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Research Article

Integrable 2D Time-Irreversible Systems with a Cubic Second Integral

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We construct a very rare integrable 2D mechanical system which admits a complementary integral of motion cubic in the velocities in the presence of conservative potential and velocity-dependent (gyroscopic) forces. Special cases are given interpretation as a motion of a particle on a sphere endowed with a Riemannian metric, a particle in the Euclidean plane, and new generalizations of two cases of motion of a rigid body with a cubic integral, known by names of Goriachev-Chaplygin and Goriachev.

1. Introduction: History and Formulation of the Problem

The search for potentials of conservative motions of a particle in the plane, so that the motion admits an integral polynomial in the velocities, was initiated by Bertrand in the middle of the nineteenth century [1, 2]. His results were developed further by Darboux [3] for the case of a quadratic integral. A large number of works were devoted to construction of integrable potentials in the plane admitting a complementary integral of degree up to 6. Notable examples are [4–9]. For a detailed account of relevant results up to 1985, see [10].

Birkhoff extended the method to accommodate general 2D mechanical systems acted upon by potential and gyroscopic forces. Those systems mostly live on Riemannian manifolds and the presence of gyroscopic forces makes their equations of motion time-irreversible. Birkhoff's procedure was completed to the end only in two cases: for reversible systems with an integral quadratic in velocities and irreversible systems with an integral linear in velocities [11].

Time-irreversible systems were considered in much fewer works. An almost complete list of those works is composed of [12–24]. Of those articles [12, 15–19, 22, 24] are exclusively devoted to irreversible systems with a quadratic complementary invariant.

An essential modification of Birkhoff's method in Yehia's work [20] significantly reduced the number of PDEs determining the system and its integral and made it possible to tackle the time-reversible and irreversible cases with a polynomial integral. The culmination of the new method was the construction and classification of 41 irreversible systems admitting a quadratic integral [22] and the construction of a gigantic reversible system involving 21 parameters called "master system" with a complementary integral quartic in velocities [25].

The new method also made it possible to construct for the first time irreversible integrable systems which admit a complementary integral cubic [21] and quartic [26] in velocities, based on the equations derived in [20].

The present paper may be regarded as a continuation of [21]. Here we study mechanical systems described by or reduced to a two-dimensional system with Lagrangian

$$L = \frac{1}{2}\left(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2\right) + a_1\dot{q}_1 + a_2\dot{q}_2 - V, \qquad (1)$$

where a_{ij}, a_j, and V are functions of q_1 and q_2 and dots denote differentiation with respect to time t. As in [21] we use a point transformation to isometric coordinates and a change of the time variable

$$dt = \Lambda d\tau, \qquad (2)$$

and one can always reduce (1) to the form

$$L = \frac{1}{2}\left(x'^2 + y'^2\right) + l_1 x' + l_2 y' + U, \quad (3)$$

where Λ, l_1, and l_2 are certain functions of x and y and primes denote derivatives with respect to τ and

$$U = \Lambda (h - V). \quad (4)$$

The equations of motion take the form

$$\begin{aligned} x'' + \Omega y' &= \frac{\partial U}{\partial x}, \\ y'' - \Omega x' &= \frac{\partial U}{\partial y}, \end{aligned} \quad (5)$$

where

$$\Omega = \frac{\partial l_1}{\partial y} - \frac{\partial l_2}{\partial x}. \quad (6)$$

This system admits the zero-value Jacobi integral

$$I_1 = \frac{1}{2}\left(x'^2 + y'^2\right) - U = 0. \quad (7)$$

The Jacobi constant h for the original system (1) enters as a parameter in the new force function (4) (see, e.g., [27]).

From the results of [20, 21] the Lagrangian and the cubic integral can be written as

$$L = \frac{1}{2}\left(x'^2 + y'^2\right) + \frac{1}{3}\left(P_2 x' - Q_2 y'\right) + U, \quad (8)$$

$$I = x'^3 + P_2 x'^2 + Q_2 x' y' + P_1 x' + Q_1 y' + R = \text{const.}, \quad (9)$$

where P_j, Q_j, and R are functions in x and y satisfying with U the nonlinear system of seven partial differential equations [21]:

$$\begin{aligned} \frac{\partial P_2}{\partial x} - \frac{\partial Q_2}{\partial y} &= 0, \\ \frac{\partial P_2}{\partial y} + \frac{\partial Q_2}{\partial x} - 3\Omega &= 0, \\ \frac{\partial P_1}{\partial x} - \frac{\partial Q_1}{\partial y} + 2\Omega Q_2 + 3\frac{\partial U}{\partial x} &= 0, \\ \frac{\partial P_1}{\partial y} + \frac{\partial Q_1}{\partial x} - 2\Omega P_2 &= 0, \\ P_1 \frac{\partial U}{\partial x} + Q_1 \frac{\partial U}{\partial y} + 2U\frac{\partial Q_1}{\partial y} - 2\Omega U Q_2 &= 0, \\ \frac{\partial R}{\partial x} + \Omega Q_1 + 2P_2 \frac{\partial U}{\partial x} + Q_2 \frac{\partial U}{\partial y} + 2U\frac{\partial Q_2}{\partial y} &= 0, \\ \frac{\partial R}{\partial y} - \Omega P_1 + Q_2 \frac{\partial U}{\partial x} &= 0. \end{aligned} \quad (10)$$

The irreversible case $\Omega \neq 0$ was considered in [21], where several parameter systems admitting a cubic integral were found under the simplifying assumption that

$$\Omega = \Omega_0 (y), \quad (11)$$

$$U = u(y) + v(y)\Phi(x). \quad (12)$$

In the present paper we will try, as in [21], to construct Lagrangian systems admitting a first integral polynomial of degree three in velocities, but instead of (11) we use the ansatz

$$\Omega = \Omega_1 (y)(a_1 \sin x + a_2 \cos x) + \lambda \Omega_0 (y). \quad (13)$$

As shown below, the problem is completely solved and an integrable system involving 15 parameters is constructed, adding two parameters to the system of [21]. Four new integrable problems are obtained as special cases of this system: a motion of a particle on a sphere endowed with a Riemannian metric, a particle in the plane, and two problems in the dynamics of a rigid body.

2. Solution of the Problem: The Conditional Integrable System

Regarding (10) and (14) and (13), a suitable ansatz for the reduced force function U has the structure

$$\begin{aligned} U = {} & u_0(y) + \left(a_1^2 + a_2^2\right) u_1(y) + \lambda^2 u_2(y) \\ & + v(y)(\mu_1 \sin(x) + \mu_2 \cos(x)) \\ & + \lambda w(y)[a_2 \cos x + a_1 \sin x] \\ & - v(y)^2 \left[\left(a_1^2 - a_2^2\right)\cos 2x - 2a_1 a_2 \sin 2x\right], \end{aligned} \quad (14)$$

where a_1, a_2, λ, μ_1, and μ_2 are arbitrary constants and u_0, u_1, u_2, v, and w are functions to be determined of the single variable y. Then the coefficients of the integral (9) should take the following forms:

$$\begin{aligned} P_2 &= f_1(y)[a_1 \sin x + a_2 \cos x] + \lambda f_0(y), \\ Q_2 &= f_2(y)[a_1 \cos(x) - a_2 \sin(x)], \\ P_1 &= f_3(y)\left[\left(a_1^2 - a_2^2\right)\cos 2x - 2a_1 a_2 \sin 2x\right] \\ &\quad + f_4(y)[\mu_1 \sin x + \mu_2 \cos x] \\ &\quad + \lambda f_5(y)[a_1 \sin x + a_2 \cos x] \\ &\quad + \left(a_1^2 + a_2^2\right) f_{6a}(y) + \lambda^2 f_{6a}(y), \\ Q_1 &= f_7(y)\left[2a_1 a_2 \cos 2x + \left(a_1^2 - a_2^2\right)\sin 2x\right] \\ &\quad - \frac{1}{2} f_2(y)(\mu_1 \cos x - \mu_2 \sin x) \\ &\quad + \lambda f_8(y)[a_1 \cos x - a_2 \sin x], \end{aligned} \quad (15)$$

in which f_i, $i = 1, \dots, 8$ are certain functions of y. Inserting those expressions in (10), we obtain the following system of ordinary differential equations:

