p/2} |u_{2n}|^{p/2} dx \\
&\leqslant \frac{3}{2p} \left(\int_\Omega |u_{1n}|^p dx + \int_\Omega |u_{2n}|^p dx\right) \\
&\leqslant \frac{c}{p} \Big(\|u_{1n}\|_{L^2(\Omega)}^{p(1-\lambda)} \|u_{1n}\|_{W^{\alpha,2}(\Omega)}^{\lambda p} + \|u_{2n}\|_{L^2(\Omega)}^{p(1-\lambda)} \\
&\cdot \|u_{2n}\|_{W^{\alpha,2}(\Omega)}^{\lambda p}\Big) \leqslant c\varepsilon,
\end{aligned} \tag{70}$$

where $\Omega = B(y_n, 2R_\varepsilon) \setminus B(y_n, R_\varepsilon)$, $\lambda = d(p-2)/2\alpha p$. From what has been discussed above, we finish the proof. □

Proof of Theorem 1. According to the definition of β in (43), using Lemmas 15, 16 and 17, we get that every minimizing sequence $\{(u_{1n}, u_{2n})\}$ for problem (5) has a subsequence and is denoted again by $\{(u_{1n}, u_{2n})\}$, satisfying

$$\begin{aligned}
&\lim_{r\to+\infty} \lim_{n\to+\infty} \sup_{y\in\mathbb{R}^d} \int_{B(y,r)} \left(|u_{1n}(x)|^2 + |u_{2n}(x)|^2\right) dx \\
&\quad = \beta = q,
\end{aligned} \tag{71}$$

which implies that, for any $\varepsilon > 0$, there exist $r_\varepsilon > 0$, $n_\varepsilon \in N^+$, and $\{y_n\} \subset \mathbb{R}^d$ such that, for every $n > n_\varepsilon$ and $r > r_\varepsilon$, we have

$$\int_{B(y_n,r)} \left(|u_{1n}(x)|^2 + |u_{2n}(x)|^2\right) dx > q - \varepsilon. \tag{72}$$

By Lemma 14, $\{u_{1n}(\cdot - y_n), u_{2n}(\cdot - y_n)\}$ is bounded in $H^\alpha(\mathbb{R}^d) \times H^\alpha(\mathbb{R}^d)$. Going if necessary to a subsequence of $\{u_{1n}(\cdot - y_n), u_{2n}(\cdot - y_n)\}$ and denoting again by $\{u_{1n}(\cdot - y_n), u_{2n}(\cdot - y_n)\}$, there exists $(g_1, g_2) \in H^\alpha(\mathbb{R}^d) \times H^\alpha(\mathbb{R}^d)$ such that

$$\begin{aligned}
&(u_{1n}(\cdot - y_n), u_{2n}(\cdot - y_n)) \rightharpoonup (g_1, g_2) \\
&\qquad \text{weekly in } H^\alpha(\mathbb{R}^d) \times H^\alpha(\mathbb{R}^d).
\end{aligned} \tag{73}$$

We can find $R_\varepsilon > r_\varepsilon$ such that $(\|g_1\|_{L^2(\mathbb{R}^d\setminus B(0,R_\varepsilon))} + \|g_2\|_{L^2(\mathbb{R}^d\setminus B(0,R_\varepsilon))}) < \varepsilon/2$. Indeed, by Lemma 8, there exists $N_\varepsilon \in N^+$ with $N_\varepsilon > n_\varepsilon$ such that, for $n > N_\varepsilon$, we have

$$\left\|(u_{1n}(\cdot - y_n), u_{2n}(\cdot - y_n)) - (g_1, g_2)\right\|_{L^2(B(0,R_\varepsilon))} < \frac{\varepsilon}{2}. \tag{74}$$

From the above, we obtain

$$\begin{aligned}
&\|g_1\|_2 + \|g_2\|_2 \geqslant (\|u_{1n}\|_2 + \|u_{2n}\|_2) \\
&- \left\|(u_{1n}(\cdot - y_n), u_{2n}(\cdot - y_n)) - (g_1, g_2)\right\|_{L^2(B(0,r_\varepsilon))} \\
&- \left\|(u_{1n}(\cdot - y_n), u_{2n}(\cdot - y_n))\right. \\
&\left.- (g_1, g_2)\right\|_{L^2(\mathbb{R}^d\setminus B(0,r_\varepsilon))} \geqslant \Big(\|u_{1n}\|_{L^2(B(y_n,r_\varepsilon))} \\
&+ \|u_{2n}\|_{L^2(B(y_n,r_\varepsilon))}\Big) - \left\|(u_{1n}(\cdot - y_n), u_{2n}(\cdot - y_n))\right. \\
&\left.- (g_1, g_2)\right\|_{L^2(B(0,r_\varepsilon))} - \Big(\|g_1\|_{L^2(B(0,r_\varepsilon))} \\
&+ \|g_2\|_{L^2(B(0,r_\varepsilon))}\Big) \geqslant \sqrt{q-\varepsilon} - \varepsilon,
\end{aligned} \tag{75}$$

which implies, by passing to the limit, $\|g_1\|_2^2 + \|g_2\|_2^2 \geqslant q$. On the other hand, the weak lower semicontinuity deduces

$$q \leqslant \left(\|g_1\|_2^2 + \|g_2\|_2^2\right) \leqslant \liminf_{n\to+\infty} \left(\|u_{1n}\|_2^2 + \|u_{2n}\|_2^2\right) = q. \tag{76}$$

