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Research Article

Twisted Frobenius Identities from Vertex Operator Superalgebras

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In consideration of the continuous orbifold partition function and a generating function for all n-point correlation functions for the rank two free fermion vertex operator superalgebra on the self-sewing torus, we introduce the twisted version of Frobenius identity.

1. Torus Intertwining n-Point Functions

1.1. Torus Intertwining n-Point Functions for $\mathcal{M}$. In this section we recall the computation [1] of the torus intertwining n-point functions for $\mathcal{M}$. Here we recall several constructions from [1]. For the notions of free fermionic VOSA and its twisted modules see Appendix. Define the square-bracket vertex operator [2] $\mathcal{Y}[u \otimes e^{\alpha}, z] = \mathcal{Y}(q_z^{L(0)}(u \otimes e^{\alpha}), q_z - 1)$, for $u \otimes e^{\alpha} \in \mathcal{M}$ and where $q_z = e^z$. As in [2] we find that $(V, \mathcal{Y}[\ ,\ ], \mathbf{1}, \tilde{\omega})$ and $(V, \mathcal{Y}(\ ,\ ), \mathbf{1}, \omega)$ are isomorphic generalized VOAs with $\tilde{\omega} = \omega - (1/24)\mathbf{1}$, a new conformal vector with vertex operator $Y[\tilde{\omega}, z] = \sum_{n\in\mathbb{Z}} L[n] z^{-n-2}$. We let $\mathrm{wt}[u \otimes e^{\alpha}] = \mathrm{wt}[u] + (1/2)\alpha^2$ denote the weight of an $L[0]$ homogeneous vector $u \otimes e^{\alpha}$. Let $M \otimes e^{\alpha}$ be an irreducible M-module for some $\alpha \in \mathbb{C}$ with torus partition function

$$Z_{\alpha}^{(1)}(q) = \mathrm{Tr}_{M\otimes e^{\alpha}} q^{L(0)-1/24} = \frac{q^{(1/2)\alpha^2}}{\eta(q)}, \tag{1}$$

where $\eta(q) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the Dedekind eta-function for modular parameter q. In general, we define the genus one intertwining n-point correlation function on $M \otimes e^{\alpha}$ for n vectors $u_1 \otimes e^{\beta_1}, \ldots, v_n \otimes e^{\beta_n} \in \mathcal{M}$ by

$$\begin{aligned} Z_{\alpha}^{(1)}&\left(u_1 \otimes e^{\beta_1}, z_1; \ldots; u_n \otimes e^{\beta_n}, z_n; q\right) \\ &= \mathrm{Tr}_{M\otimes e^{\alpha}}\left(\mathcal{Y}\left(q_1^{L(0)}\left(u_1 \otimes e^{\beta_1}\right), q_1\right)\right. \\ &\quad \left.\cdots \mathcal{Y}\left(q_n^{L(0)}\left(u_n \otimes e^{\beta_n}\right), q_n\right) q^{L(0)-1/24}\right), \end{aligned} \tag{2}$$

for formal $q_i = e^{z_i}$ with $i = 1, \ldots, n$. Since $e^{\beta \hat{q}} M \otimes e^{\alpha} = M \otimes e^{\alpha+\beta}$ it follows that the n-point function vanishes when $\sum_{i=1} \beta_i \neq 0$.

In [1] we describe a natural generalization of previous results in [3, 4]. Firstly, consider the n-point functions for n highest weight vectors $\mathbf{1} \otimes e^{\beta_i}$, which we abbreviate below to e^{β_i}, for $i = 1, \ldots, n$.

Proposition 1. *For $\sum_{i=1}^{n} \beta_i = 0$ then*

$$\begin{aligned} Z_{\alpha}^{(1)}&\left(e^{\beta_1}, z_1; \ldots; e^{\beta_n}, z_n; q\right) \\ &= \frac{q^{(1/2)\alpha^2}}{\eta(\tau)} \exp\left(\alpha \sum_{i=1}^{n} \beta_i z_i\right) \prod_{1\leq r<s\leq n} K(z_{rs}, \tau)^{\beta_r \beta_s}, \end{aligned} \tag{3}$$

where $z_{rs} = z_r - z_s$ and $K(z, \tau)$ is the genus one prime form (A.6).

It is a natural generalization of results developed in [3, 4]. In [1] generalizing results of [3] we obtain a closed form for the general n-point function (2). In particular, due to (B.16), we may apply standard genus one Zhu recursion theory [2] to reduce (2) to an explicit multiple of (3) to find the following.

Proposition 2. *For* $\sum_{i=1}^{n}\zeta_i = 0$ *then*

$$Z_{\alpha}^{(1)}\left(u_1\otimes e^{\zeta_1}, z_1;\ldots;u_n\otimes e^{\zeta_n}, z_n; q\right) = Q_{\alpha}^{\zeta_1,\ldots,\zeta_n}\left(u_1, z_1;\ldots;u_n, z_n; q\right) \cdot Z_{\alpha}^{(1)}\left(e^{\zeta_1}, z_1;\ldots;e^{\zeta_n}, z_n; q\right), \quad (4)$$

where $Q_{\alpha}^{\zeta_1,\ldots,\zeta_n}(u_1, z_1;\ldots;u_n, z_n; q)$ *is an explicit sum of elliptic and quasi-modular forms introduced in [3].*

1.2. Torus Intertwined n-Point Functions for $V_{\mathbb{Z}}$. Let g_1, h_1 be commuting automorphisms of $V_{\mathbb{Z}}$ defined by

$$\sigma f_1 = e^{2\pi i\beta_1 a(0)}, \qquad \sigma g_1 = e^{-2\pi i\alpha_1 a(0)}, \quad (5)$$

where $\sigma = e^{\pi i a(0)}$ is the fermion number automorphism. We assume that $\alpha_1, \beta_1 \in \mathbb{R}$ so that $\theta_1 = -e^{-2\pi i\beta_1}$ and $\phi_1 = -e^{2\pi i\alpha_1}$ are of unit modulus.

