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*Research Article*

# The Factorization for a Class of Hom-Coalgebras

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Let $(H, \alpha)$ be a Hom-Hopf algebra and $(C, \beta)$ a Hom-coalgebra. In this paper, we first introduce the notions of Hom-crossed coproduct $C \rtimes_\lambda H$ and cleft coextension and then discuss the equivalence between them. Furthermore, we discuss the relation between cleft coextension and Hom-module coalgebra with the Hom-Hopf module structure and obtain a Hom-coalgebra factorization for Hom-module coalgebra with the Hom-Hopf module structure.

## 1. Introduction

The motivation to introduce Hom-type algebras comes for examples related to $q$-deformations of Witt and Virasoro algebras, which play an important role in physics, mainly in conformal field theory. Hom-structures (Lie algebras, algebras, coalgebras, Hopf algebras) have been intensively investigated recently. Makhlouf and Silvestrov ([1, 2]) generalized the associativity to twisted associativity and naturally proposed the notion of Hom-associative algebras and were first to introduce Hom-coalgebras, Hom-bialgebras, Hom-Hopf algebras, and related objects.

The crossed products of algebras were independently introduced in [3, 4]. In [3], Blattner et al. showed the equivalence of crossed products and cleft extensions. In [5], Blattner and Montgomery gave several characterizations of crossed products. Lu and Wang [6] generalized the result in [3] to the case of Hom-Hopf algebras. The concept of crossed coproduct appeared as a dual version of the usual crossed product for Hopf algebras and it was studied in several papers; see [7–9]. In [8] the authors studied cleft coextension, a dual notion for that of cleft extension, and it was proved that a cleft coextension is isomorphic to a crossed coproduct. Naturally, what are the structure and the relation of crossed coproduct and cleft coextensions in the sense of Hom-structure?

In [10], Radford showed that a bialgebra with a projection had a factorization. In [11], Caenepeel et al. generalized Radford's result. It is natural to ask under what conditions a Hom-coalgebra can be factorized into Hom-crossed coproduct. These two problems motivate us writing this paper.

This paper is organized as follows.

In Section 2, we recall some basic definitions and results, such as Hom-bialgebras, Hom-Hopf algebras, Hom-(co)module, Hom-Hopf module, and Hom-module coalgebras. In Section 3, let $(H, \alpha)$ be a Hom-Hopf algebra and $(C, \beta)$ a Hom-coalgebra. We give the definition of Hom-Hopf algebra $H$ coacting weakly on Hom-coalgebra $C$ from the left, introduce the notion of Hom-crossed coproduct, and then discuss the necessary and sufficient conditions for $C \rtimes_\lambda H$ to be Hom-crossed coproduct (see Theorem 4). In Section 4, we introduce the definition of the cleft coextension and discuss the equivalence between the Hom-crossed coproducts and cleft coextensions (see Theorem 10). Furthermore, we discuss the relation between cleft coextension and Hom-module coalgebra with the Hom-Hopf module structure and obtain $(B, \cdot, \phi, \beta)$ which is a cleft coextension if and only if it is a right Hom-Hopf module (see Theorem 13). In Section 5, we obtain the Hom-coalgebra factorization for Hom-module coalgebra with the Hom-Hopf module structure (see Theorem 15).

## 2. Preliminaries

In this paper, all the vector spaces, tensor products, and homomorphisms are over a fixed field $k$. For a coalgebra $C$, we write $\Delta(c) = c_1 \otimes c_2$ for any $c \in C$ (summation omitted).

We now recall from [2, 12–14] some definitions and results about the Hom-Hopf algebras, Hom-(co)modules, and so on.

*2.1. Hom-Hopf Algebra.* A Hom-algebra is a quadruple $(A, \mu, 1_A, \alpha)$ (abbr. $(A, \alpha)$), where $A$ is a linear space, $\mu : A \otimes A \to A$ is a linear map, with notation $\mu(a \otimes a') = aa'$, $1_A \in A$, and $\alpha \in Aut_k(A)$, such that, for any $a, a', a'' \in A$,

$$\begin{gathered} \alpha\left(aa'\right) = \alpha(a)\,\alpha\left(a'\right), \\ a1_A = 1_A a = \alpha(a), \\ \alpha(a)\left(a'a''\right) = \left(aa'\right)\alpha\left(a''\right), \\ \alpha\left(1_A\right) = 1_A. \end{gathered} \tag{1}$$

A linear map $f : (A, \mu_A, 1_A, \alpha_A) \to (B, \mu_B, 1_B, \alpha_B)$ is called a morphism of Hom-algebra if $\alpha_B f = f\alpha_A$, $f(1_A) = 1_B$, and $f\mu_A = \mu_B(f \otimes f)$.

A Hom-coalgebra is a quadruple $(C, \Delta, \varepsilon, \beta)$ (abbr. $(C, \beta)$), where $C$ is a linear space, $\Delta : C \to C \otimes C$, $\varepsilon : C \to k$ are linear maps, and $\beta \in Aut_k(C)$, such that, for any $c \in C$,

$$\begin{gathered} \beta\left(c_1\right) \otimes \beta\left(c_2\right) = \beta(c)_1 \otimes \beta(c)_2, \\ \varepsilon\left(c_1\right)c_2 = c_1\varepsilon\left(c_2\right) = \beta(c), \\ \beta\left(c_1\right) \otimes c_{21} \otimes c_{22} = c_{11} \otimes c_{12} \otimes \beta\left(c_2\right), \\ \varepsilon\beta = \varepsilon. \end{gathered} \tag{2}$$

A linear map $f : (C, \Delta_C, \varepsilon_C, \beta_C) \to (D, \Delta_D, \varepsilon_D, \beta_D)$ is called a morphism of Hom-coalgebra if $\beta_D f = f\beta_C$, $\varepsilon_D f = \varepsilon_C$, and $\Delta_D f = (f \otimes f)\Delta_C$.

A Hom-bialgebra is a sextuple $(H, \mu, 1_H, \Delta, \varepsilon, \gamma)$ (abbr. $(H, \gamma)$), where $(H, \mu, 1_H, \gamma)$ is a Hom-algebra and $(H, \Delta, \varepsilon, \gamma)$ is a Hom-coalgebra, such that $\Delta$, $\varepsilon$ are morphisms of Hom-algebra; that is,

$$\begin{gathered} \Delta\left(hh'\right) = \Delta(h)\,\Delta\left(h'\right), \\ \Delta\left(1_H\right) = 1_H \otimes 1_H, \\ \varepsilon\left(hh'\right) = \varepsilon(h)\,\varepsilon\left(h'\right), \\ \varepsilon\left(1_H\right) = 1. \end{gathered} \tag{3}$$

Furthermore, if there exists a linear map $S : H \to H$ such that

$$\begin{gathered} S\left(h_1\right)h_2 = h_1 S\left(h_2\right) = \varepsilon(h)\,1_H, \\ S\left(\gamma(h)\right) = \gamma\left(S(h)\right), \end{gathered} \tag{4}$$

then we call $(H, \mu, 1_H, \Delta, \varepsilon, S, \gamma)$ (abbr. $(H, S, \gamma)$) a Hom-Hopf algebra.

Let $(H, S, \gamma)$ be a Hom-Hopf algebra; for $S$ we have the following properties for any $h, g \in H$:

$$\begin{gathered} S(h)_1 \otimes S(h)_2 = S\left(h_2\right) \otimes S\left(h_1\right), \\ S(hg) = S(g)\,S(h), \\ \varepsilon S = \varepsilon. \end{gathered} \tag{5}$$

*2.2. Hom-Hopf Module.* Let $(A, \beta)$ be a Hom-algebra; a right $(A, \beta)$-Hom-module is a triple $(M, \cdot, \alpha)$, where $M$ is a linear space, $\cdot : M \otimes A \to M$ is a linear map, and $\alpha$ is an automorphism of $M$, such that, for any $a, a' \in A$ and $m \in M$,

$$\begin{gathered} \alpha(m) \cdot \left(aa'\right) = (m \cdot a) \cdot \beta\left(a'\right), \\ m \cdot 1_A = \alpha(m), \\ \alpha(m \cdot a) = \alpha(m) \cdot \beta(a). \end{gathered} \tag{6}$$

Let $(M, \cdot_M, \alpha_M)$ and $(N, \cdot_N, \alpha_N)$ be two right $(A, \beta)$-Hom-modules. Then a linear morphism $f : M \to N$ is called a morphism of right $(A, \beta)$-Hom-modules if $f(m \cdot_M a) = f(m) \cdot_N a$ and $\alpha_M f = f\alpha_N$.

Let $(C, \beta)$ be a Hom-coalgebra; a right $(C, \beta)$-Hom-comodule is a triple $(M, \rho, \alpha)$, where $M$ is a linear space, $\rho : M \to M \otimes C$ is a linear map (write $\rho(m) = m_{(0)} \otimes m_{(1)}$), and $\alpha$ is an automorphism of $M$, such that, for any $m \in M$,

$$\begin{gathered} \alpha\left(m_{(0)}\right) \otimes m_{(1)1} \otimes m_{(1)2} = m_{(0)(0)} \otimes m_{(0)(1)} \otimes \beta\left(m_{(1)}\right), \\ m_{(0)}\varepsilon\left(m_{(1)}\right) = \alpha(m), \\ \alpha(m)_{(0)} \otimes \alpha(m)_{(1)} = \alpha\left(m_{(0)}\right) \otimes \beta\left(m_{(0)}\right). \end{gathered} \tag{7}$$

Let $(M, \rho_M, \alpha_M)$ and $(N, \rho_N, \alpha_N)$ be two right $(C, \beta)$-Hom-comodules. Then a linear morphism $f : M \to N$ is called a morphism of right $(C, \beta)$-Hom-comodules if $f(m)_{(0)} \otimes f(m)_{(1)} = f(m_{(0)}) \otimes m_{(1)}$ and $\alpha_M f = f\alpha_N$.

Let $(H, \alpha)$ be a Hom-Hopf algebra; a right $H$-Hom-Hopf module is a quadruple $(M, \cdot, \rho, \beta)$, where $M$ is a right $H$-Hom-module and a right $H$-Hom-comodule, such that, for all $m \in M$, $h \in H$,

$$\rho(m \cdot h) = m_{(0)} \cdot h_1 \otimes m_{(1)}h_2. \tag{8}$$

*2.3. The Fundamental Theorem of Hom-Hopf Module.* Let $(H, \alpha)$ be a Hom-Hopf algebra and $(B, \cdot, \rho, \beta)$ a right $H$-Hom-Hopf module, and set $A = B^{coH} = \{b \in B \mid \rho(b) = \beta(b) \otimes 1\}$, then

$$\begin{gathered} \varphi : A \otimes H \longrightarrow B, \\ b \otimes h \longmapsto b \cdot h \end{gathered} \tag{9}$$

is an isomorphism of right $H$-Hom-Hopf module.

*2.4. Hom-Module Coalgebra.* Recall from [15], let $(H, \beta)$ be a Hom-Hopf algebra and $(C, \alpha)$ a Hom-coalgebra, and if $(C, \cdot, \alpha)$ is a left $(H, \beta)$-Hom-module, for all $c \in C$, $h \in H$, the following conditions hold:

$$\begin{gathered} (h \cdot c)_1 \otimes (h \cdot c)_2 = h_1 \cdot c_1 \otimes h_2 \cdot c_2, \\ \varepsilon(h \cdot c) = \varepsilon(h)\,\varepsilon(c), \end{gathered} \tag{10}$$

then $(C, \cdot, \alpha)$ is called an $(H, \beta)$-Hom-module coalgebra.

## 3. Hom-Crossed Coproducts

Let $(H, \alpha)$ be a Hom-Hopf algebra and $(C, \beta)$ a Hom-coalgebra. In this section, we give the definition of $H$ coacting

weakly on $C$ from the left and introduce Hom-crossed coproduct. Then we discuss the necessary and sufficient conditions for $C \rtimes_\lambda H$ to be Hom-crossed coproduct and get some properties about it.