Therefore, $\|g_1\|_2^2 + \|g_2\|_2^2 = q$, and

$$\begin{aligned}
&(u_{1n}(\cdot - y_n), u_{2n}(\cdot - y_n)) \longrightarrow (g_1, g_2) \\
&\qquad \text{strongly in } L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d),
\end{aligned} \tag{77}$$

since $\{(u_{1n}(\cdot - y_n), u_{2n}(\cdot - y_n))\}$ converges weekly in $H^\alpha(\mathbb{R}^d) \times H^\alpha(\mathbb{R}^d)$. Applying an interpolation theorem, by (73) and (77), it follows immediately that

$$\begin{aligned}
&(u_{1n}(\cdot - y_n), u_{2n}(\cdot - y_n)) \longrightarrow (g_1, g_2) \\
&\qquad \text{strongly in } L^p(\mathbb{R}^d) \times L^p(\mathbb{R}^d).
\end{aligned} \tag{78}$$

Using the weak lower semicontinuity again, we have

$$\begin{aligned}
&\|g_1\|_{\dot{H}^\alpha(\mathbb{R}^d)} + \|g_2\|_{\dot{H}^\alpha(\mathbb{R}^d)} \\
&\quad \leqslant \liminf_{n\to+\infty} \left(\|u_{1n}\|_{\dot{H}^\alpha(\mathbb{R}^d)} + \|u_{2n}\|_{\dot{H}^\alpha(\mathbb{R}^d)}\right).
\end{aligned} \tag{79}$$

From (79), we obtain

$$\beta \leqslant E(g_1, g_2) \leqslant \liminf_{n\to+\infty} E(u_{1n}, u_{2n}) = \beta. \tag{80}$$

Therefore, (g_1, g_2) is a minimizer of problem (5) and

$$\begin{aligned}
&(u_{1n}(\cdot - y_n), u_{2n}(\cdot - y_n)) \longrightarrow (g_1, g_2) \\
&\qquad \text{in } H^\alpha(\mathbb{R}^d) \times H^\alpha(\mathbb{R}^d).
\end{aligned} \tag{81}$$

We next prove (8) by using an argument by contradiction. Suppose that there exist $\varepsilon_0 > 0$ and a subsequence $\{(u_{1n_k}, u_{2n_k}\}$ of $\{(u_{1n}, u_{2n})\}$ such that

$$\inf_{(g_1,g_2)\in G_q} \left\|(u_{1n_k}, u_{2n_k}) - (g_1, g_2)\right\| \geqslant \varepsilon_0 > 0. \tag{82}$$

From the result of the above proof, we know that there exists a subsequence of $\{(u_{1n_k}, u_{2n_k})\}$, denoted again by $\{(u_{1n_k}, u_{2n_k})\}$ and $\{y_{n_k}\} \in \mathbb{R}^d$ such that

$$\left(u_{1n_k}\left(\cdot - y_{n_k}\right), u_{2n_k}\left(\cdot - y_{n_k}\right)\right) \rightharpoonup (g_1, g_2) \quad \text{in } H^\alpha\left(\mathbb{R}^d\right) \times H^\alpha\left(\mathbb{R}^d\right). \tag{83}$$

Since $(g_1(\cdot + y_{n_k}), g_2(\cdot + y_{n_k})) \in G_q$, we deduce that

$$\begin{aligned}&\left\|\left(u_{1n_k}, u_{2n_k}\right) - \left(g_1\left(\cdot + y_{n_k}\right), g_2\left(\cdot + y_{n_k}\right)\right)\right\| \\ &\quad = \left\|\left(u_{1n_k}\left(\cdot - y_{n_k}\right), u_{2n_k}\left(\cdot - y_{n_k}\right)\right) - (g_1, g_2)\right\| \\ &\quad \longrightarrow 0,\end{aligned} \tag{84}$$

which contradicts (82). □

Proof of Theorem 2. Arguing by contradiction, we prove Theorem 2. Assume that the set G_q is not $H^\alpha(\mathbb{R}^d) \times H^\alpha(\mathbb{R}^d)$-stable. Then there exist $\varepsilon_0 > 0$, a sequence $(u_{1m}^{(0)}, u_{2m}^{(0)}) \subset H^\alpha(\mathbb{R}^d) \times H^\alpha(\mathbb{R}^d)$, and $t_m \in [0, T)$ such that