We then consider in [1] torus orbifold intertwining $n+2$-point functions for the σg_1-twisted module $V_{\mathbb{Z}+\alpha_1} = \oplus_{m\in\mathbb{Z}} M\otimes e^{m+\alpha_1}$ for $V_{\mathbb{Z}}$ defined by

$$\begin{aligned} &Z_{V_{\mathbb{Z}}}^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}\left(u_1\otimes e^{m_1}, z_1;\ldots,u_n\otimes e^{m_n}, z_n; v_1\otimes e^{n_1+\kappa}, w; v_2\otimes e^{n_2-\kappa}, 0; q\right) \\ &= \mathrm{Tr}_{V_{\mathbb{Z}+\alpha_1}}\left(\sigma f_1 Y\left(q_1^{L(0)}\left(u_1\otimes e^{m_1}\right), q_1\right)\cdots Y\left(q_n^{L(0)}\left(u_n\otimes e^{m_n}\right), q_n\right) \cdot \mathcal{Y}\left(q_w^{L(0)}\left(v_1\otimes e^{n_1+\kappa}\right), q_w\right)\mathcal{Y}\left(v_2\otimes e^{n_2-\kappa}, 1\right) \cdot q^{L(0)-1/24}\right) \\ &= \sum_{\mu\in\mathbb{Z}+\alpha_1} e^{2\pi i\mu\beta_1} Z_{\mu}^{(1)}\left(u_1\otimes e^{m_1}, z_1;\ldots;u_n\otimes e^{m_n}, z_n; v_1\otimes e^{n_1+\kappa}, w; v_2\otimes e^{n_2-\kappa}, 0; q\right), \end{aligned} \quad (6)$$

where $m_i, n_1,$ and $n_2 \in \mathbb{Z}$ for $\kappa \in (-1/2, 1/2)$ and $q_w = e^w$. In particular, we find

$$Z_{V_{\mathbb{Z}}}^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}\left(e^{\kappa}, w; e^{-\kappa}, 0; q\right) = \sum_{\mu\in\mathbb{Z}+\alpha_1} e^{2\pi i\mu\beta_1} Z_{\mu}^{(1)}\left(e^{\kappa}, w; e^{-\kappa}, 0; q\right) = \frac{1}{\eta(q)}\frac{\vartheta\left[\begin{smallmatrix}\alpha_1\\ \beta_1\end{smallmatrix}\right](\kappa w,\tau)}{K(w,\tau)^{\kappa^2}}, \quad (7)$$

for genus one theta series (A.1).

More generally, we define in [1] a generating function for all $n+2$-point functions (A.1) by the following formal differential form:

$$\mathcal{G}_n^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}\left(x_1, y_1,\ldots,x_n, y_n\right) \equiv Z_{V_{\mathbb{Z}}}^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}\left(\psi^+, x_1;\psi^-, y_1;\ldots;\psi^+, x_n;\psi^-, y_n; e^{\kappa}, w; e^{-\kappa}, 0; q\right)\prod_{i=1}^{n} dx_i^{1/2} dy_i^{1/2}, \quad (8)$$

for $V_{\mathbb{Z}}$ generators $\psi^{\pm} = e^{\pm 1}$ alternatively inserted at x_i, y_i for $i = 1,\ldots,n$. Recall the notion of the Szegő kernel described in the Appendix. Then we prove in [1] the following.

Proposition 3. *The generating form (8) is given by*

$$\mathcal{G}_n^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}\left(x_1, y_1,\ldots,x_n, y_n\right) = Z_{V_{\mathbb{Z}}}^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}\left(e^{\kappa}, w; e^{-\kappa}, 0; q\right)\det S_{\kappa}, \quad (9)$$

where S_{κ} *denotes the* $n\times n$ *matrix with components* $S_{\kappa}(x_i, y_j)$ *for* $i, j = 1,\ldots,n$ *for Szegő kernel (A.9).*

Finally, we obtain in [1] the following generalization of Proposition 15 of [4] concerning the generating properties of (8).

Proposition 4. $\mathcal{G}_m^{(1)}\left[\begin{smallmatrix} f_1 \\ g_1 \end{smallmatrix}\right](x_1, y_1,\ldots,x_m, y_m)$ *is a generating function for all torus orbifold intertwining* $n+2$*-point functions (A.1). In particular, for a pair of square-bracket mode twisted Fock vectors (B.10)*

$$\begin{aligned} \Psi_{\kappa}\left[\mathbf{k}_1,\mathbf{l}_2\right] &= e^{\kappa\widehat{q}}\psi^+\left[-k_{11}\right]\cdots\psi^+\left[-k_{1s_1}\right]\cdot\psi^-\left[-l_{21}\right]\cdots\psi^-\left[-l_{2t_2}\right]\mathbf{1}, \\ \Psi_{-\kappa}\left[\mathbf{k}_2,\mathbf{l}_1\right] &= e^{-\kappa\widehat{q}}\psi^+\left[-k_{21}\right]\cdots\psi^+\left[-k_{2s_2}\right]\cdot\psi^-\left[-l_{11}\right]\cdots\psi^-\left[-l_{1t_1}\right]\mathbf{1}, \end{aligned} \quad (10)$$