*Definition 1.* Let $(H, \alpha)$ be a Hom-Hopf algebra and $(C, \beta)$ a Hom-coalgebra. We say that $H$ coacts weakly on $C$ from the left if there is a linear map $C \to H \otimes C$, $c \mapsto c_{-1} \otimes c_0$ such that for all $c \in C$ the following conditions hold:

(W1) $\alpha^2(c_{-1}) \otimes c_{01} \otimes c_{02} = c_{1-1}c_{2-1} \otimes c_{10} \otimes c_{20}$,

(W2) $\varepsilon(c_0)c_{-1} = \varepsilon(c)1$,

(W3) $\varepsilon(c_{-1})c_0 = \beta(c)$.

*Definition 2.* Let $(H, \alpha)$ be a Hom-Hopf algebra and $(C, \beta)$ a Hom-coalgebra. Assume that $H$ coacts weakly on $C$ from the left. Let $\lambda \in \mathrm{Hom}(C, H \otimes H)$ be a linear map; write $\lambda(c) = \lambda_1(c) \otimes \lambda_2(c)$. Define $C \rtimes_\lambda H$, whose underlying vector space is $C \otimes H$ with the comultiplication given by

$$\begin{aligned}\Delta_\lambda (c \otimes h) = c_1 &\otimes \left(\alpha^{-4}(c_{21-1})\alpha^{-3}(\lambda_1(c_{22}))\right)\alpha^{-1}(h_1) \\ &\otimes \beta^{-2}(c_{210}) \otimes \alpha^{-2}(\lambda_2(c_{22}))\alpha^{-1}(h_2).\end{aligned} \tag{11}$$

We say that $(C \rtimes_\lambda H, \beta\otimes\alpha)$ is a Hom-crossed coproduct if $\Delta_\lambda$ is coassociative and $\varepsilon\otimes\varepsilon$ is the counit for all $c \in C$ and $h \in H$.

*Remark 3.* If the cocycle $\lambda$ is convolution invertible, we will denote its convolution inverse by $\lambda^{-1}(c) = \lambda_1^{-1}(c) \otimes \lambda_2^{-1}(c)$.

**Theorem 4.** *$(C \rtimes_\lambda H, \beta\otimes\alpha)$ is a Hom-crossed coproduct if and only if the following conditions hold.*

*(CU) Normal Cocycle Condition*

$$(id \otimes \varepsilon)\lambda = (\varepsilon \otimes id)\lambda = 1_H\varepsilon. \tag{12}$$

*(C) Cocycle Condition*

$$\begin{aligned}&\lambda_1(c_1)\alpha^{-1}(\lambda_1(c_2)_1) \otimes \lambda_2(c_1)\alpha^{-1}(\lambda_1(c_2)_2) \\ &\quad \otimes \alpha(\lambda_2(c_2)) \\ &= \alpha^{-1}(c_{1-1})\lambda_1(c_2) \otimes \alpha^{-1}(\lambda_1(c_{10}))\alpha^{-1}(\lambda_2(c_2)_1) \\ &\quad \otimes \alpha^{-1}(\lambda_2(c_{10}))\alpha^{-1}(\lambda_2(c_2)_2).\end{aligned} \tag{13}$$

*(TC) Twisted Comodule Condition*

$$\begin{aligned}&\lambda_1(c_1)\alpha^{-2}(c_{2-11}) \otimes \lambda_2(c_1)\alpha^{-2}(c_{2-12}) \otimes c_{20} \\ &\quad = \alpha^{-1}(c_{1-1})\lambda_1(c_2) \otimes \alpha^{-2}(c_{10-1})\lambda_2(c_2) \\ &\qquad \otimes \beta^{-1}(c_{100}).\end{aligned} \tag{14}$$

*Proof.* Directly computing, we can get that $\varepsilon \otimes \varepsilon$ is the counit of $(C \rtimes_\lambda H, \beta \otimes \alpha)$ if and only if (CU) holds.

Now, we prove that if $(C \rtimes_\lambda H, \beta \otimes \alpha)$ is a Hom-crossed coproduct then the conditions (C) and (TC) are satisfied. Because of coassociativity of $(C \rtimes_\lambda H, \beta \otimes \alpha)$, we can get

$$\begin{aligned}&(\Delta_\lambda \otimes (\beta \otimes \alpha))\Delta_\lambda(c \otimes h) = c_{11} \\ &\quad \otimes \left(\alpha^{-4}(c_{121-1})\alpha^{-3}(\lambda_1(c_{122}))\right) \\ &\quad \cdot \left(\left(\alpha^{-5}(c_{21-11})\alpha^{-4}(\lambda_1(c_{22})_1)\right)\alpha^{-2}(h_{11})\right) \\ &\quad \otimes \beta^{-2}(c_{1210}) \otimes \alpha^{-2}(\lambda_2(c_{122})) \\ &\quad \cdot \left(\left(\alpha^{-5}(c_{21-12})\alpha^{-4}(\lambda_1(c_{22})_2)\right)\alpha^{-2}(h_{12})\right) \\ &\quad \otimes \beta^{-1}(c_{210}) \otimes \alpha^{-1}(\lambda_2(c_{22}))h_2 = \alpha(c_1) \\ &\quad \otimes \left(\alpha^{-3}(c_{21-1})\alpha^{-2}(\lambda_1(c_{22}))\right)h_1 \otimes \beta^{-2}(c_{2101}) \\ &\quad \otimes \left(\alpha^{-6}(c_{21021-1})\alpha^{-5}(\lambda_1(c_{21022}))\right) \\ &\quad \cdot \left(\alpha^{-3}(\lambda_2(c_{22})_1)\alpha^{-2}(h_{21})\right) \otimes \beta^{-4}(c_{210210}) \\ &\quad \otimes \alpha^{-4}(\lambda_2(c_{21022}))\left(\alpha^{-3}(\lambda_2(c_{22})_2)\alpha^{-2}(h_{22})\right) \\ &\quad = ((\beta \otimes \alpha) \otimes \Delta_\lambda)\Delta_\lambda(c \otimes h).\end{aligned} \tag{15}$$

Taking $h = 1_H$ in the above equality, then applying $(\varepsilon \otimes id)^2 \otimes id \otimes \varepsilon$ to both sides, and using (CU), we obtain (TC); and applying $(\varepsilon \otimes id)^3$ to both sides, (C) is obtained too.

Conversely, suppose (C) and (TC) hold, then

$$\begin{aligned}&((\beta \otimes \alpha) \otimes \Delta_\lambda)\Delta_\lambda(c \otimes h) \overset{(W1)}{=} \alpha(c_1) \otimes \left(\left(\alpha^{-5}(c_{211-1})\right.\right. \\ &\quad \left.\left.\cdot \alpha^{-7}(c_{2121-1}c_{2122-1})\right)\alpha^{-2}(\lambda_1(c_{22}))\right)h_1 \\ &\quad \otimes \beta^{-2}(c_{2110}) \otimes \left(\alpha^{-6}(c_{21210-1})\alpha^{-5}(\lambda_1(c_{21220}))\right) \\ &\quad \cdot \left(\alpha^{-3}(\lambda_2(c_{22})_1)\alpha^{-2}(h_{21})\right) \otimes \beta^{-4}(c_{212100}) \\ &\quad \otimes \alpha^{-4}(\lambda_2(c_{21220}))\left(\alpha^{-3}(\lambda_2(c_{22})_2)\alpha^{-2}(h_{22})\right) \\ &\quad = \alpha(c_1) \otimes \left(\left(\alpha^{-5}(c_{211-1})\alpha^{-5}(c_{212-1})\right)\left(\alpha^{-5}(c_{221-1})\right.\right. \\ &\quad \left.\left.\cdot \alpha^{-4}(\lambda_1(c_{222}))\right)\right)h_1 \otimes \beta^{-2}(c_{2110}) \otimes \alpha^{-4}(c_{2120-1}) \\ &\quad \cdot \left(\left(\alpha^{-5}(\lambda_1(c_{2210}))\alpha^{-5}(\lambda_2(c_{222})_1)\right)\alpha^{-2}(h_{21})\right) \\ &\quad \otimes \beta^{-3}(c_{21200}) \otimes \left(\alpha^{-4}(\lambda_2(c_{2210}))\alpha^{-4}(\lambda_2(c_{222})_2)\right) \\ &\quad \cdot \alpha^{-1}(h_{22}) \overset{(C)}{=} \alpha(c_1) \otimes \left(\left(\alpha^{-5}(c_{211-1})\alpha^{-5}(c_{212-1})\right)\right. \\ &\quad \left.\cdot \left(\alpha^{-4}(\lambda_1(c_{221}))\alpha^{-5}(\lambda_1(c_{222})_1)\right)\right)h_1 \otimes \beta^{-2}(c_{2110}) \\ &\quad \otimes \alpha^{-4}(c_{2120-1})\left(\left(\alpha^{-4}(\lambda_2(c_{221}))\alpha^{-5}(\lambda_1(c_{222})_2)\right)\right. \\ &\quad \left.\cdot \alpha^{-2}(h_{21})\right) \otimes \beta^{-3}(c_{21200}) \otimes \alpha^{-2}(\lambda_2(c_{222})) \\ &\quad \cdot \alpha^{-1}(h_{22}) = \alpha(c_1) \otimes \left(\alpha^{-3}(c_{21-1})\right.\end{aligned}$$

$$\cdot\left(\left(\alpha^{-7}\left(c_{2211-1}\right)\alpha^{-6}\left(\lambda_1\left(c_{2212}\right)\right)\right)\alpha^{-5}\left(\lambda_1\left(c_{222}\right)_1\right)\right)$$
$$\cdot h_1\otimes\beta^{-1}\left(c_{210}\right)\otimes\left(\alpha^{-6}\left(c_{22110-1}\right)\alpha^{-4}\left(\lambda_2\left(c_{2212}\right)\right)\right)$$
$$\cdot\left(\alpha^{-4}\left(\lambda_1\left(c_{222}\right)_2\right)\alpha^{-2}\left(h_{21}\right)\right)\otimes\beta^{-4}\left(c_{221100}\right)$$
$$\otimes\alpha^{-2}\left(\lambda_2\left(c_{222}\right)\right)\alpha^{-1}\left(h_{22}\right)\overset{(\mathrm{TC})}{=}\alpha\left(c_1\right)\otimes\left(\alpha^{-3}\left(c_{21-1}\right)\right.$$
$$\cdot\left(\left(\alpha^{-6}\left(\lambda_1\left(c_{2211}\right)\right)\alpha^{-8}\left(c_{2212-11}\right)\right)\right.$$
$$\left.\cdot\alpha^{-5}\left(\lambda_1\left(c_{222}\right)_1\right)\right)\left.h_1\otimes\beta^{-1}\left(c_{210}\right)\right.$$
$$\otimes\left(\alpha^{-4}\left(\lambda_2\left(c_{2211}\right)\right)\alpha^{-6}\left(c_{2212-12}\right)\right)\left(\alpha^{-4}\left(\lambda_1\left(c_{222}\right)_2\right)\right.$$
$$\left.\cdot\alpha^{-2}\left(h_{21}\right)\right)\otimes\beta^{-3}\left(c_{22120}\right)\otimes\alpha^{-2}\left(\lambda_2\left(c_{222}\right)\right)$$
$$\cdot\alpha^{-1}\left(h_{22}\right)=\alpha\left(c_1\right)\otimes\left(\alpha^{-3}\left(c_{21-1}\right)\alpha^{-4}\left(\lambda_1\left(c_{2211}\right)\right)\right)$$
$$\cdot\left(\left(\alpha^{-7}\left(c_{2212-11}\right)\alpha^{-5}\left(\lambda_1\left(c_{222}\right)_1\right)\right)\alpha^{-2}\left(h_{11}\right)\right)$$
$$\otimes\beta^{-1}\left(c_{210}\right)\otimes\alpha^{-3}\left(\lambda_2\left(c_{2211}\right)\right)\left(\left(\alpha^{-7}\left(c_{2212-12}\right)\right.\right.$$
$$\left.\left.\cdot\alpha^{-5}\left(\lambda_1\left(c_{222}\right)_2\right)\right)\alpha^{-2}\left(h_{12}\right)\right)\otimes\beta^{-3}\left(c_{22120}\right)$$
$$\otimes\alpha^{-2}\left(\lambda_2\left(c_{222}\right)\right)h_2=\left(\Delta_\lambda\otimes\left(\beta\otimes\alpha\right)\right)\Delta_\lambda\left(c\otimes h\right). \tag{16}$$

This completes the proof. □

*Example 5.* (1) Consider the case when $\lambda$ is trivial, that is, $\lambda(c) = \varepsilon(c)1_H \otimes 1_H$ for all $c \in C$. Then the Hom-crossed coproduct is reduced to Hom-smash coproduct.