$$\inf_{(g_1, g_2) \in G_q} \left\|\left(u_{1m}^{(0)}, u_{2m}^{(0)}\right) - (g_1, g_2)\right\| < \frac{1}{m}, \tag{85}$$

$$\inf_{(g_1, g_2) \in G_q} \left\|\left(u_{1m}\left(t_m\right), u_{2m}\left(t_m\right)\right) - (g_1, g_2)\right\| \geqslant \varepsilon_0, \tag{86}$$

where $(u_{1m}, u_{2m}) \in C([0, T), H^\alpha(\mathbb{R}^d) \times H^\alpha(\mathbb{R}^d))$ are solutions to (1) with initial date $(u_{1m}(x, 0), u_{2m}(x, 0)) = (u_{1m}^{(0)}, u_{2m}^{(0)})$. From (85), we conclude that

$$\begin{aligned}&\left(\left\|u_{1m}^{(0)}\right\|_2^2 + \left\|u_{2m}^{(0)}\right\|_2^2\right) \longrightarrow q, \\ &E\left(u_{1m}^{(0)}, u_{2m}^{(0)}\right) \longrightarrow E_q, \\ &\qquad m \longrightarrow +\infty.\end{aligned} \tag{87}$$

Therefore, there exist sequences $\{\mu_{1m}\}, \{\mu_{2m}\} \in \mathbb{R}$, satisfying $\|\mu_{1m}u_{1m}^{(0)}\|_2^2 + \|\mu_{2m}u_{2m}^{(0)}\|_2^2 = q$, $\mu_{1m} \to 1$, $\mu_{2m} \to 1$, such that $\{(\mu_{1m}u_{1m}^{(0)}, \mu_{2m}u_{2m}^{(0)})\}$ is a minimizing sequence for the problem (5). According to conservation laws, it follows that

$$\begin{aligned}&\left\|\mu_{1m}u_{1m}\left(t_m\right)\right\|_2^2 + \left\|\mu_{2m}u_{2m}\left(t_m\right)\right\|_2^2 \\ &\quad = \left\|\mu_{1m}u_{1m}^{(0)}\right\|_2^2 + \left\|\mu_{2m}u_{2m}^{(0)}\right\|_2^2 = q, \\ &E\left(\mu_{1m}u_{1m}\left(t_m\right), \mu_{2m}u_{2m}\left(t_m\right)\right) \longrightarrow E_q,\end{aligned} \tag{88}$$

where $\{(\mu_{1m}u_{1m}(t_m), \mu_{2m}u_{2m}(t_m))\}$ is also a minimizing sequence for the problem (5). By Theorem 1, there exists a subsequence $\{(\mu_{1m_k}u_{1m_k}(t_{m_k}), \mu_{2m_k}u_{2m_k}(t_{m_k}))\}$ of $\{(\mu_{1m}u_{1m}(t_m), \mu_{2m}u_{2m}(t_m))\}$, and $\{(g_{1m_k}, g_{2m_k})\}$ in G_q such that

$$\begin{aligned}&\left\|\left(\mu_{1m_k}u_{1m_k}\left(t_{m_k}\right), \mu_{2m_k}u_{2m_k}\left(t_{m_k}\right)\right) - \left(g_{1m_k}, g_{2m_k}\right)\right\| \\ &\quad < \frac{\varepsilon_0}{2},\end{aligned} \tag{89}$$

for sufficiently large m_k. We get that

$$\begin{aligned}\varepsilon_0 &\leqslant \left\|\left(u_{1m_k}\left(t_{m_k}\right), u_{2m_k}\left(t_{m_k}\right)\right) - \left(g_{1m_k}, g_{2m_k}\right)\right\| \\ &\leqslant \left\|\left(u_{1m_k}\left(t_{m_k}\right), u_{2m_k}\left(t_{m_k}\right)\right)\right. \\ &\quad \left. - \left(\mu_{1m_k}u_{1m_k}\left(t_{m_k}\right), \mu_{2m_k}u_{2m_k}\left(t_{m_k}\right)\right)\right\| \\ &\quad + \left\|\left(\mu_{1m_k}u_{1m_k}\left(t_{m_k}\right), \mu_{2m_k}u_{2m_k}\left(t_{m_k}\right)\right)\right. \\ &\quad \left. - \left(g_{1m_k}, g_{2m_k}\right)\right\| \leqslant \left(\left|\mu_{1m_k} - 1\right| + \left|\mu_{2m_k} - 1\right|\right) \\ &\quad \cdot \left(\left\|u_{1m_k}\left(t_{m_k}\right)\right\| + \left\|u_{2m_k}\left(t_{m_k}\right)\right\|\right) + \frac{\varepsilon_0}{2}.\end{aligned} \tag{90}$$

This is a contradiction since $\mu_{1m} \to 1$, $\mu_{2m} \to 1$. Therefore, the set G_q is $H^\alpha(\mathbb{R}^d) \times H^\alpha(\mathbb{R}^d)$-stable with respect to (1). □

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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