$$
\begin{aligned}
&\lambda\left(3\Omega_0 - \frac{df}{dy}\right) = 0,\\
&\lambda^2\left(\frac{df_{6b}}{dy} - 2\Omega_0 f_0\right) = 0,\\
&\mu_i\left(f_2 + 2\frac{df_4}{dy}\right) = 0,\\
&\mu_i\left(6v + 2f_4 + 2\frac{df_2}{dy}\right) = 0,\\
&\left(a_1^2 + a_2^2\right)\left(\frac{df_{6a}}{dy} - f_1\Omega_1\right) = 0,\\
&a_i\left(f_1 - \frac{df_2}{dy}\right) = 0,\\
&a_i\left(3\Omega_1 - \frac{df_1}{dy} + f_2\right) = 0, \quad i = 1, 2,\\
&\left(\mu_1^2 + \mu_2^2\right)\left(f_2\frac{dv}{dy} + 2v\frac{df_2}{dy} - 2vf_4\right) = 0,\\
&\lambda a_i\left(2f_2\Omega_0 + f_5 + 3w - \frac{df_8}{dy}\right) = 0, \quad i = 1, 2,\\
&\lambda\mu_i\left[\frac{d}{dy}\left(\Omega_0 f_2 - 4vf_0\right) - 2\Omega_0 f_4\right] = 0,\\
&a_i\left(f_2\frac{d^2u_0}{dy^2} + 3\frac{df_2}{dy}\frac{du_0}{dy} + 2\frac{d^2f_2}{dy^2}u_0\right) = 0, \quad i = 1, 2,\\
&\mu_i\left(2u_0\frac{df_2}{dy} + f_2\frac{du_0}{dy}\right) = 0,\\
&\mu_i\left(2u_2\frac{df_2}{dy} + f_2\frac{du_2}{dy} - 2vf_{6b}\right) = 0, \quad i = 1, 2,\\
&\lambda a_i\left(-2u_0\frac{df_8}{dy} + 2u_0\Omega_0 f_2 - \frac{du_0}{dy}f_8\right) = 0,\\
&\lambda a_i\left(2\frac{df_5}{dy} - 2f_8 - 4\Omega_1 f_0 - 4\Omega_0 f_1\right) = 0, \quad i = 1, 2,\\
&\left(a_1^2 + a_2^2\right)\left(f_2\Omega_1 - 2f_3 - \frac{df_7}{dy} + 6v^2\right) = 0,\\
&\left(a_1^2 + a_2^2\right)\left(2f_7 + \frac{df_3}{dy} + f_1\Omega_1\right) = 0,\\
&a_2\left(3a_1^2 - a_2^2\right)\left(-2v^2\left(f_2 + 2\frac{df_1}{dy} + \frac{d^2f_2}{dy^2} + 3v^2 f_2\right)\right)\\
&\quad - \Omega_1\left(\frac{df_7}{dy} + 3f_3\right) - f_7\frac{d\Omega_1}{dy} - 2v\frac{dv}{dy}\left(3\frac{df_2}{dy}\right.\\
&\quad \left.+ 4f_1\right) - 2v\frac{d^2v}{dy^2}f_2 = 0,\\
&\mu_i a_1 a_2\left(-v^2\frac{df_2}{dy} + 2v^2 f_4 - v\Omega_1 f_2 + f_7\frac{dv}{dy} - vf_2\frac{dv}{dy}\right.\\
&\quad \left.+ 2v\frac{df_7}{dy} - vf_3\right) = 0, \quad i = 1, 2,\\
&\lambda a_i^3\left(-vf_3 - 2vf_{6a} + f_7\frac{dv}{dy} + f_2\frac{du_1}{dy} + v^2\frac{df_2}{dy} + 2v^2 f_4\right.\\
&\quad \left.+ 2u_1\frac{df_2}{dy} + v\frac{dv}{dy}f_2 - v\Omega_1 f_2 + v\frac{df_7}{dy}\right) = 0,\\
&\qquad i = 1, 2,\\
&R(x, y) = \frac{1}{6}\left(2\frac{df_2}{dy}v^2 + f_7\Omega_1 + 4v^2 f_1 + 2f_2 v\frac{dv}{dy}\right)\\
&\quad \cdot\left[a_2\left(3a_1^2 - a_2^2\right)\cos 3x + a_1\left(a_1^2 - 3a_2^2\right)\sin 3x\right]\\
&\quad - \frac{1}{8}\left(-4f_1 v - 4v\frac{df_2}{dy} + f_2\Omega_1 - 2f_2\frac{dv}{dy}\right)\left[\left(\mu_1 a_1\right.\right.\\
&\quad \left.\left.- \mu_2 a_2\right)\cos 2x - \left(\mu_1 a_2 + \mu_2 a_1\right)\sin 2x\right]\\
&\quad - \frac{\lambda}{4}\left(2w\frac{df_2}{dy} + 2\Omega_0 f_7 + f_8\Omega_1 + 2f_1 w + f_2\frac{dw}{dy}\right.\\
&\quad \left.+ 8f_0 v^2\right)\left[\left(a_1^2 - a_2^2\right)\cos 2x - 2a_1 a_2\sin 2x\right]\\
&\quad - \frac{\lambda}{2}\left(-f_2\Omega_0 + 4vf_0\right)\left[\mu_1\sin x + \mu_2\cos x\right] - \left(a_1\right.\\
&\quad \left.\cdot\sin x + a_2\cos x\right)\left[\lambda^2\left(\Omega_0 f_8 + 2f_0 w + 2u_2\frac{df_2}{dy}\right.\right.\\
&\quad \left.+ f_2\frac{du_2}{dy}\right) + f_2\frac{du_0}{dy} + 2u_0\frac{df_2}{dy} + \frac{1}{2}\left(a_1^2 + a_2^2\right)\\
&\quad \cdot\left(\Omega_1 f_7 + 4v^2 f_1 + 4u_1\frac{df_2}{dy} - 2v^2\frac{df_2}{dy} - 2vf_2\frac{dv}{dy}\right.\\
&\quad \left.\left.+ 2\frac{du_1}{dy}f_2\right)\right] + r(y),
\end{aligned}
\tag{16}
$$

where $r(y)$ is a new function determined from

$$
\begin{aligned}
\frac{dr}{dy} &= \frac{1}{2}\left(a_1\mu_1 + a_2\mu_2\right)\left(\Omega_1 f_4 - f_2 v\right) + \lambda^3\Omega_0 f_{6b}\\
&\quad + \frac{\lambda\left(a_1^2 + a_2^2\right)}{2}\left(\Omega_1 f_5 - f_2 w + 2f_{6a}\Omega_0\right),\\
&\lambda^2 a_i\left(-2u_2\frac{d^2f_2}{dy^2} - 2f_0\frac{dw}{dy} - 2w\frac{df_0}{dy} - f_8\frac{d\Omega_0}{dy}\right.\\
&\quad \left.- \Omega_0\frac{df_8}{dy} - f_2\frac{d^2u_2}{dy^2} - 3\frac{df_2}{dy}\frac{du_2}{dy} - \Omega_1 f_{6b} - \Omega_0 f_5\right)
\end{aligned}
$$

$$= 0, \quad i = 1, 2.$$

$$(\mu_2 a_1 - \mu_1 a_2)\left(2f_2\frac{dv}{dy} + 4v\frac{df_2}{dy} - 4f_1 v + f_2\Omega_1\right) = 0,$$

$$(\mu_1 a_2 + a_1\mu_2)\left[-2\frac{dv}{dy}\left(2f_1 + 3\frac{df_2}{dy}\right) - 4v\left(f_2 + \frac{df_1}{dy} - \frac{d^2 f_2}{dy^2}\right) + \Omega_1\left(\frac{df_2}{dy} - 4f_4 + f_2\right)\right] = 0,$$

$$\lambda a_1 a_2\left[-w\left(\frac{df_1}{dy} + \frac{d^2 f_2}{dy^2} + f_2\right) + \frac{dw}{dy}\left(f_1 + \frac{3}{2}\frac{df_2}{dy}\right) - \frac{f_2}{2}\frac{d^2 w}{dy^2} - \frac{\Omega_1}{2}\left(\frac{df_8}{dy} + 2f_5\right) + \Omega_0\left(2f_3 - \frac{df_7}{dy}\right) - \frac{d\Omega_0}{dy}f_7 - \frac{1}{2}\frac{d\Omega_1}{dy}f_8 - 4v^2\frac{df_0}{dy} - 8vf_0\frac{dv}{dy}\right] = 0,$$