*Proof.* If $\lambda(c) = \varepsilon(c)1_H \otimes 1_H$, $\forall c \in C$, then $(id \otimes \varepsilon)\lambda(c) = \varepsilon(c)(id\otimes\varepsilon)(1_H\otimes 1_H) = 1_H\varepsilon(c)$. Similarly we can get $(\varepsilon\otimes id)\lambda = 1_H\varepsilon$, so (CU) is satisfied.

For condition (C), the left hand side is

$$\begin{aligned}&\lambda_1\left(c_1\right)\alpha^{-1}\left(\lambda_1\left(c_2\right)_1\right)\otimes\lambda_2\left(c_1\right)\alpha^{-1}\left(\lambda_1\left(c_2\right)_2\right)\\&\quad\otimes\alpha\left(\lambda_2\left(c_2\right)\right)=\varepsilon\left(c_1\right)\varepsilon\left(c_2\right)1_H\otimes 1_H\otimes 1_H\\&\quad=\varepsilon\left(c\right)1_H\otimes 1_H\otimes 1_H,\end{aligned} \tag{17}$$

and the right hand side is

$$\begin{aligned}&\alpha^{-1}\left(c_{1-1}\right)\lambda_1\left(c_2\right)\otimes\alpha^{-1}\left(\lambda_1\left(c_{10}\right)\right)\alpha^{-1}\left(\lambda_2\left(c_2\right)_1\right)\\&\quad\otimes\alpha^{-1}\left(\lambda_2\left(c_{10}\right)\right)\alpha^{-1}\left(\lambda_2\left(c_2\right)_2\right)\\&\quad=\alpha^{-1}\left(c_{1-1}\right)\varepsilon\left(c_2\right)1_H\otimes\alpha^{-1}\left(\lambda_1\left(c_{10}\right)\right)1_H\\&\quad\otimes\alpha^{-1}\left(\lambda_2\left(c_{10}\right)\right)1_H=\alpha^{-1}\left(c_{1-1}\right)\varepsilon\left(c_2\right)1_H\\&\quad\otimes\varepsilon\left(c_{10}\right)1_H\otimes 1_H=\varepsilon\left(c\right)1_H\otimes 1_H\otimes 1_H,\end{aligned} \tag{18}$$

so (C) is satisfied.

For condition (TC), the left hand side is

$$\begin{aligned}&\lambda_1\left(c_1\right)\alpha^{-2}\left(c_{2-11}\right)\otimes\lambda_2\left(c_1\right)\alpha^{-2}\left(c_{2-12}\right)\otimes c_{20}\\&\quad=\varepsilon\left(c_1\right)\alpha^{-1}\left(c_{2-11}\right)\otimes\alpha^{-1}\left(c_{2-12}\right)\otimes c_{20}\\&\quad=c_{-11}\otimes c_{-12}\otimes\beta\left(c_0\right),\end{aligned} \tag{19}$$

the right hand side is

$$\begin{aligned}&\alpha^{-1}\left(c_{1-1}\right)\lambda_1\left(c_2\right)\otimes\alpha^{-2}\left(c_{10-1}\right)\lambda_2\left(c_2\right)\otimes\beta^{-1}\left(c_{100}\right)\\&\quad=c_{1-1}\varepsilon\left(c_2\right)\otimes\alpha^{-1}\left(c_{10-1}\right)\otimes\beta^{-1}\left(c_{100}\right)\\&\quad=\alpha\left(c_{-1}\right)\otimes c_{0-1}\otimes c_{00},\end{aligned} \tag{20}$$

and then the Hom-crossed coproduct is Hom-smash coproduct. In this case, $C$ is a left $H$-comodule coalgebra, and the condition (TC) is satisfied. □

(2) Let $H_2$ be 2-dimension Hopf group algebra with a basis $\{1, g\}$. Then $(H_2, id)$ forms a Hom-Hopf algebra. Let $H_4$ be a vector space with a basis $\{e, c, x, y\}$. Define the Hom-coalgebra structure on $H_4$ as follows.

The automorphism $\beta : H_4 \to H_4$ is given by

$$\begin{aligned}\beta\left(e\right)&=e,\\\beta\left(c\right)&=c,\\\beta\left(x\right)&=-x,\\\beta\left(y\right)&=-y;\end{aligned} \tag{21}$$

the comultiplication and counit are given by

$$\begin{aligned}\Delta\left(e\right)&=e\otimes e,\\\Delta\left(c\right)&=c\otimes c,\\\Delta\left(x\right)&=-x\otimes c-e\otimes x,\\\Delta\left(y\right)&=-y\otimes e-c\otimes y,\\\varepsilon\left(e\right)&=1,\\\varepsilon\left(c\right)&=1,\\\varepsilon\left(x\right)&=0,\\\varepsilon\left(y\right)&=0.\end{aligned} \tag{22}$$

It is easy to see that $H_4$ is a Hom-coalgebra.

Now consider the coaction $\rho^{H_4} : H_4 \to H_2 \otimes H_4$ defined by

$$\begin{aligned}e_{-1}\otimes e_0&:=1\otimes e,\\c_{-1}\otimes c_0&:=1\otimes c,\\x_{-1}\otimes x_0&:=-g\otimes x,\\y_{-1}\otimes y_0&:=-g\otimes y.\end{aligned} \tag{23}$$

Then after a direct computation, we get that $H_2$ coacts weakly on $H_4$ from the left. Further, recall from (1) that if we define $\lambda : H_4 \to H_2\otimes H_2$ by $\lambda(h) = \varepsilon(h)1\otimes 1$, then $(H_4 \rtimes_\lambda H_2, \beta\otimes id)$ is a Hom-crossed coproduct.

*Proof.* We can prove that the condition (CU) and the condition (C) hold for any $c \in H_4$ the same as Example 5(1). Then we only prove that the condition (TC) holds. By the proof of Example 5(1), for $x \in H_4$, we can get that the left hand side of the condition (TC) is

$$\begin{aligned}&\lambda_1(x_1)x_{2-11}\otimes\lambda_2(x_1)x_{2-12}\otimes x_{20}\\&\quad= x_{-11}\otimes x_{-12}\otimes\beta(x_0)=g\otimes x\otimes x,\end{aligned}\tag{24}$$

and the right hand side of the condition (TC) is

$$\begin{aligned}&x_{1-1}\lambda_1(x_2)\otimes x_{10-1}\lambda_2(x_2)\otimes\beta^{-1}(x_{100})\\&\quad= x_{-1}\otimes x_{0-1}\otimes x_{00}=g\otimes x\otimes x,\end{aligned}\tag{25}$$

so the condition (TC) holds for $x \in H_4$. Similarly, we can get the condition (TC) holds for $e, c, y \in H_4$. □

(3) Let $(H, \Delta, \mu)$ be a Hopf algebra and $(C, \Delta)$ a coalgebra, and $H$ weakly coacts on $C$ from the left. Assume that $\alpha$ is a Hopf automorphism of $H$ and $\beta$ is a coalgebra isomorphism of $C$. Then we have Hom-Hopf algebra $(H_\alpha, \Delta_\alpha, \mu_\alpha, \alpha)$ and Hom-coalgebra $(C_\beta, \Delta_\beta, \beta)$ (see [2]). Furthermore assume that $\rho(\beta(c)) = (\alpha\otimes\beta)\rho(c)$, and define the coaction $\rho(c) = (\alpha\otimes\beta)\rho(c)$, then $H_\alpha$ weakly coacts on $C_\beta$ from the left. If $C\rtimes_\lambda H$ is a crossed coproduct and $\lambda\beta = (\alpha\otimes\alpha)\lambda$, then $C_\beta\rtimes_\lambda H_\alpha$ is a Hom-crossed coproduct.

*Proof.* If $C\rtimes_\lambda H$ is a crossed coproduct, then we can get

$$(id\otimes\varepsilon)\lambda = (\varepsilon\otimes id)\lambda = 1_H\varepsilon;\tag{26}$$

$$\begin{aligned}&\lambda_1(c_1)\lambda_1(c_2)_1\otimes\lambda_2(c_1)\lambda_1(c_2)_2\otimes\lambda_2(c_2)\\&\quad= c_{1-1}\lambda_1(c_2)\otimes\lambda_1(c_{10})\lambda_2(c_2)_1\\&\qquad\otimes\lambda_2(c_{10})\lambda_2(c_2)_2;\end{aligned}\tag{27}$$

$$\begin{aligned}&\lambda_1(c_1)c_{2-11}\otimes\lambda_2(c_1)c_{2-12}\otimes c_{20}\\&\quad= c_{1-1}\lambda_1(c_2)\otimes c_{10-1}\lambda_2(c_2)\otimes c_{100}.\end{aligned}\tag{28}$$

From [2], we know that the multiplication and comultiplication of Hom-Hopf algebra $(H_\alpha, \Delta_\alpha, \mu_\alpha, \alpha)$ are, respectively, given by

$$\begin{aligned}&\mu_\alpha(h\otimes g)=\alpha(h)\alpha(g),\\&\Delta_\alpha(h)=\alpha(h_1)\otimes\alpha(h_2),\\&\qquad\qquad\forall h, g\in H_\alpha,\end{aligned}\tag{29}$$

and the comultiplication of Hom-coalgebra $(C_\beta, \Delta_\beta, \beta)$ is given by

$$\Delta_\beta(c)=\beta(c_1)\otimes\beta(c_2),\quad\forall c\in C_\beta.\tag{30}$$

First, we prove that $H_\alpha$ weakly coacts on $C_\beta$ from the left. We only prove that (W1) of Definition 1 holds; the other two are easy to get. For any $c \in C_\beta$,

$$\begin{aligned}&\alpha^2(c_{-1})\otimes c_{01}\otimes c_{02}\quad\left(\text{in } H_\alpha\otimes C_\beta\otimes C_\beta\right)\\&=\alpha^3(c_{-1})\otimes\beta^2(c_{01})\otimes\beta^2(c_{02})\quad(\text{in } H\otimes C\otimes C)\\&=\alpha^3(c_{1-1}c_{2-1})\otimes\beta^2(c_{10})\otimes\beta^2(c_{20})\quad(\text{in } H\otimes C\otimes C)\\&=c_{1-1}c_{2-1}\otimes c_{10}\otimes c_{20},\quad\left(\text{in } H_\alpha\otimes C_\beta\otimes C_\beta\right),\end{aligned}\tag{31}$$

so $H_\alpha$ weakly coacts on $C_\beta$ from the left.