for $p = s_1 + s_2 = t_1 + t_2 > 0$*; then*

$$Z_{V_{\mathbb{Z}}}^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}\left(\Psi_{\kappa}\left[\mathbf{k}_1,\mathbf{l}_2\right], w;\Psi_{-\kappa}\left[\mathbf{k}_2,\mathbf{l}_1\right], 0; q\right) = \epsilon Z_{V_{\mathbb{Z}}}^{(1)}\begin{bmatrix} f_1 \\ g_1 \end{bmatrix}\left(e^{\kappa}, w; e^{-\kappa}, 0; q\right)\det C_{ab}\left(\mathbf{k}_a,\mathbf{l}_b\right), \quad (11)$$

where $\epsilon = (-1)^{(t_1+s_2)t_2+\lfloor (1/2)p\rfloor} e^{i\pi B\kappa(s_2-t_1)}$*, for the odd integer* B *fixed in (B.18) and*

$$C_{ab}\left(\mathbf{k}_a,\mathbf{l}_b\right) = \begin{bmatrix} C_{11}\left(\mathbf{k}_1,\mathbf{l}_1\right) & C_{12}\left(\mathbf{k}_1,\mathbf{l}_2\right) \\ C_{21}\left(\mathbf{k}_2,\mathbf{l}_1\right) & C_{22}\left(\mathbf{k}_2,\mathbf{l}_2\right) \end{bmatrix} \quad (12)$$

is a $p\times p$ *block matrix with components* $C_{ab}(k_{ai_a}, l_{bj_b})$ *for* $i_a = 1,\ldots,s_a$ *and* $j_b = 1,\ldots,t_b$ *for* $a, b = 1, 2$ *for* S_{κ} *expansion coefficients (A.11).*

2. Twisted Frobenius Identity

In this subsection we finally derive the main formula of this paper, the twisted Frobenius identity. Recall the Frobenius identity (e.g., [4, 5])

$$\frac{\vartheta\left[\begin{smallmatrix}\alpha_1\\ \beta_1\end{smallmatrix}\right]\left(\sum_{m=1}^{n}(x_m-y_m),\tau\right)}{\vartheta\left[\begin{smallmatrix}\alpha_1\\ \beta_1\end{smallmatrix}\right](0,\tau)} \cdot \frac{\prod_{1\le i<j\le n}K\left(x_i-x_j,\tau\right)K\left(y_i-y_j,\tau\right)}{\prod_{1\le i,j\le n}K\left(x_i-y_j,\tau\right)} = \det P_1, \tag{13}$$

where P_1 denotes the $n\times n$ matrix with twisted Weierstrass function components $P_1\left[\begin{smallmatrix}\theta_1\\ \phi_1\end{smallmatrix}\right](x_i,y_j)$ of (A.8) for $i,j = 1,\ldots,n$.

The trivial form of the twisted Frobenius identity follows from the consideration of the generating function for all $n+2$-point functions

$$\begin{aligned}&\frac{\vartheta\left[\begin{smallmatrix}\alpha_1\\ \beta_1\end{smallmatrix}\right]\left(\sum_{m=1}^{n}(x_m-y_m)+\kappa w,\tau\right)}{\vartheta\left[\begin{smallmatrix}\alpha_1\\ \beta_1\end{smallmatrix}\right](\kappa w,\tau)}\\ &\cdot\frac{\prod_{1\le i<j\le n}K\left(x_i-x_j,\tau\right)K\left(y_i-y_j,\tau\right)}{\prod_{1\le i,j\le n}K\left(x_i-y_j,\tau\right)}\\ &\cdot\prod_{1\le i,j\le n}\left[\frac{K\left(x_i-w,\tau\right)K\left(y_j,\tau\right)}{K\left(x_i,\tau\right)K\left(y_j-w,\tau\right)}\right]^{\kappa}dx_i^{1/2}dy_j^{1/2}\\ &=\det S_\kappa.\end{aligned} \tag{14}$$

We obtain the following.

Proposition 5. *The twisted two-point Frobenius identity is given by*

$$\begin{aligned}&\sum_{\mu\in\mathbb{Z}+\alpha_1}e^{2\pi i\mu\beta_1}Q_\mu^{\zeta}\left(\Psi_\kappa\left[\mathbf{k}_1,\mathbf{l}_2\right],w;\Psi_{-\kappa}\left[\mathbf{k}_2,\mathbf{l}_1\right],0;v_1\right.\\ &\left.\otimes e^{n_1+\kappa},w;v_2\otimes e^{n_2-\kappa},0;q\right)\cdot q^{(1/2)\mu^2}\exp\left(\mu\sum_{i=1}\zeta_i z_i\right)\\ &\cdot K(w,\tau)^{\kappa^2}\prod_{r<s}K\left(z_{rs},\tau\right)^{\zeta_r\zeta_s}=\vartheta\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}(\kappa w,\tau)\\ &\cdot\det C_{ab}\left(\mathbf{k}_a,\mathbf{l}_b\right),\end{aligned} \tag{15}$$

for $\zeta = \zeta(\Psi_\kappa[\mathbf{k}_1,\mathbf{l}_2];\Psi_{-\kappa}[\mathbf{k}_2,\mathbf{l}_1];v_1\otimes e^{n_1+\kappa};\ v_2\otimes e^{n_2-\kappa})$ *(i.e., values of exponents in module elements), and* $z=\{w,0,w,0\}$ *correspondingly.*