$$a_i^3\left[-2\frac{d^2 f}{y^2}u_1 - \frac{f_7}{2}\frac{d\Omega_1}{dy} - 3\frac{df_2}{dy}\frac{du_1}{dy} - f_2\frac{d^2 u_1}{dy^2} + v^2\left(f_2 + \frac{d^2 f_2^{\circ\circ}}{dy^2} - 2\frac{df_1}{dy}\right) + \frac{1}{2}\Omega_1\left(f_3 - \frac{df_7}{dy} - 2f_{6a}\right) + v\frac{dv}{dy}(3f_2 - 4f_1) + \left(\frac{dv}{dy}\right)^2 f_2 + v\frac{d^2 v}{dy^2}f_2\right] = 0, \quad i = 1, 2,$$

$$\left(a_1^2 + 2a_1a_2 - a_2^2\right)\left(a_1^2 - 2a_1a_2 - a_2^2\right) \cdot \left[v\left(-2\frac{df_7}{dy} + \Omega_1 f_2 + 2f_3\right) - 2f_7\frac{dv}{dy}\right] = 0,$$

$$\lambda a_1 a_2\left[v^2\left(-3f_2\Omega_0 + 3f_5\right) - \frac{3w}{2}\left(\Omega_1 f_2 + f_3 + \frac{1}{2}\frac{df_7}{dy}\right) + \frac{3f_7}{2}\frac{dw}{dy} + 3vf_8\frac{dv}{dy}\right] = 0,$$

$$\left(a_1^2 + a_2^2\right)\left[f_7\frac{du_0}{dy} + 2u_0\frac{df_7}{dy} - f_2\Omega_1 u_0\right] = 0,$$

$$\left(a_1^4 + a_2^4\right)\left[u_1\left(2\frac{df_7}{dy} - \Omega_1 f_2\right) + 2v^2 f_{6a} + f_7\frac{du_1}{dy}\right] = 0,$$

$$\lambda(\mu_1 a_1 + \mu_2 a_2)\left[2w\left(f_4 - \frac{df_2}{dy}\right) + 2v\left(2\frac{df_8}{dy} + f_5 - 2f_2\Omega_0 + f_8\right) - f_2\frac{dw}{dy}\right] = 0,$$

$$\mu_i\left(a_1^2 + a_2^2\right)\left[v\left(\Omega_1 f_2 - f_3 - 2\frac{df_7}{dy}\right) - v^2\left(\frac{df_2}{dy} - 2f_4\right) - \frac{dv}{dy}(vf_2 - f_7)\right] = 0,$$

$$\lambda^3 a_i\left[2u_0\left(\Omega_0 f_2 - \frac{df_8}{dy}\right) - f_{6b}w - f_8\frac{du_2}{dy}\right] = 0, \quad i = 1, 2,$$

$$\lambda a_1 a_2\left[2f_8\frac{du_1}{dy} + 4u_1\frac{df_8}{dy} - w\left(\Omega_1 f_2 - 2f_{6a} - f_3 - 2\frac{df_7}{dy}\right) + 2v^2\left(f_5 + f_2\Omega_0 - \frac{df_8}{dy}\right) + 4u_1 f_2\Omega_0 - 2v\frac{dv}{dy}f_8 + f_7\frac{dw}{dy}\right] = 0. \tag{17}$$

Building on the solution of the less general system of [21] and after some tedious manipulations, the solution of (16)-(17) was constructed. For convenience we introduce a new variable v defined by the following relation [20]:

$$y = \int \frac{\sqrt{9\alpha^2 + 12\beta v - 36\alpha v^2 - 12v^4}}{4v^3 + 6\alpha v - \beta}dv. \tag{18}$$

We give here only the final form of the Lagrangian and the complementary cubic integral in the following form:

$$L_0 = \frac{1}{2}\left[x'^2 - \frac{3F}{F_1^2}v'^2\right] + \frac{1}{3}\left(P_2 x' - \overline{Q}_2 v'\right) + \frac{F_1}{F}\left[-\frac{v^6 + 24\alpha v^4 - 16\beta v^3 - 54\alpha^2 v^2 + 12\alpha\beta v - \beta^2}{4F} + \frac{4\rho_1^2 v + 3\rho_2^2\left(4v^3 - \beta\right) + 6\rho_1\rho_2\left(2v^2 - \alpha\right)}{6F} + \rho_3 v + \rho_4\right] + \frac{F_1^{3/2}}{F}\left\{\rho_5 + \frac{1}{F}\left[\rho_2\left(2v^3 - 3\alpha v + \beta\right) + \rho_1\left(2v^2 + \alpha\right)\right] \cdot (c\sin x + d\cos x)\right\} - \frac{F_1^3}{4F^2}\left[\left(c^2 - d^2\right)\cos 2x - 2cd \cdot \sin 2x\right] - \frac{F_1^{3/2}}{4F}(a\sin x + b\cos x), \tag{19}$$

$$I_2 = x'^3 + P_2 x'^2 + \overline{Q}_2 x' v' + P_1 x' + \overline{Q}_1 v' + R, \tag{20}$$

where $F = 4v^4 + 12\alpha v^2 - 4\beta v - 3\alpha^2$, $F_1 = 4v^3 + 6\alpha v - \beta$, $F_2 = 2v^2 + \alpha$, and ρ_i ($i = 1, 2, \ldots, 5$), a, b, c, and d are arbitrary parameters, introduced instead of the original parameters C_i and a_i for convenience, and

$$P_2 = \frac{3}{F}\left[\rho_1 F_2 + \rho_2\left(2\nu^3 - 3\alpha\nu + \beta\right)\right] + 3 \cdot \frac{8\nu^6 + 12\alpha\nu^4 + 8\beta\nu^3 + 54\alpha^2\nu^2 - 12\alpha\beta\nu + 9\alpha^3 + 2\beta^2}{2F\sqrt{F_1}}(c \cdot \sin x + d\cos x),$$

$$\overline{Q}_2 = \frac{9F}{2F_1^{3/2}}(d\sin x - c\cos x),$$

$$\overline{Q}_1 = -\frac{9}{4}\left[2cd\cos 2x + 2\left(c^2 - d^2\right)\sin 2x\right] + \frac{4}{2\sqrt{F_1^3}}\left\{\rho_5 F\left[(d+b)\sin x - (a+c)\cos x\right] + \left[\rho_1 F_2 + \left(2\nu^3 - 3\alpha\nu + \beta\right)\rho_2\right](d\sin x - c\cos x)\right\},$$

$$P_1 = \frac{3F_1}{2F^2}\left[8\nu^6 + 36\alpha\nu^4 - 16\beta\nu^3 - 18\alpha^2\nu^2 - 9\alpha^3 - \beta^2\right]\left[\left(c^2 - d^2\right)\cos 2x - 2cd\sin 2x\right] + \frac{9F_2}{2\sqrt{F_1}}(a\sin x + b\cos x) - \frac{3}{2F^2\sqrt{F_1}}\left[\rho_1\left(16\nu^8 + 32\alpha\nu^6 - 64\beta\nu^5 - 120\alpha^2\nu^4 - 64\alpha\beta\nu^3 - 216\alpha^3\nu^2 + 8\beta^2\nu^2 + 48\beta\alpha^2\nu - 9\alpha^4 - 4\alpha\beta^2\right) + \rho_2\left(16\nu^9 + 288\alpha\nu^7 - 144\beta\nu^6 + 216\alpha\nu^4(\alpha\nu - \beta) + 24\nu^3\left(\beta^2 + 3\alpha^3\right) - 108\beta\alpha^2\nu^2 + 81\alpha^4\nu + 36\alpha\beta^2\nu - 18\beta\alpha^3 - 4\beta^3\right) + \frac{\rho_5}{4}F^2F_1\right](c\sin x + d\cos x) + \frac{3\left(c^2 + d^2\right)}{2F^2}\left[16\nu^9 + 96\alpha\nu^7 - 24\beta\nu^6 + 216\alpha^2\nu^5 - 156\alpha\beta\nu^4 + 36\left(\beta^2 - 2\alpha^3\right)\nu^3 + 54\beta\alpha^2\nu^2 + 3\alpha\left(2\beta^2 + 27\alpha^3\right) - 9\beta\alpha^3 - \beta^3\right],$$