Then we prove that $C_\beta\rtimes_\lambda H_\alpha$ is a Hom-crossed coproduct. It is easy to see that the condition (CU) holds. For the condition (C),

$$\begin{aligned}&\lambda_1(c_1)\alpha^{-1}(\lambda_1(c_2)_1)\otimes\lambda_2(c_1)\alpha^{-1}(\lambda_1(c_2)_2)\\&\quad\otimes\alpha(\lambda_2(c_2))\quad(\text{in } H_\alpha\otimes H_\alpha\otimes H_\alpha)\\&=\alpha^2(\lambda_1(c_1)\lambda_1(c_2)_1)\otimes\alpha^2(\lambda_2(c_1)\lambda_1(c_2)_2)\\&\quad\otimes\alpha^2(\lambda_2(c_2))\quad(\text{in } H\otimes H\otimes H)\\&\overset{(27)}{=}\alpha^2(c_{1-1}\lambda_1(c_2))\otimes\alpha^2(\lambda_1(c_{10})\lambda_2(c_2)_1)\\&\quad\otimes\alpha^2(\lambda_2(c_{10})\lambda_2(c_2)_2)\quad(\text{in } H\otimes H\otimes H)\\&=\alpha^{-1}(c_{1-1})\lambda_1(c_2)\otimes\alpha^{-1}(\lambda_1\lambda_1(c_{10}))\alpha^{-1}(\lambda_2(c_2)_1)\\&\quad\otimes\alpha^{-1}(\lambda_2(c_{10}))\alpha^{-1}(\lambda_2(c_2)_2),\\&\qquad\qquad\left(\text{in } H_\alpha\otimes H_\alpha\otimes H_\alpha\right)\end{aligned}\tag{32}$$

so the condition (CU) holds. Similarly we can prove that the condition (TC) holds by (28). □

For a Hom-crossed coproduct, we can get the following properties, which are useful for the latter conclusions.

**Lemma 6.** *Let $C\rtimes_\lambda H$ be a Hom-crossed coproduct with invertible cocycle $\lambda$. Then the following equalities hold for any $c \in C$:*

(i)

$$\begin{aligned}&c_{-1}\otimes\lambda_1(c_0)\otimes\lambda_2(c_0)\\&\quad=\left(\alpha^{-3}(\lambda_1(c_{11}))\alpha^{-4}(\lambda_1(c_{12})_1)\right)\alpha^{-1}\left(\lambda_1^{-1}(c_2)\right)\\&\qquad\otimes\left(\alpha^{-3}(\lambda_2(c_{11}))\alpha^{-4}(\lambda_1(c_{12})_2)\right)\alpha^{-2}\left(\lambda_2^{-1}(c_2)_1\right)\\&\qquad\otimes\alpha^{-2}(\lambda_2(c_{12}))\alpha^{-2}\left(\lambda_2^{-1}(c_2)_2\right);\end{aligned}\tag{33}$$

(ii)

$$\begin{aligned}&c_{-1}\otimes\lambda_1^{-1}(c_0)\otimes\lambda_2^{-1}(c_0)\\&\quad=\alpha^{-1}(\lambda_1(c_1))\left(\alpha^{-4}\left(\lambda_1^{-1}(c_{21})_1\right)\alpha^{-3}\left(\lambda_1^{-1}(c_{22})\right)\right)\\&\qquad\otimes\alpha^{-2}(\lambda_2(c_1)_1)\left(\alpha^{-4}\left(\lambda_1^{-1}(c_{21})_2\right)\alpha^{-3}\left(\lambda_2^{-1}(c_{22})\right)\right)\\&\qquad\otimes\alpha^{-2}(\lambda_2(c_1)_2)\alpha^{-2}\left(\lambda_2^{-1}(c_{21})\right).\end{aligned}\tag{34}$$

*Proof.* Applying $\alpha^{-3}\otimes\alpha^{-3}\otimes\alpha^{-3}$ to both sides of (C) and then multiplying it (by convolution) to the right by the following map, we can get (i):

$$c \longmapsto \alpha^{-1}\lambda_1^{-1}(c)\otimes\alpha^{-2}\left(\lambda_2^{-1}(c)_1\right)\otimes\alpha^{-2}\left(\lambda_2^{-1}(c)_2\right). \quad (35)$$

Let us denote by $F_1, F_2, F_3 : C \to H\otimes H\otimes H$, where the map is defined by

$$F_1(c) = c_{-1}\otimes\lambda_1(c_0)\otimes\lambda_2(c_0). \quad (36)$$

We get from (i) that a right convolution inverse for $F_1$ is the map $F_2$, which is given by

$$\begin{aligned} &F_2(c) \\ &= \alpha^{-1}(\lambda_1(c_1))\left(\alpha^{-4}\left(\lambda_1^{-1}(c_{21})_1\right)\alpha^{-3}\left(\lambda_1^{-1}(c_{22})\right)\right) \\ &\quad\otimes\alpha^{-2}(\lambda_2(c_1)_1)\left(\alpha^{-4}\left(\lambda_1^{-1}(c_{21})_2\right)\alpha^{-3}\left(\lambda_2^{-1}(c_{22})\right)\right) \\ &\quad\otimes\alpha^{-2}(\lambda_2(c_1)_2)\,\alpha^{-2}\left(\lambda_2^{-1}(c_{21})\right). \end{aligned} \quad (37)$$

Let $F_3$ be given by $F_3(c) = c_{-1}\otimes\lambda_1^{-1}(c_0)\otimes\lambda_2^{-1}(c_0)$. Since $(F_3 * F_1)(c) = \varepsilon(c)1\otimes 1\otimes 1$. We get that $F_3$ is a left convolution inverse of $F_1$, so $F_3 = F_2$. Therefore, we prove that (ii) holds. □

*Remark 7.* Note that if $(C\rtimes_\lambda H, \beta\otimes\alpha)$ is a Hom-crossed coproduct, then the map $\pi : C\rtimes_\lambda H \to C$, defined by $\pi(c\otimes h) = \varepsilon(h)\beta^{-1}(c)$, is a Hom-coalgebra map, and we will also define the map $\theta : C\rtimes_\lambda H \to H$, $\theta(c\otimes h) = \varepsilon(c)\alpha^{-1}(h)$.

The following result is the generalization of Proposition 2.1 in [8].

**Proposition 8.** *Let $B = C\rtimes_\lambda H$ be a Hom-crossed coproduct. Then $\theta$ is convolution invertible in $Hom(B,H)$ if and only if $\lambda$ is convolution invertible in $Hom(C, H\otimes H)$.*

*Proof.* Assume first that $\lambda$ is convolution invertible; for any $c\rtimes h\in B$, define $\mu : B\to H$, by

$$\mu(c\rtimes h) = S\left(\alpha^{-3}\left(\lambda_1^{-1}(c)\right)\alpha^{-3}(h)\right)\alpha^{-2}\left(\lambda_2^{-1}(c)\right). \quad (38)$$

Now we show that $\mu$ is a convolution inverse of $\theta$.

$$\begin{aligned} (\mu*\theta)(c\rtimes h) &= \mu\left(c_1\otimes\left(\alpha^{-4}(c_{21-1})\alpha^{-3}(\lambda_1(c_{22}))\right)\right. \\ &\quad\left.\cdot\alpha^{-1}(h_1)\right)\theta\left(\beta^{-2}(c_{210})\otimes\alpha^{-2}(\lambda_2(c_{22}))\alpha^{-1}(h_2)\right) \\ &= S\left(\alpha^{-3}\left(\lambda_1^{-1}(c_1)\right)\right. \\ &\quad\cdot\alpha^{-3}\left(\left(\alpha^{-4}(c_{21-1})\alpha^{-3}(\lambda_1(c_{22}))\right)\alpha^{-1}(h_1)\right)\Big) \\ &\quad\cdot\alpha^{-2}\left(\lambda_2^{-1}(c_1)\right)\varepsilon(c_{210})\left(\alpha^{-3}(\lambda_2(c_{22}))\alpha^{-2}(h_2)\right) \\ &= S\left(\alpha^{-3}\left(\lambda_1^{-1}(c_1)\right)\left(\alpha^{-4}(\lambda_1(c_2))\alpha^{-4}(h_1)\right)\right) \\ &\quad\cdot\alpha^{-2}\left(\lambda_2^{-1}(c_1)\right)\left(\alpha^{-2}(\lambda_2(c_2))\alpha^{-2}(h_2)\right) \\ &= S\left(\alpha^{-3}\left(\lambda_1^{-1}(c_1)\lambda_1(c_2)\right)\alpha^{-2}(h_1)\right) \\ &\quad\cdot\left(\alpha^{-3}\left(\lambda_2^{-1}(c_1)\lambda_2(c_2)\right)\alpha^{-2}(h_2)\right) = \varepsilon(c)\varepsilon(h). \end{aligned} \quad (39)$$

So $\mu$ is a left inverse for $\theta$; in order to show that $\mu$ is a right inverse we compute as follows:

$$\begin{aligned} (\theta*\mu)(c\rtimes h) &= \theta\left(c_1\otimes\left(\alpha^{-4}(c_{21-1})\alpha^{-3}(\lambda_1(c_{22}))\right)\right. \\ &\quad\left.\cdot\alpha^{-1}(h_1)\right)\mu\left(\beta^{-2}(c_{210})\otimes\alpha^{-2}(\lambda_2(c_{22}))\alpha^{-1}(h_2)\right) \\ &= \varepsilon(c_1)\left(\alpha^{-5}(c_{21-1})\alpha^{-4}(\lambda_1(c_{22}))\right)\alpha^{-2}(h_1) \\ &\quad\cdot\left(S\left(\alpha^{-5}\left(\lambda_1^{-1}(c_{210})\right)\left(\alpha^{-5}(\lambda_2(c_{22}))\alpha^{-4}(h_2)\right)\right)\right. \\ &\quad\left.\cdot\alpha^{-4}\left(\lambda_2^{-1}(c_{210})\right)\right) = \left(\alpha^{-4}(c_{1-1})\alpha^{-3}(\lambda_1(c_2))\right) \\ &\quad\cdot\alpha^{-2}(h_1)\left(S\left(\left(\alpha^{-5}\left(\lambda_1^{-1}(c_{10})\right)\alpha^{-4}(\lambda_2(c_2))\right)\right.\right. \\ &\quad\left.\left.\cdot\alpha^{-3}(h_2)\right)\alpha^{-3}\left(\lambda_2^{-1}(c_{10})\right)\right) = \left(\alpha^{-3}(c_{1-1})\right. \\ &\quad\left.\cdot\alpha^{-2}(\lambda_1(c_2))\right)\left(\left(\alpha^{-3}(h_1)S\left(\alpha^{-3}(h_2)\right)\right)\right. \\ &\quad\cdot\left(S\left(\alpha^{-5}\left(\lambda_1^{-1}(c_{10})\right)\alpha^{-4}(\lambda_2(c_2))\right)\right. \\ &\quad\left.\left.\cdot\alpha^{-4}\left(\lambda_2^{-1}(c_{10})\right)\right)\right) = \left(\alpha^{-3}(c_{1-1})\alpha^{-2}(\lambda_1(c_2))\right) \\ &\quad\cdot\varepsilon(h)\left(S\left(\alpha^{-4}\left(\lambda_1^{-1}(c_{10})\right)\alpha^{-3}(\lambda_2(c_2))\right)\right. \\ &\quad\left.\cdot\alpha^{-3}\left(\lambda_2^{-1}(c_{10})\right)\right) \overset{(34)}{=} \left(\left(\alpha^{-4}(\lambda_1(c_{11}))\right.\right. \\ &\quad\left.\left.\cdot\left(\alpha^{-7}\left(\lambda_1^{-1}(c_{121})_1\right)\alpha^{-6}\left(\lambda_1^{-1}(c_{122})\right)\right)\right)\right. \\ &\quad\left.\cdot\alpha^{-2}(\lambda_1(c_2))\right)\varepsilon(h)\left(S\left(\left(\alpha^{-6}(\lambda_2(c_{11})_1)\right.\right.\right. \\ &\quad\left.\left.\cdot\left(\alpha^{-8}\left(\lambda_1^{-1}(c_{121})_2\right)\alpha^{-7}\left(\lambda_2^{-1}(c_{122})\right)\right)\right)\right. \\ &\quad\left.\cdot\alpha^{-3}(\lambda_2(c_2))\right)\left(\alpha^{-5}(\lambda_2(c_{11})_2)\alpha^{-5}\left(\lambda_2^{-1}(c_{121})\right)\right)\Big) \\ &= \left(\alpha^{-3}(\lambda_1(c_{11}))\left(\alpha^{-5}\left(\lambda_1^{-1}(c_{12})_1\right)\varepsilon(c_2)1\right)\right)\varepsilon(h) \\ &\quad\cdot\left(S\left(\alpha^{-5}(\lambda_2(c_{11})_1)\left(\alpha^{-6}\left(\lambda_1^{-1}(c_{12})_2\right)1\right)\right)\right. \\ &\quad\left.\cdot\left(\alpha^{-5}(\lambda_2(c_{11})_2)\alpha^{-4}\left(\lambda_2^{-1}(c_{12})\right)\right)\right) \\ &= \left(\alpha^{-2}(\lambda_1(c_1))\alpha^{-3}\left(\lambda_1^{-1}(c_2)_1\right)\right)\varepsilon(h) \\ &\quad\cdot\left(S\left(\alpha^{-4}(\lambda_2(c_1)_1)\alpha^{-4}\left(\lambda_1^{-1}(c_2)_2\right)\right)\right. \\ &\quad\left.\cdot\left(\alpha^{-4}(\lambda_2(c_1)_2)\alpha^{-3}\left(\lambda_2^{-1}(c_2)\right)\right)\right) = \varepsilon(c)\varepsilon(h). \end{aligned} \quad (40)$$

So $\theta$ is convolution invertible.