Proof. Consider the expression for the torus orbifold intertwining two-point function (11). Using (4) we derive the following:

$$\begin{aligned}&Z_{V_{\mathbb{Z}}}^{(1)}\begin{bmatrix}f_1\\ g_1\end{bmatrix}\left(\Psi_\kappa\left[\mathbf{k}_1,\mathbf{l}_2\right],w;\Psi_{-\kappa}\left[\mathbf{k}_2,\mathbf{l}_1\right],0;v_1\right.\\ &\left.\otimes e^{n_1+\kappa},w;v_2\otimes e^{n_2-\kappa},0;q\right)\\ &=\sum_{\mu\in\mathbb{Z}+\alpha_1}e^{2\pi i\mu\beta_1}Z_\mu^{(1)}\left(\Psi_\kappa\left[\mathbf{k}_1,\mathbf{l}_2\right],w;\Psi_{-\kappa}\left[\mathbf{k}_2,\mathbf{l}_1\right],0;v_1\right.\\ &\left.\otimes e^{n_1+\kappa},w;v_2\otimes e^{n_2-\kappa},0;q\right)\\ &=\sum_{\mu\in\mathbb{Z}+\alpha_1}e^{2\pi i\mu\beta_1}Q_\mu^{\zeta}\left(\Psi_\kappa\left[\mathbf{k}_1,\mathbf{l}_2\right],w;\Psi_{-\kappa}\left[\mathbf{k}_2,\mathbf{l}_1\right],0;v_1\right.\\ &\left.\otimes e^{n_1+\kappa},w;v_2\otimes e^{n_2-\kappa}\right)Z_\mu^{(1)}\left(e^{\zeta_1},z_1;\ldots;e^{\zeta_n},z_n;q\right)\\ &=\sum_{\mu\in\mathbb{Z}+\alpha_1}e^{2\pi i\mu\beta_1}Q_\mu^{\zeta}\left(\Psi_\kappa\left[\mathbf{k}_1,\mathbf{l}_2\right],w;\Psi_{-\kappa}\left[\mathbf{k}_2,\mathbf{l}_1\right],0;v_1\right.\\ &\left.\otimes e^{n_1+\kappa},w;v_2\otimes e^{n_2-\kappa},0;q\right)\cdot\frac{q^{(1/2)\alpha^2}}{\eta(\tau)}\exp\left(\mu\sum_{i=1}\zeta_i z_i\right)\\ &\cdot\prod_{1\le r<s\le n}K\left(z_{rs},\tau\right)^{\beta_r\beta_s}.\end{aligned} \tag{16}$$

Thus we obtain the result. □

Appendix

A. The Szegő Kernel on a Riemann Surface

Consider a compact connected Riemann surface Σ of genus g with canonical homology cycle bases a_i, b_i for $i=1,\ldots,g$. Let ν_i be a basis of holomorphic one-form with normalization $\oint_{a_i}\nu_j = 2\pi i\delta_{ij}$ and period matrix $\Omega_{ij} = (1/2\pi i)\oint_{b_i}\nu_j \in \mathbb{H}_g$, the Siegel upper half plane. Define the theta function with real characteristics [5–7]

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z\mid\Omega)=\sum_{n\in\mathbb{Z}^g}e^{i\pi(n+\alpha)\cdot\Omega\cdot(n+\alpha)+(n+\alpha)\cdot(z+2\pi i\beta)}, \tag{A.1}$$

for $\alpha=(\alpha_j)$, $\beta=(\beta_j)\in\mathbb{R}^g$, and $z=(z_j)\in\mathbb{C}^g$ for $j=1,\ldots,g$. The Szegő kernel is defined for $\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](0\mid\Omega)\neq 0$ by [5, 8, 9]

$$S\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x,y\mid\Omega)=\frac{\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\left(\int_y^x\nu\mid\Omega\right)}{\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](0\mid\Omega)E(x,y)}, \tag{A.2}$$

for $x,y\in\Sigma$ and where $\theta=(\theta_j)$, and $\phi=(\phi_j)\in U(1)^n$ for

$$\begin{aligned}\theta_j&=-e^{-2\pi i\beta_j},\\ \phi_j&=-e^{2\pi i\alpha_j},\\ j&=1,\ldots,g\end{aligned} \tag{A.3}$$

(where the -1 factors are included for later convenience) and $E(x,y)$ is the prime form [5, 6]. We use the convention $E(x,y) \sim (x-y)dx^{-1/2}dy^{-1/2}$ for $x \sim y$. The Szegő kernel is periodic in x along the a_i and b_j cycles with multipliers $-\phi_i$ and $-\theta_j$, respectively, and is a meromorphic $(1/2,1/2)$-form (and is thus necessarily defined on a double-cover $\widetilde{\Sigma}$ of the Riemann surface) satisfying

$$S\begin{bmatrix}\theta\\\phi\end{bmatrix}(x,y) \sim \frac{1}{x-y}dx^{1/2}dy^{1/2} \quad \text{for } x \sim y,$$
$$S\begin{bmatrix}\theta\\\phi\end{bmatrix}(x,y) = -S\begin{bmatrix}\theta^{-1}\\\phi^{-1}\end{bmatrix}(y,x), \tag{A.4}$$

where $\theta^{-1} = (\theta_i^{-1})$ and $\phi^{-1} = (\phi_i^{-1})$.