$$R = \frac{9\sqrt{F_1}}{F}\left[\frac{2\rho_1}{3}\nu + \rho_2\left(\nu^2 - \frac{\alpha}{2}\right)\right](a\sin x + b\cos x) + \frac{\sqrt{F_1^5}}{8F^3}\left\{40\nu^6 + 156\alpha\nu^4 - 56\beta\nu^3 - 18\alpha^2\nu^2 - 12\alpha\beta\nu - 27\alpha^3 - 2\beta^2\right\}\left[\left(3c^2 d - d^3\right)\cos 3x + \left(c^3 - 3cd\right)\sin 3x\right] + \left\{\frac{-8\sqrt{F_1}\rho_5}{27F}\left[4\nu\rho_1 + 3\rho_2\left(2\nu^2 - \alpha\right)\right] - \frac{8\rho_3\sqrt{F_1^3}}{27} + \frac{2\sqrt{F_1^3}\left(c^2 + d^2\right)}{27}\left[32\nu^9 + 288\alpha\nu^7 - 120\beta\nu^6 + 576\alpha^2\nu^5 - 612\alpha\beta\nu^4 - 648\alpha^3\nu^3 + 144\beta^2\nu^3 + 342\alpha^2\beta\nu^2 + 270\alpha^4\nu - 27\alpha^3\beta - 2\beta^3\right] - \frac{8\sqrt{F_1}}{27F^3}\left[\rho_1\left(16\alpha\nu^3 - 12\beta\nu^2 - 24\alpha^2\nu + 2\alpha\beta\right) + \rho_2\left(8\nu^6 + 60\alpha\nu^4 - 28\beta\nu^3 - 18\alpha^2\nu^2 - 6\alpha\beta\nu + 9\alpha^3 + 2\beta^2\right)\right]\left[\rho_1 F_2 + \rho_2\left(2\nu^3 - 3\alpha\nu + \beta\right)\right]\right\}[c\sin x + d\cos x] + 16F_1^2\left\{\frac{6\rho_5 F_2}{27F} + \left[\rho_1\left(80\nu^8 + 256\alpha\nu^6 - 96\beta\nu^5 + 120\alpha^2\nu^4 - 128\alpha\beta\nu^3 - 144\alpha^3\nu^2 + 12\beta^2\nu^2 + 24\alpha^2\beta\nu - 9\alpha^4 - 2\alpha\beta^2\right) + \rho_2\left(80\nu^9 + 384\alpha\nu^7 - 144\beta\nu^6 + 72\alpha^2\nu^5 - 120\alpha\beta\nu^4 + 12\nu^3\left(\beta^2 - 12\alpha^3\right) + 81\alpha^4\nu + 6\alpha\beta^2\nu - 18\beta\alpha^3 - 2\beta^3\right)\right]\frac{2}{27F^3}\right\}\left[\left(c^2 - d^2\right) \cdot \cos 2x - 2cd\sin 2x\right] + \frac{64\left(c^2 + d^2\right)}{927F}\left\{\rho_5\left[8\alpha\nu^3 - 6\beta\nu^2 - 12\alpha^2\nu + \alpha\beta\right] - 16\left[\left(4\alpha\rho_1 + \beta\rho_2\right)\nu^9 - 48\left(6\alpha^2\rho_2 + \beta\rho_1\right)\nu^8 + 576\alpha\beta\rho_2\nu^7 - 192\rho_2\left(\beta^2 - 6\alpha^3\right)\nu^6 + \left(288\alpha^3\rho_1 - 504\alpha^2\beta\rho_2 - 48\beta^2\rho_1\right)\nu^5 - 72\alpha\left(-6\alpha^3\rho_2 + 7\alpha\beta\rho_1 + \beta^2\rho_2\right)\nu^4 + \left(12\beta^2\left(\rho_2 + 8\alpha\rho_1\right) - 576\alpha^3\left(\alpha\rho_1 + \beta\rho_2\right)\right)\nu^3 - 12\beta\left(\beta^2\rho_1 - 12\alpha^2\left(\alpha\rho_1 + \beta\rho_2\right)\right)\nu^2 - 9\alpha\left(\beta\rho_2\left(3\alpha^2 + \beta^2\right) + 4\alpha\rho_1\left(3\alpha^3 + \beta^2\right)\right)\nu + 2\rho_2\left(27\alpha^6 + 18\alpha^3\beta^2 + 2\beta^4\right) + \alpha\beta\rho_1\left(2\beta^2 + 9\alpha^3\right)\right]\right\} - \frac{\rho_3}{4F}\left[\frac{F^{\circ\circ}}{4}\rho_1 + 6\left(\nu^3 - 3\alpha\nu + \beta\right) \cdot \rho_2\right] + \frac{1}{F^3}\left[\rho_1 F_2 + \rho_2\left(\nu^3 - 3\alpha\nu + \beta\right)\right]^3 - \frac{9(ac + bd)}{4F}\left[8\alpha\nu^3 - 6\beta\nu^2 - 12\alpha^2\nu + \alpha\beta\right]. \quad (21)$$

3. The Generic Unconditional System

The Lagrangian (19) describes a system integrable on its zero-level of Jacobi's integral I_1. Following the method devised by Yehia [21, 25] (for a detailed account of this method, see

[28]), we now proceed to construct the corresponding unconditional system by performing the inverse of time transformation (2). Our conditional system involves 4 energy-type parameters ρ_3, ρ_4, a, and b. We first express those parameters in terms of nine new parameters

$$\begin{aligned} \rho_3 &= -\frac{1}{2}\left(h_1 + \varepsilon_1 h\right), \\ \rho_4 &= -\left(h_2 + \varepsilon_2 h\right), \\ a &= h_3 + \varepsilon_3 h, \\ b &= h_4 + \varepsilon_4 h \end{aligned} \tag{22}$$

and then we can perform the time transformation (2) with

$$\begin{aligned} \Lambda = {} & \frac{\left(4\nu^3 + 6\alpha\nu - \beta\right)}{\left(3\alpha^2 + 4\beta\nu - 12\alpha\nu^2 - 4\nu^4\right)} \Big[\left(\varepsilon_1 \nu + \varepsilon_2\right) \\ & + \sqrt{4\nu^3 + 6\alpha\nu - \beta}\left(\varepsilon_3 \sin x + \varepsilon_4 \cos x\right)\Big] \end{aligned} \tag{23}$$

to the above system. Thus we obtain the Lagrangian

$$\begin{aligned} L = {} & \frac{1}{2}\left[\varepsilon_1 \nu + \varepsilon_2 + \sqrt{4\nu^3 + 6\alpha\nu - \beta}\left(\varepsilon_3 \sin x + \varepsilon_4 \cos x\right)\right] \\ & \cdot \left[\frac{\left(4\nu^3 + 6\alpha\nu - \beta\right)\dot{x}^2}{3\alpha^2 + 4\beta\nu - 12\alpha\nu^2 - 4\nu^4} + \frac{3\dot{\nu}^2}{\left(4\nu^3 + 6\alpha\nu - \beta\right)}\right] \\ & + \frac{1}{3}\left(P_2 \dot{x} - \overline{Q}_2 \dot{\nu}\right) \\ & + \frac{1}{\varepsilon_1 \nu + \varepsilon_2 + \sqrt{4\nu^3 + 6\alpha\nu - \beta}\left(\varepsilon_3 \sin x + \varepsilon_4 \cos x\right)} \Bigg\{\frac{h_1}{2} \\ & \cdot \nu + h_2 + \sqrt{4\nu^3 + 6\alpha\nu - \beta}\Bigg(h_3 \sin x + h_4 \cos x \\ & - \left[\frac{\rho_1\left(2\nu^2 + \alpha\right) + \rho_2\left(2\nu^3 - 3\alpha\nu + \beta\right)}{4\nu^4 + 12\alpha\nu^2 - 4\beta\nu - 3\alpha^2} + \rho_5\right](c \sin x \\ & + d \cos x)\Bigg) + \frac{1}{12\left(4\nu^4 + 12\alpha\nu^2 - 4\beta\nu - 3\alpha^2\right)}\Big[3\Big(4\nu^3 \\ & + 6\alpha\nu - \beta\Big)^2\left[\left(c^2 - d^2\right)\cos 2x - 2cd \sin 2x\right] - 8\rho_1^2 \nu \\ & - 12\rho_1\rho_2\left(\alpha - 2\nu^2\right) - 6\left(4\nu^3 - \beta\right)\rho_2^2 + 3\left(c^2 + d^2\right) \\ & \cdot \left(8\nu^6 + 24\alpha\nu^4 - 16\beta\nu^3 - 54\alpha^2\nu^2 + 12\alpha\beta\nu - \beta^2\right)\Big]\Bigg\} \\ & + h. \end{aligned} \tag{24}$$