Next we prove that if $\theta$ is convolution invertible, then $\lambda$ is also convolution invertible. Recalling

$$\begin{aligned} \pi &: B \longrightarrow C, \\ c\otimes h &\longmapsto \varepsilon(h)\beta^{-1}(c), \end{aligned} \quad (41)$$

we prove that $\lambda\pi$ is convolution invertible.

Firstly, we prove that

$$\begin{aligned} (\lambda\pi)(b) &= \alpha^{-3}(\theta(b_{11}))\alpha^{-3}\left(\theta^{-1}(b_2)_1\right) \\ &\quad\otimes\alpha^{-3}(\theta(b_{12}))\alpha^{-3}\left(\theta^{-1}(b_2)_2\right). \end{aligned} \quad (42)$$

Observe that if $b = c \otimes h$, then the equality $\theta(b_1)\theta^{-1}(b_2) = \varepsilon(b)1$ becomes

$$\begin{aligned}&\left(\left(\alpha^{-4}\left(c_{1-1}\right)\alpha^{-3}\left(\lambda_1\left(c_2\right)\right)\right)\alpha^{-2}\left(h_1\right)\right)\\&\quad\cdot\theta^{-1}\left(\beta^{-1}\left(c_{10}\right)\otimes\alpha^{-1}\left(\lambda_2\left(c_2\right)\right)\alpha^{-1}\left(h_2\right)\right)=\varepsilon\left(c\right)\\&\quad\cdot\varepsilon\left(h\right)1.\end{aligned}\tag{43}$$

By applying $\Delta_H$ we get

$$\begin{aligned}&\left(\left(\alpha^{-4}\left(c_{1-1}\right)_1\alpha^{-3}\left(\lambda_1\left(c_2\right)\right)_1\right)\alpha^{-2}\left(h_1\right)_1\right)\\&\quad\cdot\theta^{-1}\left(\beta^{-1}\left(c_{10}\right)\otimes\alpha^{-1}\left(\lambda_2\left(c_2\right)\right)\alpha^{-1}\left(h_2\right)\right)_1\\&\quad\otimes\left(\left(\alpha^{-4}\left(c_{1-1}\right)_2\alpha^{-3}\left(\lambda_1\left(c_2\right)\right)_2\right)\alpha^{-2}\left(h_1\right)_2\right)\\&\quad\cdot\theta^{-1}\left(\beta^{-1}\left(c_{10}\right)\otimes\alpha^{-1}\left(\lambda_2\left(c_2\right)\right)\alpha^{-1}\left(h_2\right)\right)_2=\varepsilon\left(c\right)\\&\quad\cdot\varepsilon\left(h\right)1\otimes 1.\end{aligned}\tag{44}$$

At the same time, we have

$$\begin{aligned}b_{11}\otimes b_{12}\otimes b_2&=c_{11}\otimes\left(\alpha^{-4}\left(c_{121-1}\right)\alpha^{-3}\left(\lambda_1\left(c_{122}\right)\right)\right)\\&\quad\cdot\left(\left(\alpha^{-5}\left(c_{21-11}\right)\alpha^{-4}\left(\lambda_1\left(c_{22}\right)_1\right)\right)\alpha^{-2}\left(h_{11}\right)\right)\\&\quad\otimes\beta^{-2}\left(c_{1210}\right)\otimes\alpha^{-2}\left(\lambda_2\left(c_{122}\right)\right)\\&\quad\cdot\left(\left(\alpha^{-5}\left(c_{21-12}\right)\alpha^{-4}\left(\lambda_1\left(c_{22}\right)_2\right)\right)\alpha^{-2}\left(h_{12}\right)\right)\\&\quad\otimes\beta^{-2}\left(c_{210}\right)\otimes\alpha^{-2}\left(\lambda_2\left(c_{22}\right)\right)\alpha^{-1}\left(h_2\right).\end{aligned}\tag{45}$$

So we get

$$\begin{aligned}&\alpha^{-3}\left(\theta\left(b_{11}\right)\right)\alpha^{-3}\left(\theta^{-1}\left(b_2\right)_1\right)\otimes\alpha^{-3}\left(\theta\left(b_{12}\right)\right)\alpha^{-3}\left(\theta^{-1}\left(b_2\right)_2\right)\\&\quad=\alpha^{-3}\left(\lambda_1\left(c_1\right)\right)\underline{\left(\left(\left(\alpha^{-9}\left(c_{21-11}\right)\alpha^{-8}\left(\lambda_1\left(c_{22}\right)_1\right)\right)\alpha^{-6}\left(h_{11}\right)\right)\right)}\,\underline{\theta^{-1}\left(\beta^{-6}\left(c_{210}\right)\otimes\alpha^{-6}\left(\lambda_2\left(c_{22}\right)\right)\alpha^{-5}\left(h_2\right)\right)_1}\\&\qquad\otimes\alpha^{-3}\left(\lambda_2\left(c_1\right)\right)\underline{\left(\left(\left(\alpha^{-9}\left(c_{21-12}\right)\alpha^{-8}\left(\lambda_1\left(c_{22}\right)_2\right)\right)\alpha^{-6}\left(h_{12}\right)\right)\right)}\,\underline{\theta^{-1}\left(\beta^{-6}\left(c_{210}\right)\otimes\alpha^{-6}\left(\lambda_2\left(c_{22}\right)\right)\alpha^{-5}\left(h_2\right)\right)_2}\\&\quad\overset{(44)}{=}\alpha^{-2}\left(\lambda_1\left(c_1\right)\right)\otimes\alpha^{-2}\left(\lambda_2\left(c_1\right)\right)\varepsilon\left(c_2\right)\varepsilon\left(h\right)=\alpha^{-1}\left(\lambda_1\left(c\right)\right)\otimes\alpha^{-1}\left(\lambda_2\left(c\right)\right)\varepsilon\left(h\right)=\alpha^{-1}\left(\lambda\left(c\right)\right)\varepsilon\left(h\right)=\left(\lambda\pi\right)\left(b\right).\end{aligned}\tag{46}$$

Secondly, let $\omega : B \to H \otimes H$, which is given by

$$\begin{aligned}\omega\left(b\right)&=\alpha^{-4}\left(\theta\left(b_{11}\right)_1\right)\alpha^{-2}\left(\theta^{-1}\left(b_2\right)\right)\\&\quad\otimes\alpha^{-4}\left(\theta\left(b_{11}\right)_2\right)\alpha^{-3}\left(\theta^{-1}\left(b_{12}\right)\right),\end{aligned}\tag{47}$$

and we prove that $\omega$ is the convolution inverse of $\lambda\pi$.

On the one hand,

$$\begin{aligned}&\left(\lambda\pi*\omega\right)\left(b\right)=\left(\lambda\pi\right)\left(b_1\right)\omega\left(b_2\right)=\left(\alpha^{-3}\left(\theta\left(b_{111}\right)\right)\right.\\&\quad\cdot\alpha^{-3}\left(\theta^{-1}\left(b_{12}\right)_1\right)\left)\left(\alpha^{-4}\left(\theta\left(b_{211}\right)_1\right)\alpha^{-2}\left(\theta^{-1}\left(b_{22}\right)\right)\right)\right.\\&\quad\otimes\left(\alpha^{-3}\left(\theta\left(b_{112}\right)\right)\alpha^{-3}\left(\theta^{-1}\left(b_{12}\right)_2\right)\right)\left(\alpha^{-4}\left(\theta\left(b_{211}\right)_2\right)\right.\\&\quad\cdot\alpha^{-3}\left(\theta^{-1}\left(b_{212}\right)\right)\left)=\alpha^{-1}\left(\theta\left(b_{11}\right)\right)\right.\\&\quad\cdot\left(\left(\alpha^{-5}\left(\theta^{-1}\left(b_{211}\right)_1\right)\alpha^{-5}\left(\theta\left(b_{212}\right)_1\right)\right)\right.\\&\quad\cdot\alpha^{-3}\left(\theta^{-1}\left(b_{222}\right)\right)\left)\otimes\alpha^{-1}\left(\theta\left(b_{12}\right)\right)\right.\\&\quad\cdot\left(\left(\alpha^{-5}\left(\theta^{-1}\left(b_{211}\right)_2\right)\alpha^{-5}\left(\theta\left(b_{212}\right)_2\right)\right)\right.\\&\quad\cdot\alpha^{-3}\left(\theta^{-1}\left(b_{221}\right)\right)\left)=\alpha^{-1}\left(\theta\left(b_{11}\right)\right)\alpha^{-2}\left(\theta^{-1}\left(b_{222}\right)\right)\right.\\&\quad\cdot\varepsilon\left(b_{21}\right)\otimes\alpha^{-1}\left(\theta\left(b_{12}\right)\right)\alpha^{-2}\left(\theta^{-1}\left(b_{221}\right)\right)=\theta\left(b_1\right)\\&\quad\cdot\alpha^{-1}\left(\theta^{-1}\left(b_{22}\right)\right)\otimes\alpha^{-2}\left(\theta\left(b_{211}\right)\right)\alpha^{-2}\left(\theta^{-1}\left(b_{222}\right)\right)\\&\quad=\theta\left(b_1\right)\alpha^{-1}\left(\theta^{-1}\left(b_{22}\right)\right)\otimes 1\varepsilon\left(b_{21}\right)=\theta\left(b_1\right)\theta^{-1}\left(b_2\right)\\&\quad\otimes 1=1\otimes 1\varepsilon\left(b\right).\end{aligned}\tag{48}$$