A.1. The Genus Two Szegő Kernel in the ρ-Formalism. Now we recall the construction of the Szegő kernel on a genus two Riemann surface in the ρ-formalism [10]. The genus one prime form for $x, y \in \mathbb{C}$ and $\tau \in \mathbb{H}_1$ is

$$E(x,y) = K(x-y,\tau)\,dx^{-1/2}dy^{-1/2}, \tag{A.5}$$

where

$$K(z,\tau) = \frac{\vartheta_1(z,\tau)}{\partial_z\vartheta_1(0,\tau)}, \quad \vartheta_1(z,\tau) = \vartheta\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}(z,\tau). \tag{A.6}$$

Let $(\theta_1,\phi_1) \in U(1)\times U(1)$ with $(\theta_1,\phi_1) \neq (1,1)$. The genus one Szegő kernel is

$$S^{(1)}\begin{bmatrix}\theta_1\\\phi_1\end{bmatrix}(x,y\mid\tau) = P_1\begin{bmatrix}\theta_1\\\phi_1\end{bmatrix}(x-y,\tau)\,dx^{1/2}dy^{1/2}, \tag{A.7}$$

where

$$P_1\begin{bmatrix}\theta_1\\\phi_1\end{bmatrix}(z,\tau) = \frac{\vartheta\left[\begin{smallmatrix}\alpha_1\\\beta_1\end{smallmatrix}\right](z,\tau)}{\vartheta\left[\begin{smallmatrix}\alpha_1\\\beta_1\end{smallmatrix}\right](0,\tau)}\frac{1}{K(z,\tau)}, \tag{A.8}$$

where $-\phi_1 = \exp(2\pi i\alpha_1)$ and $-\theta_1 = \exp(-2\pi i\beta_1)$ are the periodicities of $S^{(1)}\left[\begin{smallmatrix}\theta_1\\\phi_1\end{smallmatrix}\right](x,y\mid\tau)$ in x on the standard a and b cycles, respectively.

It is convenient to define $\kappa \in (1/2,1/2)$ by $\phi_2 = -e^{2\pi i\kappa}$ (i.e., $\kappa = \alpha_2^{(2)} \bmod 1$) and introduce [1]

$$S_\kappa(x,y) = \left(\frac{\vartheta_1(x-w,\tau)\,\vartheta_1(y,\tau)}{\vartheta_1(x,\tau)\,\vartheta_1(y-w,\tau)}\right)^{\kappa}$$
$$\cdot\frac{\vartheta\left[\begin{smallmatrix}\alpha_1\\\beta_1\end{smallmatrix}\right](x-y+\kappa w,\tau)}{\vartheta\left[\begin{smallmatrix}\alpha_1\\\beta_1\end{smallmatrix}\right](\kappa w,\tau)\,K(x-y,\tau)}dx^{1/2}dy^{1/2}, \tag{A.9}$$

for $\kappa \neq -1/2$ (with a different expression when $\kappa = -1/2$ given in [10]). We will assume $\kappa \neq -1/2$ throughout this paper. Note also that $S_{\kappa=0}(x,y) = S^{(1)}\left[\begin{smallmatrix}\theta_1\\\phi_1\end{smallmatrix}\right](x,y)$, the genus one Szegő kernel.

$S_\kappa(x,y)$ has an expansion in the neighborhood of the punctures at 0, w in terms of local coordinates $x_1 = x$ and $y_1 = y$, and $x_1 = x - w$ and $y_2 = y - w$ as follows [11]:

$$S_\kappa(x_{\bar{a}},y_b) = \left[\delta_{\bar{a},b}\frac{1}{x_b-y_b}\left(\frac{x_b}{y_b}\right)^{\kappa(-1)^b}\right.$$
$$\left.+\sum_{k,l\geq1}C_{ab}(k,l)\,x_{\bar{a}}^{k_a-1}y_b^{l_b-1}\right]dx_{\bar{a}}^{1/2}dy_b^{1/2}, \tag{A.10}$$

where $C_{ab}(k,l) = C_{ab}\left[\begin{smallmatrix}\theta_1\\\phi_1\end{smallmatrix}\right](k,l\mid\tau,w,\kappa)$ and $k_a = k + \kappa(-1)^{\bar{a}}$ for integer $k \geq 1$ and $a = 1,2$. We may invert this to obtain the infinite block moment matrix

$$C_{ab}(k,l) = \frac{1}{(2\pi i)^2}\oint_{\mathscr{C}_{\bar{a}}(x_{\bar{a}})}\oint_{\mathscr{C}_b(y_b)}(x_{\bar{a}})^{-k_a}(y_b)^{-l_b}$$
$$\cdot S_\kappa(x_{\bar{a}},y_b)\,dx_{\bar{a}}^{1/2}dy_b^{1/2}. \tag{A.11}$$

B. The Free Fermion VOSA and Its Twisted Modules

In this Appendix we recall [1] the notion of the free fermionic VOSA and its twisted modules.