The presence of the arbitrary parameter h in the last Lagrangian as an additive constant is insignificant and it can be ignored, as it does not contribute to the equations of motion. The same arbitrary constant h is now interpreted as the value of the Jacobi integral. Thus, we have the unconditional Jacobi integral

$$\begin{aligned} I_1 = {} & \frac{1}{2}\left[\varepsilon_1 \nu + \varepsilon_2 + \sqrt{4\nu^3 + 6\alpha\nu - \beta}\left(\varepsilon_3 \sin x + \varepsilon_4 \cos x\right)\right] \\ & \cdot \left[\frac{\left(4\nu^3 + 6\alpha\nu - \beta\right)\dot{x}^2}{3\alpha^2 + 4\beta\nu - 12\alpha\nu^2 - 4\nu^4} + \frac{3\dot{\nu}^2}{\left(4\nu^3 + 6\alpha\nu - \beta\right)}\right] \\ & - \frac{1}{\varepsilon_1 \nu + \varepsilon_2 + \sqrt{4\nu^3 + 6\alpha\nu - \beta}\left(\varepsilon_3 \sin x + \varepsilon_4 \cos x\right)} \Bigg\{\frac{h_1}{2} \\ & \cdot \nu + h_2 + \sqrt{4\nu^3 + 6\alpha\nu - \beta}\Bigg(h_3 \sin x + h_4 \cos x \\ & - \left[\frac{\rho_1\left(2\nu^2 + \alpha\right) + \rho_2\left(2\nu^3 - 3\alpha\nu + \beta\right)}{4\nu^4 + 12\alpha\nu^2 - 4\beta\nu - 3\alpha^2} + \rho_5\right](c \sin x \\ & + d \cos x)\Bigg) + \frac{1}{12\left(4\nu^4 + 12\alpha\nu^2 - 4\beta\nu - 3\alpha^2\right)}\Big[3\Big(4\nu^3 \\ & + 6\alpha\nu - \beta\Big)^2\left[\left(c^2 - d^2\right)\cos 2x - 2cd \sin 2x\right] - 8\rho_1^2 \nu \\ & - 12\rho_1\rho_2\left(\alpha - 2\nu^2\right) - 6\left(4\nu^3 - \beta\right)\rho_2^2 + 3\Big(8\nu^6 + 24\alpha\nu^4 \\ & - 16\beta\nu^3 - 54\alpha^2\nu^2 + 12\alpha\beta\nu - \beta^2\Big)\left(c^2 + d^2\right)\Big]\Bigg\} = h. \end{aligned} \tag{25}$$

The final form of the second integral can be obtained by replacing (x', ν') in (20) by $(\Lambda\dot{x}, \Lambda\dot{\nu})$. The Lagrangian (24) characterizes a new integrable system. It contains fifteen arbitrary parameters $\alpha, \beta, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \rho_1, \rho_2, \rho_5, c, d, h_1, h_2, h_3$, and h_4. Note that the angle variable x can be shifted by a phase angle in such a way to make one of the four parameters ε_3, ε_4, c, and d equal zero. The last system is an extension of the two systems with a cubic integral obtained in [21, 28] by adding the parameters c and d which invoke a part of the gyroscopic (irreversible) and potential terms.

4. Applications

In its full capacity, the fifteen-parameter system with the Lagrangian (24) has not yet found a mechanical interpretation for the full range of values of the parameters. In this section we provide four applications as special cases of that system: one integrable system on the sphere, one in the Euclidean plane, and two new integrable cases in rigid body dynamics. Those special cases indicate the richness of this system.

4.1. An Integrable System on the Sphere. The metric

$$\begin{aligned} ds^2 = {} & \Big[\varepsilon_1 \nu + \varepsilon_2 \\ & + \sqrt{4\nu^3 + 6\alpha\nu - \beta}\left(\varepsilon_3 \sin x + \varepsilon_4 \cos x\right)\Big] \\ & \cdot \left[\frac{\left(4\nu^3 + 6\alpha\nu - \beta\right)dx^2}{3\alpha^2 + 4\beta\nu - 12\alpha\nu^2 - 4\nu^4} + \frac{3d\nu^2}{\left(4\nu^3 + 6\alpha\nu - \beta\right)}\right] \end{aligned} \tag{26}$$

of the configuration space of the system described by (24) was considered in [28] and sufficient conditions for it to be Riemannian and well defined on S^2 were found. Regarding this result we formulate the following.

Theorem 1. *Suppose that* $8\alpha^3 + \beta^2 < 0$ *and let* ν_1, ν_2, *and* ν_3 *such that* $\nu_1 < \nu_2 < \nu_3$, $\nu_1 + \nu_2 + \nu_3 = 0$ *be three real roots of the cubic polynomial* $4\nu^3 + 6\alpha\nu - \beta$. *Let also* $\varepsilon_1 + \varepsilon_2\nu_1 > \sqrt{\varepsilon_3^2 + \varepsilon_4^2}[(-2\alpha)^{3/2} - \beta]$. *Then the Lagrangian (24) for* $\nu \in [\nu_1, \nu_2]$ *describes an integrable time-irreversible system on* S^2.

4.2. A New Integrable System in the Plane. As in [21], the Lagrangian (24) acquires the simplest form when one sets $\alpha = \beta = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0$, $\varepsilon_1 = 1$. Then, introducing the change of variables $x \to i\sqrt{3}x$, $\nu \to e^{2y}$, we reduce the Lagrangian (24) to the form

$$\begin{aligned}
L &= \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + P_1\dot{x} - V + h, \\
P_1 &= \alpha_1 e^{-4y} + \alpha_2 e^{-2y} + \alpha_3 e^{y+\sqrt{3}x} + \alpha_4 e^{y-\sqrt{3}x}, \\
V &= -\frac{2\alpha_1^2}{3}e^{-8y} - 2\alpha_1\alpha_2 e^{-6y} - 2\alpha_2^2 e^{-4y} - \alpha_6 e^{-2y} \\
&\quad - \alpha_1 e^{-3y}\left[\alpha_3 e^{\sqrt{3}x} + \alpha_4 e^{-\sqrt{3}x}\right] \\
&\quad - \alpha_2 e^{-y}\left[\alpha_3 e^{\sqrt{3}x} + \alpha_4 e^{-\sqrt{3}x}\right] \\
&\quad + \frac{2e^{y}}{27}\left[\alpha_4 e^{\sqrt{3}x} - \alpha_5 e^{-\sqrt{3}x}\right] \\
&\quad - \frac{e^{2y}}{2}\left[\alpha_3^2 e^{2\sqrt{3}x} + \alpha_4^2 e^{-2\sqrt{3}x} - \alpha_3\alpha_4\right],
\end{aligned} \tag{27}$$

where α_i, $i = 1, 2, \ldots, 6$ are arbitrary parameters, introduced instead of the original parameters for convenience:

$$\begin{aligned}
\ddot{x} + \Omega\dot{y} &= -\frac{\partial V}{\partial x}, \\
\ddot{y} - \Omega\dot{x} &= -\frac{\partial V}{\partial y}, \\
\Omega &= -4\alpha_1 e^{-4y} - 2\alpha_2 e^{-2y} + \alpha_3 e^{y+\sqrt{3}x} \\
&\quad + \alpha_4 e^{y-\sqrt{3}x}.
\end{aligned} \tag{28}$$