So $\omega$ is a left convolution inverse for $\lambda\pi$. On the other hand,

$$\begin{aligned}&\left(\omega*\lambda\pi\right)\left(b\right)=\omega\left(b_1\right)\lambda\pi\left(b_2\right)=\left(\alpha^{-4}\left(\theta\left(b_{111}\right)_1\right)\right.\\&\quad\cdot\alpha^{-2}\left(\theta^{-1}\left(b_{12}\right)\right)\left)\left(\alpha^{-3}\left(\theta\left(b_{211}\right)\right)\alpha^{-3}\left(\theta^{-1}\left(b_{22}\right)_1\right)\right)\right.\\&\quad\otimes\left(\alpha^{-4}\left(\theta\left(b_{111}\right)_2\right)\alpha^{-3}\left(\theta^{-1}\left(b_{112}\right)\right)\right)\left(\alpha^{-3}\left(\theta\left(b_{212}\right)\right)\right.\\&\quad\cdot\alpha^{-3}\left(\theta^{-1}\left(b_{22}\right)_2\right)\left)=\alpha^{-2}\left(\theta\left(b_{11}\right)_1\right)\right.\\&\quad\cdot\left(\left(\alpha^{-4}\left(\theta^{-1}\left(b_{211}\right)\right)\alpha^{-4}\left(\theta\left(b_{212}\right)\right)\right)\alpha^{-4}\left(\theta^{-1}\left(b_{222}\right)_1\right)\right)\\&\quad\otimes\alpha^{-2}\left(\theta\left(b_{11}\right)_2\right)\left(\left(\alpha^{-3}\left(\theta^{-1}\left(b_{12}\right)\right)\alpha^{-4}\left(\theta\left(b_{221}\right)\right)\right)\right.\\&\quad\cdot\alpha^{-4}\left(\theta^{-1}\left(b_{222}\right)_2\right)\left)=\alpha^{-2}\left(\theta\left(b_{11}\right)_1\right)\right.\\&\quad\cdot\alpha^{-3}\left(\theta^{-1}\left(b_{222}\right)_1\right)\varepsilon\left(b_{21}\right)\otimes\alpha^{-2}\left(\theta\left(b_{11}\right)_2\right)\\&\quad\cdot\left(\left(\alpha^{-3}\left(\theta^{-1}\left(b_{12}\right)\right)\alpha^{-4}\left(\theta\left(b_{221}\right)\right)\right)\alpha^{-4}\left(\theta^{-1}\left(b_{222}\right)_2\right)\right)\\&\quad=\alpha^{-2}\left(\theta\left(b_{11}\right)_1\right)\alpha^{-2}\left(\theta^{-1}\left(b_{22}\right)_1\right)\otimes\alpha^{-2}\left(\theta\left(b_{11}\right)_2\right)\\&\quad\cdot\left(\left(\alpha^{-3}\left(\theta^{-1}\left(b_{12}\right)\right)\alpha^{-3}\left(\theta\left(b_{21}\right)\right)\right)\alpha^{-3}\left(\theta^{-1}\left(b_{22}\right)_2\right)\right)\\&\quad=\alpha^{-1}\left(\theta\left(b_1\right)_1\right)\alpha^{-2}\left(\theta^{-1}\left(b_{22}\right)_1\right)\otimes\alpha^{-1}\left(\theta\left(b_1\right)_2\right)\end{aligned}$$

$$\begin{aligned}
&\cdot\left(\left(\alpha^{-4}\left(\theta^{-1}\left(b_{211}\right)\right)\alpha^{-4}\left(\theta\left(b_{212}\right)\right)\right)\alpha^{-3}\left(\theta^{-1}\left(b_{22}\right)_2\right)\right)\\
&=\alpha^{-1}\left(\theta\left(b_1\right)_1\right)\alpha^{-2}\left(\theta^{-1}\left(b_{22}\right)_1\right)\otimes\alpha^{-1}\left(\theta\left(b_1\right)_2\right)\\
&\cdot\alpha^{-2}\left(\theta^{-1}\left(b_{22}\right)_2\right)\varepsilon\left(b_{21}\right)=\alpha^{-1}\left(\theta\left(b_1\right)_1\right)\\
&\cdot\alpha^{-1}\left(\theta^{-1}\left(b_2\right)_1\right)\otimes\alpha^{-1}\left(\theta\left(b_1\right)_2\right)\alpha^{-1}\left(\theta^{-1}\left(b_2\right)_2\right)\\
&=1\otimes 1\varepsilon\left(d\right).
\end{aligned}\tag{49}$$

Thirdly, let $j : C \to B$, $j(c) = c \otimes 1$, then we have

$$\begin{aligned}
(\pi\otimes id)\,\Delta\left(j\left(c\right)\right)&=(\pi\otimes id)\,\Delta\left(\beta\left(c\right)\otimes 1\right)\\
&=\pi\left(\beta\left(c_1\right)\otimes\alpha^{-2}\left(c_{21-1}\right)\alpha^{-1}\left(\lambda_1\left(c_{22}\right)\right)\right)\\
&\quad\otimes\beta^{-1}\left(c_{210}\right)\otimes\lambda_2\left(c_{22}\right)\\
&=c_1\varepsilon\left(c_{21-1}\right)\varepsilon\left(\lambda_1\left(c_{22}\right)\right)\otimes\beta^{-1}\left(c_{210}\right)\otimes\lambda_2\left(c_{22}\right)\\
&=c_1\otimes c_{21}\otimes\varepsilon\left(c_{22}\right)1=c_1\otimes\beta\left(c_2\right)\otimes 1\\
&=c_1\otimes j\left(c_2\right).
\end{aligned}\tag{50}$$

Since $(\lambda\pi) * \omega = 1$, we obtain that

$$\begin{aligned}
(\lambda\pi*\omega)\left(j\left(c\right)\right)&=(\lambda\pi)\left(j\left(c\right)_1\right)\omega\left(j\left(c\right)_2\right)\\
&=\lambda\left(c_1\right)\left(\omega j\right)\left(c_2\right)=\varepsilon\left(c\right)1.
\end{aligned}\tag{51}$$

So $\lambda$ has a right convolution inverse $\omega j$ in $\mathrm{Hom}(C, H \otimes H)$. On the other hand,

$$\begin{aligned}
\left(\lambda\pi*\omega j\right)\pi\left(b\right)&=\lambda\left(\pi\left(b_1\right)\right)\left(\omega j\right)\left(\pi\left(b_2\right)\right)\\
&=\lambda\left(\pi\left(b\right)_1\right)\left(\omega j\right)\left(\pi\left(b\right)_2\right)\\
&=\varepsilon\left(\pi\left(b\right)\right)1=\varepsilon\left(b\right)1.
\end{aligned}\tag{52}$$

It shows that $\omega j\pi$ is a right convolution inverse for $\lambda\pi$ in $\mathrm{Hom}(B, H \otimes H)$; thus $\omega j\pi = \omega$.

Finally, we denote that $\lambda' = \omega j$, because we have that $(\lambda\pi) * (\lambda'\pi) = (\lambda'\pi) * (\lambda\pi) = 1$ in $\mathrm{Hom}(B, H \otimes H)$ and $\pi$ is a surjective Hom-coalgebra map, and we obtain $\lambda * \lambda' = \lambda' * \lambda = 1$ in $\mathrm{Hom}(C, H \otimes H)$. So $\lambda'$ is the convolution inverse of $\lambda$. □

## 4. Two Equivalent Characterizations of Cleft Coextensions

In this section, we introduce the notion of the cleft coextension and then discuss two equivalent characterizations of cleft coextensions.

Let $(H, \alpha)$ be a Hom-Hopf algebra and $(B, \beta)$ a right $H$-Hom-module coalgebra. Let $H^+$ denote the augmentation Hom-ideal $\ker\varepsilon$ which is a Hom-Hopf ideal, then $BH^+$ is a Hom-coideal of $B$, and $B/BH^+$ is a Hom-coalgebra with a trivial right $H$-Hom-module structure.

*Definition 9.* Let $(H, \alpha)$ be a Hom-Hopf algebra and $(B, \beta)$ a right $H$-Hom-module coalgebra. Denote by $C = B/BH^+$; if there exists a right $H$-Hom-module map $\phi : B \to H$ which is convolution invertible, then $B/C$ is said to be a cleft coextension and write $(B, \cdot, \phi, \beta)$.

By Proposition 8, we get the main result of this section, which is the generalization of Theorem 2.10 in [8].

**Theorem 10.** *The coextension $B/C$ is cleft if and only if $B$ is isomorphic to a Hom-crossed coproduct $C \rtimes_\lambda H$ with invertible cocycle $\lambda$.*

*Proof.* If $B$ is isomorphic to the Hom-crossed coproduct $C \rtimes_\lambda H$, we prove that the coextension $B/C$ is cleft. First, we show that $\phi : C \rtimes_\lambda H \to H$ is a right $H$-Hom-module map. For any $h, g \in H$, we get

$$\phi\left(c\otimes h\right)\cdot g=\phi\left(c\otimes h\right)g=\varepsilon\left(c\right)\alpha^{-1}\left(h\right)g.\tag{53}$$

On the other hand,

$$\begin{aligned}
\phi\left(\left(c\otimes h\right)\cdot g\right)&=\phi\left(c\cdot g_1\otimes hg_2\right)=\varepsilon\left(c\cdot g_1\right)\alpha^{-1}\left(hg_2\right)\\
&=\varepsilon\left(c\right)\alpha^{-1}\left(h\right)g,
\end{aligned}\tag{54}$$

and by Proposition 8, $\lambda$ is an invertible cocycle, so $\phi$ is convolution invertible. Hence $C \rtimes_\lambda H/C$ is cleft, so $B/C$ is cleft too.

Conversely, if the coextension $B/C$ is cleft, let us denote by $\bar{c}$ the image in $C$ of the element $c \in B$. Define a weak coaction of $H$ on $C$ by

$$\begin{aligned}
&\rho : C\longrightarrow H\otimes C,\\
&\rho\left(\bar{c}\right)=\alpha^{-2}\left(\phi\left(c_{11}\right)\right)\alpha^{-1}\left(\phi^{-1}\left(c_2\right)\right)\otimes\beta^{-1}\left(\bar{c}_{12}\right).
\end{aligned}\tag{55}$$

The definition is well defined. The conditions (W1), (W2), and (W3) hold which can be directly verified.

Define $\lambda : C \to H \otimes H$, which is given by

$$\begin{aligned}
\lambda\left(\bar{c}\right)&=\alpha^{-3}\left(\phi\left(c_{11}\right)\right)\alpha^{-3}\left(\phi^{-1}\left(c_2\right)_1\right)\\
&\quad\otimes\alpha^{-3}\left(\phi\left(c_{12}\right)\right)\alpha^{-3}\left(\phi^{-1}\left(c_2\right)_2\right),
\end{aligned}\tag{56}$$

and $\omega : C \to H \otimes H$ is given by

$$\begin{aligned}
\omega\left(\bar{c}\right)&=\alpha^{-4}\left(\phi\left(c_{11}\right)_1\right)\alpha^{-2}\left(\phi^{-1}\left(c_2\right)\right)\\
&\quad\otimes\alpha^{-4}\left(\phi\left(c_{11}\right)_2\right)\alpha^{-3}\left(\phi^{-1}\left(c_{12}\right)\right).
\end{aligned}\tag{57}$$

Similar to the proof of Proposition 8, we can obtain that $\lambda * \omega = \omega * \lambda = 1$, that is, $\omega$ is the convolution inverse of $\lambda$. With convolution inverse $\lambda$ and weak coaction of $H$ on $C$ as the above, we can define a comultiplication $\Delta$ and a counit $\varepsilon$ as the Definition 2. The comultiplication $\Delta$ is given by

$$\begin{aligned}
\Delta\left(\bar{c}\otimes h\right)&=\bar{c}_1\otimes\left(\alpha^{-4}\left(\bar{c}_{21-1}\right)\alpha^{-3}\left(\lambda_1\left(\bar{c}_{22}\right)\right)\right)\alpha^{-1}\left(h_1\right)\\
&\quad\otimes\beta^{-2}\left(\bar{c}_{210}\right)\otimes\alpha^{-2}\left(\lambda_2\left(\bar{c}_{22}\right)\right)\alpha^{-1}\left(h_2\right)=\bar{c}_1\\
&\quad\otimes\left(\left(\alpha^{-6}\left(\phi\left(c_{2111}\right)\right)\alpha^{-5}\left(\phi^{-1}\left(c_{212}\right)\right)\right)\right.
\end{aligned}$$

$$\cdot\left(\alpha^{-6}\left(\phi\left(c_{2211}\right)\right)\alpha^{-6}\left(\phi^{-1}\left(c_{222}\right)_1\right)\right)\alpha^{-1}\left(h_1\right)$$
$$\otimes\beta^{-3}\left(\bar{c}_{2112}\right)\otimes\left(\alpha^{-5}\left(\phi\left(c_{2212}\right)\right)\alpha^{-5}\left(\phi^{-1}\left(c_{222}\right)_2\right)\right)$$
$$\cdot\alpha^{-1}\left(h_2\right), \tag{58}$$

and the counit $\varepsilon$ is given by

$$\varepsilon\left(\bar{c}\otimes h\right)=\varepsilon\left(c\right)\varepsilon\left(h\right), \tag{59}$$

for any $c\in B, h\in H$.