B.1. The Free Fermion VOSA. We consider in this paper the rank two free fermion vertex operator superalgebra (VOSA) $V = V(H,\mathbb{Z}+1/2)^{\otimes 2}$ of central charge 1 (e.g., see [4, 12] for details). The weight $1/2$ space is spanned by ψ^+, ψ^- with vertex operator modes which satisfy the anticommutation relations

$$[\psi^+(m),\psi^-(n)] = \delta_{m,-n-1},$$
$$[\psi^+(m),\psi^+(n)] = [\psi^-(m),\psi^-(n)] = 0. \tag{B.1}$$

V is spanned by Fock vectors of the form

$$\Psi(\mathbf{k},\mathbf{l})$$
$$\equiv \psi^+(-k_1)\cdots\psi^+(-k_s)\,\psi^-(-l_1)\cdots\psi^-(-l_t)\,\mathbf{1}, \tag{B.2}$$

for distinct $0 < k_1 < \cdots < k_s$ and $0 < l_1 < \cdots < l_t$ and of Virasoro weight

$$\mathrm{wt}(\Psi(\mathbf{k},\mathbf{l})) = \sum_{i=1}^{s}\left(k_i-\frac{1}{2}\right)+\sum_{j=1}^{t}\left(l_j-\frac{1}{2}\right), \tag{B.3}$$

with Virasoro vector

$$\omega = \frac{1}{2}\left(\psi^+(-2)\,\psi^-(-1)+\psi^-(-2)\,\psi^+(-1)\right)\mathbf{1}. \tag{B.4}$$

The weight 1 space is spanned by $a = \psi^+(-1)\psi^-$ whose modes obey the Heisenberg commutation relations

$$[a(m),a(n)] = m\delta_{m,-n}. \tag{B.5}$$

As is well known, we may decompose V into irreducible M-modules $M\otimes e^m$ with $a(0)$ eigenvalue $m \in \mathbb{Z}$ so that $V = V(H,\mathbb{Z}+1/2)^{\otimes 2} \cong V_{\mathbb{Z}} = \oplus_{m\in\mathbb{Z}}M\otimes e^m$, the lattice VOSA for the $\mathbb{Z}$-lattice with trivial cocycle structure.

B.2. g-Twisted $V_{\mathbb{Z}}$-Modules and a Generalized VOA. $a(0)$ generates continuous winding $V_{\mathbb{Z}}$-automorphism $g = e^{-2\pi i\alpha a(0)}$ for $\alpha \in \mathbb{C}$. In particular, the fermion number involution is $\sigma = e^{\pi i a(0)}$. We define for all $u \in M$ the following operators:

$$\begin{aligned} Y_g\left(u\otimes e^m, z\right) &= Y\left(\Delta\left(\alpha, z\right)\left(u\otimes e^m\right), z\right),\\ \Delta\left(\alpha, z\right) &= z^{\alpha a(0)} Y_+\left(\alpha, -z\right),\\ Y_\pm\left(\alpha, z\right) &= \exp\left(\mp\alpha\sum_{n>0}\frac{a\left(\pm n\right)}{n} z^{\mp n}\right). \end{aligned} \tag{B.6}$$

Then we have [13] the following.

Proposition B.1. *$(V_{\mathbb{Z}}, Y_g)$ is a g-twisted $V_{\mathbb{Z}}$-module.*

In Section 5 of [11] an isomorphic construction is described whereby the g-twisted module is determined by the action of the original vertex operators on a twisted vector space $V_{\mathbb{Z}+\alpha} = e^{\alpha\widehat{q}}V_{\mathbb{Z}} = \oplus_{m\in\mathbb{Z}} M\otimes e^{m+\alpha}$, where

$$Y_g\left(u\otimes e^m, z\right) = e^{-\alpha\widehat{q}} Y\left(u\otimes e^m, z\right) e^{\alpha\widehat{q}}, \tag{B.7}$$

where $\widehat{q}$ is defined by

$$\left[a\left(n\right), \widehat{q}\right] = \delta_{n,0}. \tag{B.8}$$

In particular $Y_g(\omega, z)$ determines the g-twisted grading operator

$$L_g\left(0\right) = L\left(0\right) + \alpha a\left(0\right) + \frac{1}{2}\alpha^2. \tag{B.9}$$

Hence $\Psi(\mathbf{k}, \mathbf{l}) \in M\otimes e^{s-t}$ of (B.2) has g-twisted Virasoro weight $\mathrm{wt}(\Psi(\mathbf{k}, \mathbf{l})) + \alpha(s-t) + (1/2)\alpha^2$ which is equal to the $L(0)$ weight of the $V_{\mathbb{Z}+\alpha}$ twisted Fock vector

$$\Psi_\alpha\left(\mathbf{k}, \mathbf{l}\right) \equiv e^{\alpha\widehat{q}}\Psi\left(\mathbf{k}, \mathbf{l}\right) \in M\otimes e^{s-t+\alpha}. \tag{B.10}$$

In [11] we describe a generalized VOA with vector space

$$\mathscr{M} = \bigoplus_{\alpha\in\mathbb{C}} M\otimes e^\alpha, \tag{B.11}$$

formed as a direct sum of the Heisenberg VOA with all of its irreducible modules. $\mathscr{M}$ is spanned by $\Psi_\alpha(\mathbf{k}, \mathbf{l})$ for all $\alpha \in \mathbb{C}$. The generalized vertex operators are

$$\mathscr{Y}\left(u\otimes e^\alpha, z\right) = e^{\alpha\widehat{q}} Y_-\left(\alpha, z\right) Y\left(u, z\right) Y_+\left(\alpha, z\right) z^{\alpha a(0)}. \tag{B.12}$$

Equation (B.12) reduces to the usual bosonized form of the vertex operators for $V_{\mathbb{Z}}$ for $\alpha \in \mathbb{Z}$. A similar construction also appears in [14].

The generalized VOA leads to more general notions of locality, skew-symmetry, associativity, and commutivity than those for a VOSA as follows [1].