Jacobi's integral for this motion is

$$I_1 = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + V = h \tag{29}$$

and the cubic integral can be written as

$$\begin{aligned}
I_2 &= \dot{x}^3 - 3\dot{x}\dot{y}^2 + 9\left(\alpha_1 e^{-4y} + \alpha_2 e^{-2y}\right)\dot{x}^2 \\
&\quad + 3\sqrt{3}e^{y}\left[\alpha_3 e^{\sqrt{3}x} - \alpha_4 e^{-\sqrt{3}x}\right]\dot{x}\dot{y} - 3\left[\alpha_1 e^{-4y}\right. \\
&\quad \left. + \alpha_2 e^{-2y} + e^{y}\left(\alpha_3 e^{\sqrt{3}x} + \alpha_4 e^{-\sqrt{3}x}\right)\right]\dot{y}^2 \\
&\quad + \left[3e^{2y}\left(\alpha_3\alpha_4 - \alpha_3^2 e^{2\sqrt{3}x} - \alpha_4^2 e^{-2\sqrt{3}x}\right)\right. \\
&\quad + \frac{2e^{y}}{9}\left(\alpha_4 e^{\sqrt{3}x} - \alpha_5 e^{-\sqrt{3}x}\right) \\
&\quad + 3\alpha_2 e^{-y}\left(\alpha_3 e^{\sqrt{3}x} + \alpha_4 e^{-\sqrt{3}x}\right) \\
&\quad + 9\alpha_1 e^{-3y}\left(\alpha_3 e^{\sqrt{3}x} + \alpha_4 e^{-\sqrt{3}x}\right) + 6\alpha_6 e^{-2y} \\
&\quad \left. + 24\alpha_2^2 e^{-4y} + 16\alpha_1^2 e^{-8y} + 36\alpha_1\alpha_2 e^{-6y}\right]\dot{x} \\
&\quad + \left[-3\sqrt{3}e^{2y}\left(\alpha_4^2 e^{-2\sqrt{3}x} - \alpha_3^2 e^{2\sqrt{3}x}\right)\right. \\
&\quad - \frac{2\sqrt{3}e^{y}}{9}\left(\alpha_4 e^{\sqrt{3}x} + \alpha_5 e^{-\sqrt{3}x}\right) \\
&\quad + 3\sqrt{3}\alpha_2 e^{-y}\left(\alpha_3 e^{\sqrt{3}x} - \alpha_4 e^{-\sqrt{3}x}\right) \\
&\quad \left. + 3\sqrt{3}\alpha_1 e^{-3y}\left(\alpha_3 e^{\sqrt{3}x} - \alpha_4 e^{-\sqrt{3}x}\right)\right]\dot{y} \\
&\quad + 10\alpha_1^2 e^{-7y}\left(\alpha_3 e^{\sqrt{3}x} + \alpha_4 e^{-\sqrt{3}x}\right) \\
&\quad + 18\alpha_1\alpha_2 e^{-5y}\left(\alpha_3 e^{\sqrt{3}x} + \alpha_4 e^{-\sqrt{3}x}\right) + \frac{4}{9} \\
&\quad \cdot e^{-3y}\left[e^{\sqrt{3}x}\left(\alpha_1\alpha_4 + 27\alpha_2^2\alpha_3\right)\right. \\
&\quad \left. + e^{-\sqrt{3}x}\left(27\alpha_2^2\alpha_4 - \alpha_1\alpha_5\right)\right] \\
&\quad - \frac{2e^{2y}}{9}\left[\alpha_4\left(\alpha_5 e^{-2\sqrt{3}x} - \alpha_3 e^{2\sqrt{3}x}\right) - 2\alpha_3\alpha_5 + 2\alpha_4^2\right] \\
&\quad + \frac{2e^{-y}}{9}\left[e^{\sqrt{3}x}\left(4\alpha_2\alpha_4 + 27\alpha_3\alpha_6\right)\right. \\
&\quad \left. + e^{-\sqrt{3}x}\left(27\alpha_4\alpha_6 - 4\alpha_2\alpha_5\right)\right] \\
&\quad - e^{3y}\left[2\left(\alpha_4^3 e^{-3\sqrt{3}x} + \alpha_3^3 e^{3\sqrt{3}x}\right)\right. \\
&\quad \left. - 3\alpha_3\alpha_4\left(\alpha_3 e^{\sqrt{3}x} + \alpha_4 e^{-\sqrt{3}x}\right)\right] + 2e^{-6y}\left(8\alpha_2^3\right. \\
&\quad \left. + 3\alpha_1\alpha_6\right) - 6\alpha_2\left(\alpha_4^2 e^{-2\sqrt{3}x} + \alpha_3^2 e^{2\sqrt{3}x}\right) + 8\alpha_1^3 e^{-12y} \\
&\quad + 28\alpha_2\alpha_1^2 e^{-10y} + 6\alpha_2\alpha_6 e^{-4y} + 36\alpha_1\alpha_2^2 e^{-8y} \\
&\quad + 9\alpha_1\alpha_3\alpha_4 e^{-2y}.
\end{aligned} \tag{30}$$

This integrable system can be viewed as a generalization of a previously known one due to Yehia [21] by the introduction of two constants α_3 and α_4 to equations of motion. It also generalizes the reversible Toda-like system obtained by Hall [13] by the presence of the four parameters $\alpha_1, \alpha_2, \alpha_3$, and α_4.

4.3. Applications to Rigid Body Dynamics. The problem of motion of a rigid body whose principal moments of inertia are A, A, and C, about a fixed point under forces with a scalar potential $V(\gamma)$ and vector potential $\mathbf{l} = (0, 0, l_3)$, reduces after ignoring the cyclic angle of precession ψ to the Routhian

$$R = \frac{1}{2}\left[\frac{\dot{\gamma}_3^2}{1-\gamma_3^2} + \frac{C\left(1-\gamma_3^2\right)\dot{\varphi}^2}{A-(A-C)\gamma_3^2}\right] + \frac{\left(fC\gamma_3 + Al_3\left(1-\gamma_3^2\right)\right)\dot{\varphi}}{A\left[A-(A-C)\gamma_3^2\right]} - \frac{1}{A}\left\{V + \frac{\left(f-l_3\gamma_3\right)^2}{2\left[A-(A-C)\gamma_3^2\right]}\right\}, \quad (31)$$

where $\gamma_3 = \cos(\theta)$, θ is the nutation angle, φ is the angle of proper rotation, and f is the value of cyclic integral. For more details see [21].

As in [21], the Lagrangian (24) can be identified with the Routhian (31) in the following two cases.

4.3.1. Case (a): $A = 4C$, $\alpha = -1/2$, $\beta = 1$, $\varepsilon_1 = -1/3$, $\varepsilon_2 = -1/6$, and $\varepsilon_3 = \varepsilon_4 = 0$. In this case, using the substitution $\nu = (1/2)(3\gamma_3^2 - 1)$, the Lagrangian (24) can be identified with the Routhian (31) if we assume that the moments of inertia satisfy $A = 4C$, set the cyclic constant $f = 0$, and choose

$$\begin{aligned}
l_1 &= 4Cn\gamma_1,\\
l_2 &= 4Cn\gamma_2,\\
l_3 &= C\left[k + e_0\left(\frac{2}{\gamma_3^4} - \frac{1}{\gamma_3^2}\right) + \frac{e_1}{\gamma_1^2+\gamma_2^2} + e_2\gamma_1 + e_3\gamma_2 + n\gamma_3\right],\\
V &= C\left[e_5\gamma_1 + e_6\gamma_2 + \frac{\varepsilon}{\gamma_3^2} + e_0^2\left(\frac{4}{\gamma_3^6} - \frac{2}{\gamma_3^8} - \frac{5}{2\gamma_3^4}\right) - \frac{e_1\left(k+e_0-2e_1\right)}{\gamma_1^2+\gamma_2^2} - \frac{e_1^2}{2\left(\gamma_1^2+\gamma_2^2\right)^2} - \frac{e_3^2}{2}\left(\gamma_2^2+\gamma_3^2\right) \right.\\
&\quad - e_2e_3\gamma_1\gamma_2 + \frac{e_2^2}{2}\gamma_2^2 + \frac{3n^2}{2}\gamma_3^2 + \left\{\frac{e_0\left(\gamma_3^2-2\right)}{\gamma_3^4} - n\gamma_3 - \frac{e_1}{\gamma_1^2+\gamma_2^2}\right\}\left(e_2\gamma_1+e_3\gamma_2\right)\\
&\quad \left. + n\left\{\frac{e_0\left(\gamma_3^2-2\right)}{\gamma_3^3} - \gamma_3\left(k + \frac{e_1}{\gamma_1^2+\gamma_2^2}\right)\right\}\right].
\end{aligned} \quad (32)$$