Recalling from [16], the isomorphism is given by

$$F: B\longrightarrow C\otimes H,$$
$$F\left(c\right)=\bar{c}_1\otimes\phi\left(c_2\right), \tag{60}$$

with inverse

$$F^{-1}\left(\bar{c}\otimes h\right)=c_{(1)}\cdot\phi^{-1}\left(c_{(2)}\right)h. \tag{61}$$

Now, for any $c\in B, h\in H$, we have

$$\begin{aligned}
\Delta F\left(c\right)&=\Delta\left(\bar{c}_1\otimes\phi\left(c_2\right)\right)=\bar{c}_{11}\\
&\otimes\left(\left(\alpha^{-6}\left(\phi\left(c_{12111}\right)\right)\alpha^{-5}\left(\phi^{-1}\left(c_{1212}\right)\right)\right)\right.\\
&\left.\cdot\left(\alpha^{-6}\left(\phi\left(c_{12211}\right)\right)\alpha^{-6}\left(\phi^{-1}\left(c_{1222}\right)_1\right)\right)\right)\\
&\cdot\alpha^{-1}\left(\phi\left(c_2\right)_1\right)\otimes\beta^{-3}\left(\bar{c}_{12112}\right)\otimes\left(\alpha^{-5}\left(\phi\left(c_{12212}\right)\right)\right.\\
&\left.\cdot\alpha^{-5}\left(\phi^{-1}\left(c_{1222}\right)_2\right)\right)\alpha^{-1}\left(\phi\left(c_2\right)_2\right)=\bar{c}_{11}\\
&\otimes\left(\alpha^{-4}\left(\phi\left(c_{1211}\right)\right)\varepsilon\left(c_{1221}\right)\alpha^{-6}\left(\phi^{-1}\left(c_{12222}\right)_1\right)\right)\\
&\cdot\alpha^{-1}\left(\phi\left(c_2\right)_1\right)\otimes\beta^{-2}\left(\bar{c}_{1212}\right)\otimes\left(\alpha^{-5}\left(\phi\left(c_{12221}\right)\right)\right.\\
&\left.\cdot\alpha^{-6}\left(\phi^{-1}\left(c_{12222}\right)_2\right)\right)\alpha^{-1}\left(\phi\left(c_2\right)_2\right)=\bar{c}_{11}\\
&\otimes\left(\alpha^{-4}\left(\phi\left(c_{1211}\right)\right)\alpha^{-5}\left(\phi^{-1}\left(c_{1222}\right)_1\right)\right)\alpha^{-1}\left(\phi\left(c_2\right)_1\right)\\
&\otimes\beta^{-2}\left(\bar{c}_{1212}\right)\otimes\left(\alpha^{-4}\left(\phi\left(c_{1221}\right)\right)\alpha^{-5}\left(\phi^{-1}\left(c_{1222}\right)_2\right)\right)\\
&\cdot\alpha^{-1}\left(\phi\left(c_2\right)_2\right)=\alpha^{-1}\left(\bar{c}_{111}\right)\otimes\left(\alpha^{-4}\left(\phi\left(c_{1121}\right)\right)\right.\\
&\left.\cdot\alpha^{-3}\left(\phi^{-1}\left(c_{21}\right)_1\right)\right)\alpha^{-2}\left(\phi\left(c_{22}\right)_1\right)\otimes\beta^{-2}\left(\bar{c}_{1122}\right)\\
&\otimes\left(\alpha^{-2}\left(\phi\left(c_{12}\right)\right)\alpha^{-3}\left(\phi^{-1}\left(\bar{c}_{21}\right)_2\right)\right)\alpha^{-2}\left(\phi\left(c_{22}\right)_2\right)\\
&=\alpha^{-1}\left(\bar{c}_{111}\right)\otimes\alpha^{-2}\left(\phi\left(c_{1121}\right)\right)\varepsilon\left(c_2\right)\otimes\beta^{-2}\left(\bar{c}_{1122}\right)\\
&\otimes\phi\left(c_{12}\right)=\bar{c}_{11}\otimes\alpha^{-1}\left(\phi\left(c_{121}\right)\right)\otimes\beta^{-1}\left(\bar{c}_{122}\right)\\
&\otimes\alpha\left(\phi\left(c_2\right)\right)=\bar{c}_{11}\otimes\phi\left(c_{12}\right)\otimes\bar{c}_{21}\otimes\phi\left(c_{22}\right)=(F\\
&\otimes F)\,\Delta\left(c\right).
\end{aligned} \tag{62}$$

So we can obtain that $(F\otimes F)\Delta=\Delta F$ and $\varepsilon=\varepsilon F$. Since $B$ is a Hom-coalgebra, $C\otimes H$ is a Hom-crossed coproduct and $F$ is the required Hom-coalgebra isomorphism. Thus the proof is complete. □

Now we give a lemma, which will be used in the sequel.

**Lemma 11.** *Let $(H,\alpha)$ be a Hom-Hopf algebra and $(B,\cdot,\rho,\beta)$ a right $H$-Hom-Hopf module; set $A=B^{coH}=\{b\in B\mid\rho(b)=\beta(b)\otimes 1\}$ and $C=B/BH^+$. Then*

*(1) $C\simeq A$ as vector spaces, and the isomorphism is $\psi: C\to A, \psi(\bar{b})=b_{(0)}\cdot S(b_{(1)})$.*

*(2) If $(B,\cdot)$ is a right $H$-Hom-module coalgebra, then there exists a Hom-coalgebra structure on $A$ such that $C\simeq A$ as Hom-coalgebras.*

*Proof.* (1) It is easy to see that $\psi$ is an isomorphism of vector spaces.

(2) In order to prove that $C\simeq A$ as Hom-coalgebras, we first give the Hom-coalgebra structure on $A$. The comultiplication on $A$ is given by

$$\begin{aligned}\Delta\left(a\right)&=\beta^{-3}\left(a_{1(0)(0)}\right)\cdot S\left(\alpha^{-3}\left(a_{1(0)(1)}\right)\right)\\&\otimes\beta^{-3}\left(a_{2(0)(0)}\right)\cdot S\left(\alpha^{-3}\left(a_{2(0)(1)}\right)\right),\end{aligned} \tag{63}$$

for any $a\in A$. With the above defined comultiplication, we can prove that $A$ is a Hom-coalgebra. Furthermore we can get that $\psi$, which is as similar as defined in (1), is also a Hom-coalgebra morphism. So $C\simeq A$ as Hom-coalgebras. □

Then we discuss the relation between cleft coextension and Hom-module coalgebra with the Hom-Hopf module structure.

**Proposition 12.** *Let $(B,\cdot,\phi,\beta)$ be a cleft coextension. One assumes that $H$ coacts on $B$ to the right by*

$$\begin{aligned}\rho\left(b\right)&=\beta^{-2}\left(b_{11}\right)\cdot\alpha^{-3}\left(\phi^{-1}\left(b_{12}\right)\phi\left(b_2\right)_1\right)\\&\otimes\alpha^{-1}\left(\phi\left(b_2\right)_2\right),\end{aligned} \tag{64}$$

*then*

(1) *$(B,\rho,\beta)$ is a right $H$-Hom-comodule;*

(2) *$(B,\cdot,\rho,\beta)$ is a right $H$-Hom-Hopf module.*

*Proof.* Note that $\phi$ is convolution invertible right $H$-Hom-module map, where the convolution inverse of $\phi$ is denoted by $\phi^{-1}$. We can get

$$h_1\alpha^{-1}\phi^{-1}\left(b\cdot h_2\right)=\varepsilon\left(h\right)\alpha\left(\phi^{-1}\left(b\right)\right). \tag{65}$$

In fact

$$\begin{aligned}\varepsilon\left(h\right)\phi^{-1}\left(\beta^5\left(b\right)\right)&=\phi^{-1}\left(\beta^4\left(b_1\right)\right)\varepsilon\left(b_2\right)\varepsilon\left(h\right)\\
&=\phi^{-1}\left(\beta^3\left(b_1\right)\right)\left(\phi*\phi^{-1}\left(b_2\cdot h\right)\right)\\
&=\phi^{-1}\left(\beta^3\left(b_1\right)\right)\left(\phi\left(b_{21}\cdot h_1\right)\phi^{-1}\left(b_{22}\cdot h_2\right)\right)\\
&=\left(\phi^{-1}\left(\beta^2\left(b_1\right)\right)\left(\phi\left(b_{21}\right)h_1\right)\right)\alpha\left(\phi^{-1}\left(b_{22}\cdot h_2\right)\right)\\
&=\left(\left(\phi^{-1}\left(\beta\left(b_1\right)\right)\phi\left(b_{21}\right)\right)\alpha\left(h_1\right)\right)\alpha\left(\phi^{-1}\left(b_{22}\cdot h_2\right)\right)\\
&=\left(\left(\phi^{-1}\left(b_{11}\right)\right)\phi\left(b_{12}\right)\right)\alpha\left(h_1\right)\alpha\left(\phi^{-1}\left(\beta\left(b_2\right)\cdot h_2\right)\right)\end{aligned}$$

$$= \varepsilon(b_1)\alpha^2(h_1)\alpha\left(\phi^{-1}\left(\beta(b_2)\cdot h_2\right)\right)$$
$$= \alpha^2(h_1)\alpha\left(\phi^{-1}\left(\beta^2(b)\cdot h_2\right)\right). \tag{66}$$

(1) Now we prove that $(B,\rho,\beta)$ is a right $H$-Hom-comodule. For any $b\in B$,