Proposition B.2. *For $u\otimes e^\alpha$, $v\otimes e^\beta$, $w\otimes e^\gamma \in \mathscr{M}$, and for integer $N \gg 0$*

$$\begin{aligned} &\left(z_1 - z_2\right)^N \left(z_1 - z_2\right)^{-\alpha\beta} \mathscr{Y}\left(u\otimes e^\alpha, z_1\right)\mathscr{Y}\left(v\otimes e^\beta, z_2\right)\\ &\quad = \left(z_1 - z_2\right)^N \left(z_2 - z_1\right)^{-\alpha\beta} \mathscr{Y}\left(v\otimes e^\beta, z_2\right)\\ &\quad\cdot \mathscr{Y}\left(u\otimes e^\alpha, z_1\right), \end{aligned} \tag{B.13}$$

$$\begin{aligned} &z^{-\alpha\beta}\mathscr{Y}\left(u\otimes e^\alpha, z\right) v\otimes e^\beta = (-z)^{-\alpha\beta}\\ &\quad\cdot e^{zL(-1)}\mathscr{Y}\left(v\otimes e^\beta, -z\right) u\otimes e^\alpha, \end{aligned} \tag{B.14}$$

$$\begin{aligned} &\left(z_0 + z_2\right)^{N-\alpha\gamma} \mathscr{Y}\left(u\otimes e^\alpha, z_0 + z_2\right)\mathscr{Y}\left(v\otimes e^\beta, z_2\right) w\\ &\quad\otimes e^\gamma = \left(z_2 + z_0\right)^{N-\alpha\gamma}\\ &\quad\cdot \mathscr{Y}\left(\mathscr{Y}\left(u\otimes e^\alpha, z_0\right)\left(v\otimes e^\beta\right), z_2\right) w\otimes e^\gamma, \end{aligned} \tag{B.15}$$

$$\begin{aligned} &\left[u\left(k\right), \mathscr{Y}\left(v\otimes e^\beta, z\right)\right]\\ &\quad = \sum_{j\geq 0}\binom{k}{j}\mathscr{Y}\left(u\left(j\right) v\otimes e^\beta, z\right) z^{k-j}, \quad u\in M. \end{aligned} \tag{B.16}$$

It is convenient to define for formal parameter z and $\chi \in \mathbb{C}$

$$(-z)^\chi = e^{i\pi B\chi} z^\chi, \tag{B.17}$$

where we choose, once and for all, an *odd integer B* parametrizing the formal branch cut. Note some notational changes from [11]. Then generalized locality and skew-symmetry can be rewritten as

$$\begin{aligned} &\left(z_1 - z_2\right)^{N-\alpha\beta} \mathscr{Y}\left(u\otimes e^\alpha, z_1\right)\mathscr{Y}\left(v\otimes e^\beta, z_2\right)\\ &\quad = e^{-i\pi B\alpha\beta}\left(z_1 - z_2\right)^{N-\alpha\beta} \mathscr{Y}\left(v\otimes e^\beta, z_2\right)\\ &\quad\cdot \mathscr{Y}\left(u\otimes e^\alpha, z_1\right), \end{aligned} \tag{B.18}$$

$$\begin{aligned} &\mathscr{Y}\left(u\otimes e^\alpha, z\right) v\otimes e^\beta = e^{-i\pi B\alpha\beta} e^{zL(-1)}\mathscr{Y}\left(v\otimes e^\beta, -z\right)\\ &\quad\cdot u\otimes e^\alpha. \end{aligned} \tag{B.19}$$

B.3. An Invariant Form on $\mathscr{M}$. In [11] we introduced an invariant bilinear form $\langle\cdot,\cdot\rangle$ on $\mathscr{M}$ associated with the Möbius map [8, 15, 16]

$$\begin{pmatrix} 0 & \lambda\\ -e^{i\pi B}\lambda^{-1} & 0 \end{pmatrix} : z \longmapsto -\frac{\lambda^2}{e^{i\pi B} z}, \tag{B.20}$$

for $\lambda \neq 0$. We will later choose

$$\lambda = e^{(1/2)i\pi B}\rho^{1/2}, \tag{B.21}$$

for the odd integer B of (B.17). Thus we reformulate the sewing relationship as $z_1 = -\lambda^2/z_2$ so that we get $dz_1^{1/2} = \xi\rho^{1/2}/z_2 dz_2^{1/2}$ for $\xi = e^{(1/2)i\pi B}$.

Define the adjoint vertex operator

$$\mathscr{Y}^{\dagger}\left(u\otimes e^{\alpha},z\right) = \mathscr{Y}\left(e^{-z\lambda^{-2}L(1)}\left(\frac{\lambda}{e^{i\pi B}z}\right)^{2L(0)}\left(u\otimes e^{\alpha}\right),\frac{\lambda^{2}}{e^{i\pi B}z}\right). \tag{B.22}$$

A bilinear form $\langle\cdot,\cdot\rangle_{\lambda}$ on $\mathscr{M}$ is said to be invariant if for all $u\otimes e^{\alpha}$, $v\otimes e^{\beta}$, and $w\otimes e^{\gamma}\in\mathscr{M}$ we have

$$\left\langle \mathscr{Y}\left(u\otimes e^{\alpha},z\right)\left(v\otimes e^{\beta}\right),w\otimes e^{\gamma}\right\rangle_{\lambda} = e^{-i\pi B\alpha\beta}\left\langle v\otimes e^{\beta},\mathscr{Y}^{\dagger}\left(u\otimes e^{\alpha},z\right)w\otimes e^{\gamma}\right\rangle_{\lambda}. \tag{B.23}$$