The cyclic integral can be written in the form

$$I_1 = 4p\gamma_1 + 4q\gamma_2 + \left[r + k + e_2\gamma_1 + e_3\gamma_2 + \frac{e_1}{\gamma_1^2+\gamma_2^2} + e_0\left(\frac{2}{\gamma_3^4} - \frac{1}{\gamma_3^2}\right)\right]\gamma_3 + n\left[4\left(\gamma_1^2+\gamma_2^2\right) + \gamma_3^2\right] = 0. \quad (33)$$

The complementary cubic integral is

$$\begin{aligned}
I_2 &= \left[r - k + e_2\gamma_1 + e_3\gamma_2 + n\gamma_3 + \frac{e_0\left(2-\gamma_3^2\right)}{\gamma_3^4} - \frac{e_1\left(8\gamma_1^2-1\right)}{\left(\gamma_1^2+\gamma_2^2\right)}\right]\left[\left(p + n\gamma_1 + \frac{e_2}{2}\gamma_3\right)^2\right.\\
&\quad + \left(q + n\gamma_2 + \frac{e_3\gamma_3}{2}\right)^2 + \frac{\varepsilon}{2\gamma_3^2} + k\left(\frac{e_0}{\gamma_3^4} - \frac{e_1}{2}\right)\\
&\quad - \left(\frac{e_1}{2} + \frac{e_0\left(2-\gamma_3^2\right)}{2\gamma_3^4}\right)\left(r + n\gamma_3\right)\\
&\quad + \left(\frac{e_1}{2} + \frac{e_0\left(2-\gamma_3^2\right)}{2\gamma_3^4}\right)\left(e_2\gamma_1 + e_3\gamma_2\right)\\
&\quad - \frac{e_0^2\left(3\gamma_3^4 - 6\gamma_3^2 + 4\right)}{2\gamma_3^8} + \frac{e_1}{\gamma_1^2+\gamma_2^2}\left(\frac{e_1\left(1-8\gamma_1^2\right)}{2}\right.\\
&\quad \left.\left. + \frac{8\gamma_1^2\left(\gamma_3^2-2\right) - 2\gamma_3^4 + 9\gamma_3^2 - 8}{2\gamma_3^4}\right)\right]\\
&\quad - \gamma_3\left[\left(2e_1e_2 - e_2k + e_5\right)\left(p + n\gamma_1 + \frac{e_2\gamma_3}{2}\right)\right.\\
&\quad \left. + \left(2e_1e_3 - e_3k + e_6\right)\left(q + n\gamma_2 + \frac{e_3}{2}\gamma_3\right)\right]\\
&\quad + k\left[\left(\frac{e_0}{\gamma_3^2} + \frac{e_1\left(1-2\gamma_3^2\right)}{\gamma_1^2+\gamma_2^2}\right)\left(e_2\gamma_1 + e_3\gamma_2\right)\right.\\
&\quad + \frac{4e_1\gamma_1^2\left(e_0 - 2e_1\gamma_3^2\right)}{\gamma_3^2\left(\gamma_1^2+\gamma_2^2\right)}\\
&\quad \left. + -\frac{4e_1\gamma_3}{\gamma_1^2+\gamma_2^2}\left[\left(p + n\gamma_1\right)\gamma_1 + \left(q + n\gamma_2\right)\gamma_2\right]\right]\\
&\quad - \frac{8e_0e_1e_3\gamma_2\gamma_1^2}{\gamma_3^2\left(\gamma_1^2+\gamma_2^2\right)} + \left(\frac{e_0}{\gamma_3^2} - \frac{e_1}{\gamma_1^2+\gamma_2^2}\right)\left(e_5\gamma_1 + e_6\gamma_2\right)\\
&\quad + 4e_1\gamma_1^2\left(\frac{2e_0^2}{\gamma_3^6} + \frac{\varepsilon}{\gamma_3^2\left(\gamma_1^2+\gamma_2^2\right)}\right) - \frac{8e_1e_2\gamma_1^3}{\gamma_1^2+\gamma_2^2}\left(e_1\right.\\
&\quad \left. + \frac{e_0}{\gamma_3^2}\right) + \frac{4e_0e_1^2\gamma_1^2}{\gamma_3^4\left(\gamma_1^2+\gamma_2^2\right)^2}\left[8\gamma_1^2\left(\gamma_3^2-2\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&-\left(\gamma_1^2+\gamma_2^2\right)\left(\gamma_3^2-4\right)\Big]+\frac{e_1\left(e_3^2+e_2^2\right)}{9\left(\gamma_1^2+\gamma_2^2\right)}\left[18\gamma_1^2\gamma_3^2\right.\\
&\left.-9\gamma_3^4+13\gamma_3^2-4\right]+\frac{8e_1^3\gamma_1^3\left(1-4\gamma_1^2\right)}{\left(\gamma_1^2+\gamma_2^2\right)^2}\\
&+\frac{8e_1\left(\gamma_1^2-\gamma_2^2\right)}{\gamma_1^2+\gamma_2^2}\left(q+n\gamma_2\right)^2\\
&+\frac{2e_1\gamma_3}{\gamma_1^2+\gamma_2^2}\Big[\left(q+n\gamma_2\right)\left(e_3\left(5\gamma_1^2-\gamma_2^2\right)-2e_2\gamma_1\gamma_2\right)\\
&+\left(p+n\gamma_1\right)\left(e_2\left(3\gamma_1^2+\gamma_2^2\right)-2e_3\gamma_1\gamma_2\right)\Big]\\
&-\frac{16e_1\gamma_1\gamma_2}{\gamma_1^2+\gamma_2^2}\left[\left(q+n\gamma_2\right)\left(p+n\gamma_1\right)-\frac{e_1e_3}{2}\gamma_1\right],
\end{aligned}
\tag{34}
$$

where k, ε, n, and e_i $(i = 0, 1, \ldots, 6)$ are arbitrary parameters, introduced instead of the original parameters. This choice (32) characterizes a new integrable problem in the dynamics of a rigid body, which generalizes all previously known integrable cases with a cubic integral in this field, as in Table 1.

4.3.2. Case (b): $A = (4/3)C$, $\alpha = 0$, $\beta = 1$, and $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0$. For this case we use the substitution $\nu = \gamma_3^{2/3}$. We construct the integrable case of a rigid body dynamics in which $A = B = 4/3$, $C = 1$, and

$$
\begin{aligned}
l_3 &= n+\frac{1}{\gamma_1^2+\gamma_2^2}\left[3n+e_0\gamma_3^{2/3}+\frac{e_1\left(2+\gamma_3^2\right)}{\gamma_3^{2/3}}\right]\\
&\quad+\frac{e_2\gamma_1+e_3\gamma_2}{\gamma_3^{2/3}},\\
V &= \frac{e_4\gamma_1+e_5\gamma_2+e_6}{\gamma_3^{2/3}}\\
&\quad+\frac{\left(e_3^2-e_2^2\right)\left(\gamma_1^2-\gamma_2^2\right)-4e_2e_3\gamma_1\gamma_2}{4\gamma_3^{4/3}}\\
&\quad+\frac{1}{\left(\gamma_1^2+\gamma_2^2\right)^2}\left[\frac{e_0^2\left(4-7\gamma_3^2\right)}{6\gamma_3^{2/3}}-e_0e_1\left(5\gamma_3^2-2\right)\right.\\
&\quad-3e_0n\gamma_3^{8/3}-\frac{e_1^2\left(13\gamma_3^4-8\gamma_3^2+4\right)}{2\gamma_3^{4/3}}\\
&\quad\left.-3ne_1\gamma_3^{4/3}\left(\gamma_3^2+2\right)+\frac{3n^2\gamma_3^2}{2}\left(\gamma_3^2-4\right)\right]\\
&\quad-\frac{e_2^2+e_3^2}{4\gamma_3^{4/3}}\left(5\gamma_3^2+1\right)-\frac{e_2\gamma_1+e_3\gamma_2}{3\left(\gamma_1^2+\gamma_2^2\right)}\left[3e_0+3e_1\gamma_3^{2/3}\right.\\
&\quad\left.+\left(e_7+9n\right)\gamma_3^{4/3}+\frac{6e_1}{\gamma_3^{4/3}}-\frac{e_7}{\gamma_3^{2/3}}\right],
\end{aligned}
\tag{35}
$$

Table 1

Conditions	Authors	Reference
$e_0 = e_1 = \varepsilon = n = 0$	Sokolov and Tsiganov	[32] 2002
$e_2 = e_3 = 0$	Yehia	[21] 2002
$e_0 = e_1 = e_2 = e_3 = 0$	Yehia	[33] 1996
$e_0 = e_1 = e_2 = e_3 = \varepsilon = n = 0$	Sretensky	[34] 1963
$e_0 = e_1 = e_2 = e_3 = k = n = 0$	Goriachev	[29] 1915
$e_0 = e_1 = e_2 = e_3 = k = \varepsilon = n = 0$	Goriachev	[35] 1900

where n and e_i, $i = 0, 1, \ldots, 7$, are arbitrary constant. This choice (35) gives a new integrable case in a rigid body dynamics. This case adds two arbitrary parameters e_2 and e_3 to the case found by Yehia [21] and has five arbitrary parameters n, e_0, e_1, e_2, and e_3, more than the original case found by Goriachev [29] in 1915.

Although having no obvious physical meaning, Goriachev's case has received a growing interest in the last years [30, 31]. It turns out to be the first example of a mechanical system whose complex invariant varieties are strata of Jacobians of a nonhyperelliptic curve, here a trigonal curve of genus 3 [31].

Competing Interests

The authors declare that they have no competing interests.

References

[1] J. Bertrand, "Sur les intégrales commune à plusieurs problémes de mécanique," *Journal de Mathématiques Pures et Appliquées*, vol. 17, pp. 121–174, 1852.

[2] J. Bertrand, "Memoire sur quelque-unes des formes les plus simple que puissent présenter les intégrals des équations différentielles du movement d'un point matériel," *Journal de Mathématiques Pures et Appliquées, Série II*, vol. 2, pp. 113–140, 1857.

[3] G. Darboux, "Sur un problème de Mécanique," *Archives Néerlandaises*, vol. 6, no. 2, pp. 371–376, 1901.

[4] G. Bozis, "Compatibility conditions for a nonquadratic integral of motion," *Celestial Mechanics*, vol. 28, no. 4, pp. 367–380, 1982.

[5] B. Grammaticos, B. Dorizzi, and A. Ramani, "Integrability of Hamiltonians with third- and fourth-degree polynomial potentials," *Journal of Mathematical Physics*, vol. 24, no. 9, pp. 2289–2295, 1983.

[6] G. Thompson, "Polynomial constants of motion in flat space," *Journal of Mathematical Physics*, vol. 25, no. 12, pp. 3474–3478, 1984.

[7] M. Wojciechowska and S. Wojciechowski, "New integrable potentials of two degrees of freedom," *Physics Letters A*, vol. 105, no. 1-2, pp. 11–14, 1984.

[8] S. Wojciechowski, "Construction of integrable systems by dressing a free motion with a potential," *Physics Letters A*, vol. 96, no. 8, pp. 389–392, 1983.

[9] B. C. Xanthopoulos, "Integrals of motion and analytic functions," *Journal of Physics A: Mathematical and General*, vol. 17, no. 1, pp. 87–94, 1984.

[10] J. Hietarinta, "Direct methods for the search of the second invariant," *Physics Reports A: Review Section of Physics Letters*, vol. 147, no. 2, pp. 87–154, 1987.

[11] G. D. Birkhoff, *Dynamical Systems*, American Mathematical Society, New York, NY, USA, 1927.

[12] P. O. Vandervoort, "Isolating integrals of the motion for stellar orbits in a rotating galactic bar," *The Astrophysical Journal*, vol. 232, pp. 91–105, 1979.

[13] L. S. Hall, "A theory of exact and approximate configurational invariant," *Physica D*, vol. 8, no. 1-2, pp. 90–116, 1983.

[14] W. Sarlet, P. G. L. Leach, and F. Cantrijn, "First integrals versus configurational invariants and a weak form of complete integrability," *Physica D: Nonlinear Phenomena*, vol. 17, no. 1, pp. 87–98, 1985.

[15] B. Dorizzi, B. Grammaticos, A. Ramani, and P. Winternitz, "Integrable Hamiltonian systems with velocity-dependent potentials," *Journal of Mathematical Physics*, vol. 26, no. 12, pp. 3070–3079, 1985.

[16] E. V. Ferapontov and A. P. Fordy, "Non-homogeneous systems of hydrodynamic type, related to quadratic Hamiltonians with electromagnetic term," *Physica D: Nonlinear Phenomena*, vol. 108, no. 4, pp. 350–364, 1997.

[17] E. V. Ferapontov and A. P. Fordy, "Commuting quadratic Hamiltonians with velocity-dependent potentials," *Reports on Mathematical Physics*, vol. 44, pp. 71–80, 1999.

[18] V. G. Marikhin and V. V. Sokolov, "Pairs of commuting Hamiltonians that are quadratic in momenta," *Theoretical and Mathematical Physics*, vol. 149, no. 2, pp. 1425–1436, 2006.

[19] V. G. Marikhin and V. V. Sokolov, "On quasi-Stäckel Hamiltonians," *Russian Mathematical Surveys*, vol. 60, no. 5, pp. 981–983, 2005.

[20] H. M. Yehia, "On the integrability of certain problems in particle and rigid body dynamics," *Journal of Theoretical and Applied Mechanics*, vol. 5, no. 1, pp. 55–71, 1986.

[21] H. M. Yehia, "On certain two-dimensional conservative mechanical systems with a cubic second integral," *Journal of Physics A: Mathematical and General*, vol. 35, no. 44, pp. 9469–9487, 2002.

[22] H. M. Yehia, "Atlas of two-dimensional irreversible conservative Lagrangian mechanical systems with a second quadratic integral," *Journal of Mathematical Physics*, vol. 48, no. 8, Article ID 082902, 32 pages, 2007.

[23] H. M. Yehia and A. A. Elmandouh, "New integrable systems with a quartic integral and new generalizations of Kovalevskaya's and Goriatchev's cases," *Regular and Chaotic Dynamics*, vol. 13, no. 1, pp. 57–69, 2008.

[24] H. M. Yehia, "Further classification of 2D integrable mechanical systems with quadratic invariants," *Regular and Chaotic Dynamics. International Scientific Journal*, vol. 14, no. 4-5, pp. 571–579, 2009.

[25] H. M. Yehia, "The master integrable two-dimensional system with a quartic second integral," *Journal of Physics A: Mathematical and General*, vol. 39, no. 20, pp. 5807–5824, 2006.

[26] A. A. Elmandouh, "New integrable problems in the dynamics of particle and rigid body," *Acta Mechanica*, vol. 226, no. 11, pp. 3749–3762, 2015.

[27] L. A. Pars, *A Treatise on Analytical Dynamics*, Heinemann, London, UK, 1964.

[28] H. M. Yehia, "Completely integrable 2D Lagrangian systems and related integrable geodesic flows on various manifolds," *Journal of Physics. A: Mathematical and Theoretical*, vol. 46, no. 32, Article ID 325203, 22 pages, 2013.

[29] D. N. Goriachev, "New cases of motion of a rigid body about a fixed point," *Warshav. Univ. Izvest.*, no. 3, pp. 1–11, 1915.

[30] A. V. Tsiganov, "On a family of integrable systems on S^2 with a cubic integral of motion," *Journal of Physics A: Mathematical and General*, vol. 38, no. 4, pp. 921–927, 2005.

[31] H. W. Braden, V. Z. Enolski, and Y. N. Fedorov, "Dynamics on strata of trigonal Jacobians and some integrable problems of rigid body motion," *Nonlinearity*, vol. 26, no. 7, pp. 1865–1889, 2013.

[32] V. V. Sokolov and A. V. Tsiganov, "Commutative Poisson subalgebras for sklyanin brackets and deformations of some known integrable models," *Theoretical and Mathematical Physics*, vol. 133, pp. 1730–1743, 2002.

[33] H. M. Yehia, "On a generalization of certain results of Goriatchev, Chaplygin and Sretensky in the dynamics of rigid bodies," *Journal of Physics A: Mathematical and General*, vol. 29, no. 24, pp. 8159–8161, 1996.

[34] L. N. Sretensky, "On certain cases of motion of a heavy rigid body with a gyroscope," *Vestnik Moskovskogo Universiteta*, no. 3, pp. 60–71, 1963.

[35] D. N. Goriachev, "On the motion of a rigid body about a fixed point in the case $A = B = 4C$," *Matemat Sbornik Kruzhka Lyubitelei Mat. Nauk*, vol. 21, no. 3, pp. 431–438, 1900.