Equation (B.22) reduces to the usual definition for a VOSA when $\alpha,\beta,\gamma\in\mathbb{Z}$ [8, 16]. Choosing the normalization $\langle\mathbf{1},\mathbf{1}\rangle = 1$ then $\langle\cdot,\cdot\rangle_{\lambda}$ on $\mathscr{M}$ is symmetric, unique, and invertible with [11]

$$\left\langle u\otimes e^{\alpha},v\otimes e^{\beta}\right\rangle_{\lambda} = \lambda^{-\alpha^{2}}\delta_{\alpha,-\beta}\left\langle u\otimes e^{0},v\otimes e^{0}\right\rangle_{\lambda}. \tag{B.24}$$

The dual of the Fock vector $\Psi=\Psi(\mathbf{k},\mathbf{l})$ with respect to $\langle\cdot,\cdot\rangle_{\lambda}$, which we refer to as the λ-dual, is

$$\overline{\Psi}\left(\mathbf{k},\mathbf{l}\right) = (-1)^{st+\lfloor\mathrm{wt}(\Psi)\rfloor}\lambda^{2\mathrm{wt}(\Psi)}\Psi\left(\mathbf{l},\mathbf{k}\right), \tag{B.25}$$

where $\lfloor x\rfloor$ denotes the integer part of x [16]. Applying (B.21) and (B.24) it follows that $\Psi_{\alpha}=\Psi_{\alpha}(\mathbf{k},\mathbf{l})$ of (B.10) has λ-dual

$$\begin{aligned}\overline{\Psi}_{\alpha}\left(\mathbf{k},\mathbf{l}\right) &= (-1)^{st+\lfloor\mathrm{wt}(\Psi)\rfloor}\lambda^{2\mathrm{wt}(\Psi_{\alpha})}\Psi_{-\alpha}\left(\mathbf{l},\mathbf{k}\right)\\ &= (-1)^{st+\lfloor\mathrm{wt}(\Psi)\rfloor}e^{i\pi B\mathrm{wt}(\Psi_{\alpha})}\rho^{\mathrm{wt}(\Psi_{\alpha})}\Psi_{-\alpha}\left(\mathbf{l},\mathbf{k}\right).\end{aligned} \tag{B.26}$$

Conflicts of Interest

The author declares that they have no conflicts of interest.

References

[1] M. P. Tuite and A. Zuevsky, "Genus two partition and correlation functions for fermionic vertex operator superalgebras II," *Communications in Mathematical Physics*, vol. 306, no. 2, pp. 419–447, 2013, https://arxiv.org/abs/1308.2441.

[2] Y. Zhu, "Modular invariance of characters of vertex operator algebras," *Journal of the American Mathematical Society*, vol. 9, no. 1, pp. 237–302, 1996.

[3] G. Mason and M. P. Tuite, "Free bosonic vertex operator algebras on genus two Riemann surfaces I," *Communications in Mathematical Physics*, vol. 300, no. 3, pp. 673–713, 2010.

[4] G. Mason, M. P. Tuite, and A. Zuevsky, "Torus n–point functions for R-graded vertex operator superalgebras and continuous fermion orbifolds," *Communications in Mathematical Physics*, vol. 283, no. 2, pp. 305–342, 2008.

[5] J. D. Fay, *Theta Functions on Riemann Surfaces*, vol. 352 of *Lecture Notes in Mathematics*, Springer, New York, NY, USA, 1973.

[6] D. Mumford, *Tata Lectures on Theta I and II*, vol. 28 of *Progress in Mathematics*, Birkhäuser Boston, Inc., Boston, MA, USA, 1983.

[7] H. M. Farkas and I. Kra, *Theta Constants, Riemann Surfaces and the Modular Group*, vol. 37 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, USA, 2001.

[8] N. R. Scheithauer, "Vertex algebras, Lie algebras, and superstrings," *Journal of Algebra*, vol. 200, no. 2, pp. 363–403, 1998.

[9] N. S. Hawley and M. Schiffer, "Half-order differentials on Riemann surfaces," *Acta Mathematica*, vol. 115, pp. 199–236, 1966.

[10] M. P. Tuite and A. Zuevsky, "The Szegö kernel on a sewn Riemann surface," *Communications in Mathematical Physics*, vol. 306, no. 3, pp. 617–645, 2011.

[11] M. P. Tuite and A. Zuevsky, "A generalized vertex operator algebra for Heisenberg intertwiners," *Journal of Pure and Applied Algebra*, vol. 216, no. 6, pp. 1442–1453, 2012.

[12] V. Kac, *Vertex Algebras for Beginners*, vol. 10 of *University Lecture Series*, American Mathematical Society, Providence, RI, USA, 1998.

[13] H. S. Li, "Symmetric invariant bilinear forms on vertex operator algebras," *Journal of Pure and Applied Algebra*, vol. 96, no. 3, pp. 2790–2799, 1994.

[14] B. Bakalov and V. Kac, "Generalized vertex algebras," *Math*, QA/0602072, 2006.

[15] I. B. Frenkel, Y.-Z. Huang, and J. Lepowsky, "On axiomatic approaches to vertex operator algebras and modules," *Memoirs of the American Mathematical Society*, vol. 104, no. 494, 1993.

[16] M. P. Tuite and A. Zuevsky, "Genus two partition and correlation functions for fermionic vertex operator superalgebras I," *Communications in Mathematical Physics*, vol. 306, no. 2, pp. 419–447, 2011.