$$\begin{aligned}
(\rho\otimes\alpha)\rho(b) &= (\rho\otimes\alpha)\left(\beta^{-2}(b_{11})\right.\\
&\quad\cdot\alpha^{-3}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)\otimes\alpha^{-1}\left(\phi(b_2)_2\right)\Big)\\
&= \left(\beta^{-4}(b_{1111})\cdot\alpha^{-5}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)_{11}\right)\\
&\quad\cdot\alpha^{-3}\phi^{-1}\left(\beta^{-2}(b_{1112})\cdot\alpha^{-3}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)_{12}\right)\\
&\quad\cdot\alpha^{-3}\phi\left(\beta^{-2}(b_{112})\right.\\
&\quad\cdot\alpha^{-3}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)_2\Big)_1\otimes\alpha^{-1}\phi\left(\beta^{-2}(b_{112})\right.\\
&\quad\cdot\alpha^{-3}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)_2\Big)_2\otimes\phi(b_2)_2 = \beta^{-3}(b_{1111})\\
&\quad\cdot\left(\alpha^{-6}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)_{11}\right.\\
&\quad\cdot\alpha^{-4}\phi^{-1}\left(\beta^{-2}(b_{1112})\cdot\alpha^{-3}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)_{12}\right)\Big)\\
&\quad\cdot\alpha^{-3}\phi\left(\beta^{-2}(b_{112})\right.\\
&\quad\cdot\alpha^{-3}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)_2\Big)_1\otimes\alpha^{-1}\phi\left(\beta^{-2}(b_{112})\right.\\
&\quad\cdot\alpha^{-3}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)_2\Big)_2\otimes\phi(b_2)_2\\
&\overset{(65)}{=} \beta^{-3}(b_{1111})\cdot\varepsilon\left(\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)_1\right)\\
&\quad\cdot\phi^{-1}\left(\beta^{-4}(b_{1112})\right)\alpha^{-3}\phi\left(\beta^{-2}(b_{112})\right.\\
&\quad\cdot\alpha^{-3}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)_2\Big)_1\otimes\alpha^{-1}\phi\left(\beta^{-2}(b_{112})\right.\\
&\quad\cdot\alpha^{-3}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)_2\Big)_2\otimes\phi(b_2)_2 = \beta^{-3}(b_{1111})\\
&\quad\cdot\phi^{-1}\left(\beta^{-4}(b_{1112})\right)\alpha^{-3}\phi\left(\beta^{-2}(b_{112})\right.\\
&\quad\cdot\alpha^{-2}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)\Big)_1\otimes\alpha^{-1}\phi\left(\beta^{-2}(b_{112})\right.\\
&\quad\cdot\alpha^{-2}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)\Big)_2\otimes\phi(b_2)_2 = \beta^{-3}(b_{1111})\\
&\quad\cdot\phi^{-1}\left(\beta^{-4}(b_{1112})\right)\left(\alpha^{-5}\phi(b_{112})\right.\\
&\quad\cdot\alpha^{-5}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)\Big)_1\otimes\left(\alpha^{-3}\phi(b_{112})\right.\\
&\quad\cdot\alpha^{-3}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)\Big)_2\otimes\phi(b_2)_2 = \beta^{-3}(b_{1111})\\
&\quad\cdot\phi^{-1}\left(\beta^{-4}(b_{1112})\right)\left(\left(\alpha^{-6}(\phi(b_{112}))\alpha^{-5}\left(\phi^{-1}(b_{12})\right)\right)\right.\\
&\quad\cdot\alpha^{-4}\phi(b_2)_1\Big)_1\otimes\left(\left(\alpha^{-4}(\phi(b_{112}))\alpha^{-3}\left(\phi^{-1}(b_{12})\right)\right)\right.\\
&\quad\cdot\alpha^{-2}\phi(b_2)_1\Big)_2\otimes\phi(b_2)_2 = \beta^{-1}(b_{11})\\
&\quad\cdot\phi^{-1}\left(\beta^{-2}(b_{12})\right)\varepsilon(b_{21})\alpha^{-4}\left(\phi(b_{22})_{11}\right)\\
&\quad\otimes\alpha^{-2}\left(\phi(b_{22})_{12}\right)\otimes\phi(b_2)_2 = \beta^{-1}(b_{11})\\
&\quad\cdot\phi^{-1}\left(\beta^{-2}(b_{12})\right)\alpha^{-3}\left(\phi(b_2)_{11}\right)\otimes\alpha^{-1}\left(\phi(b_2)_{12}\right)\\
&\quad\otimes\phi(b_2)_2 = \beta^{-1}(b_{11})\cdot\phi^{-1}\left(\beta^{-2}(b_{12})\right)\alpha^{-2}\left(\phi(b_2)_1\right)\\
&\quad\otimes\alpha^{-1}\left(\phi(b_2)_{21}\right)\otimes\alpha^{-1}\left(\phi(b_2)_{22}\right) = (\alpha\otimes\Delta)\rho(b),
\end{aligned} \tag{67}$$

and $(\beta\otimes\varepsilon)\rho(b) = b$; thus $(B,\rho,\beta)$ is a right $H$-Hom-comodule.

(2) For all $b\in B$ and $h\in H$, we can calculate

$$\begin{aligned}
\rho(b\cdot h) &= \beta^{-2}\left((b\cdot h)_{11}\right)\\
&\quad\cdot\alpha^{-3}\left(\phi^{-1}\left((b\cdot h)_{12}\right)\phi\left((b\cdot h)_2\right)_1\right)\\
&\quad\otimes\alpha^{-1}\left(\phi\left((b\cdot h)_2\right)_2\right) = \beta^{-2}(b_{11}\cdot h_{11})\\
&\quad\cdot\alpha^{-3}\left(\phi^{-1}(b_{12}\cdot h_{12})\phi(b_2\cdot h_2)_1\right)\\
&\quad\otimes\alpha^{-1}\left(\phi(b_2\cdot h_2)_2\right) = \beta^{-1}(b_{11})\\
&\quad\cdot\left(\alpha^{-3}(h_{11})\alpha^{-4}\left(\phi^{-1}(b_{12}\cdot h_{12})\right)\right)\alpha^{-3}\left(\phi(b_2\cdot h_2)_1\right)\\
&\quad\otimes\alpha^{-1}\left(\phi(b_2\cdot h_2)_2\right)\overset{(65)}{=}\beta^{-1}(b_{11})\\
&\quad\cdot\varepsilon(h_1)\alpha^{-2}\left(\phi^{-1}(b_{12})\right)\alpha^{-3}\left(\phi(b_2\cdot h_2)_1\right)\\
&\quad\otimes\alpha^{-1}\left(\phi(b_2\cdot h_2)_2\right) = \beta^{-1}(b_{11})\\
&\quad\cdot\alpha^{-2}\left(\phi^{-1}(b_{12})\left(\alpha^{-3}\left(\phi(b_2)_1\right)\right)\alpha^{-2}(h)_1\right)\\
&\quad\otimes\alpha^{-1}\left(\phi(b_2)_2\right)h_2\\
&= \left(\beta^{-2}(b_{11})\cdot\alpha^{-3}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)\right)\cdot h_1\\
&\quad\otimes\alpha^{-1}\left(\phi(b_2)_2\right)h_2 = b_{(0)}h_1\otimes b_{(1)}h_1;
\end{aligned} \tag{68}$$

thus $(B,\cdot,\rho,\beta)$ is a right $H$-Hom-Hopf module. □

**Theorem 13.** *Let $(H,\alpha)$ be a Hom-Hopf algebra and $(B,\cdot,\beta)$ a right $H$-Hom-module coalgebra, if $C = B/BH^+$, then $B/C$ is a cleft coextension if and only if $(B,\cdot,\rho,\beta)$ is a right $H$-Hom-Hopf module, and there exists a convolution invertible linear map $\phi: B\to H$ satisfying $\phi(b) = \varepsilon(b_{(0)})\beta^{-1}(b_{(1)})$.*

*Proof.* Assume that $B$ is a cleft coextension, then there exists a convolution invertible Hom-module map $\phi: B\to H$ such that $\varepsilon\phi = \varepsilon$, so we have $\varepsilon\phi^{-1} = \varepsilon$. Define a map,

$$\begin{aligned}
&\rho: B\longrightarrow B\otimes H,\\
&\rho(b) = \beta^{-2}(b_{11})\cdot\alpha^{-3}\left(\phi^{-1}(b_{12})\phi(b_2)_1\right)\\
&\qquad\otimes\alpha^{-1}\left(\phi(b_2)_2\right),\\
&\qquad\qquad\forall b\in B;
\end{aligned} \tag{69}$$

clearly, $\phi(b) = \varepsilon(b_{(0)})\beta^{-1}(b_{(1)})$. By Proposition 12, we see that $(B,\cdot,\rho,\beta)$ is a right $H$-Hom-Hopf module.

Conversely, it is sufficient to show that $\phi$ is a right $H$-Hom-module map. Using the fact that $(B,\cdot,\rho,\beta)$ is a right $H$-Hom-Hopf module and $\phi(b)=\varepsilon(b_{(0)})\beta^{-1}(b_{(1)})$, for any $b\in B$ and $h\in H$, we get

$$\begin{aligned}\phi(b\cdot h)&=\varepsilon\left((b\cdot h)_{(0)}\right)\beta^{-1}\left((b\cdot h)_{(1)}\right)\\&=\varepsilon\left(b_{(0)}\right)\varepsilon\left(h_1\right)\beta^{-1}\left(b_{(1)}\right)\alpha^{-1}\left(h_2\right)=\phi(b)h.\end{aligned}\tag{70}$$

The proof is complete. □

## 5. The Factorization of Hom-Module Coalgebra with Hopf Module Structure

In this section, we always assume that $(H,\alpha)$ is a Hom-Hopf algebra and $(B,\cdot,\phi,\beta)$ is a cleft coextension. Based on this assumption, $B$ can not only form a right $H$-Hom-comodule, but also form a right $H$-Hom-Hopf module. Furthermore, we obtain the factorization of Hom-coalgebra $B$.

Now, we can obtain a Hom-coalgebra factorization for Hom-module coalgebra with the Hom-Hopf module structure.

**Theorem 14.** *Let $(H,\alpha)$ be a Hom-Hopf algebra and $(B,\cdot,\rho,\beta)$ a right $H$-Hom-Hopf module. One assumes that $(B,\cdot,\beta)$ is a right $H$-Hom-module coalgebra and set $A=B^{coH}=\{a\in B\mid \rho(a)=\beta(a)\otimes 1\}$. The comultiplication on $A\otimes H$ can be defined*

$$\begin{aligned}\Delta(a\otimes h)&=\beta^{-3}\left(a_{1(0)(0)}\right)\cdot S\left(\alpha^{-3}\left(a_{1(0)(1)}\right)\right)\\&\quad\otimes\alpha^{-2}\left(a_{1(1)}\right)\alpha^{-1}\left(h_1\right)\otimes\beta^{-3}\left(a_{2(0)(0)}\right)\\&\quad\cdot S\left(\alpha^{-3}\left(a_{2(0)(1)}\right)\right)\otimes\alpha^{-2}\left(a_{2(1)}\right)\alpha^{-1}\left(h_2\right),\end{aligned}\tag{71}$$

*and the counit can be defined*

$$\varepsilon(a\otimes h)=\varepsilon(a)\varepsilon(h),\tag{72}$$

*for any $a\in A$, $h\in H$, then $(A\otimes H,\alpha\otimes\beta)$ is a Hom-coalgebra, and one writes it as $A\,\dot{\square}\,H$. Moreover, $B\simeq A\,\dot{\square}\,H$ as Hom-coalgebras.*

*Proof.* Since $(B,\cdot,\rho)$ is a right $H$-Hom-Hopf module, $\Delta$ is well defined and the map

$$\begin{aligned}\varphi: A\otimes H&\longrightarrow B,\\a\otimes h&\longmapsto a\cdot h\end{aligned}\tag{73}$$

is an isomorphism of right $H$-Hom-Hopf module.

We first verify that $(A\otimes H,\Delta,\varepsilon,\beta\otimes\alpha)$ is a Hom-coalgebra. It is easy to see that $(\varepsilon\otimes(\beta\otimes\alpha))\Delta=((\beta\otimes\alpha)\otimes\varepsilon)\Delta$. It is sufficient to show the coassociativity. For any $a\in A$, $h\in H$, we calculate as follows:

$$\begin{aligned}&(\Delta\otimes(\beta\otimes\alpha))\Delta(a\otimes h)\\&=\left(\beta^{-6}\left(a_{1(0)(0)1(0)(0)}\right)\cdot S\left(\alpha^{-6}\left(a_{1(0)(1)222}\right)\right)\right)\\&\cdot S\left(\alpha^{-6}\left(a_{1(0)(0)1(0)(1)}\right)S\left(\alpha^{-6}\left(a_{1(0)(1)221}\right)\right)\right)\\&\otimes\left(\alpha^{-5}\left(a_{1(0)(0)1(1)}\right)S\left(\alpha^{-5}\left(a_{1(0)(1)21}\right)\right)\right)\\&\cdot\left(\alpha^{-3}\left(a_{1(1)1}\right)\alpha^{-2}\left(h_{11}\right)\right)\\&\otimes\left(\beta^{-6}\left(a_{1(0)(0)2(0)(0)}\right)\cdot S\left(\alpha^{-6}\left(a_{1(0)(1)122}\right)\right)\right)\\&\cdot S\left(\alpha^{-6}\left(a_{1(0)(0)2(0)(1)}\right)S\left(\alpha^{-6}\left(a_{1(0)(1)121}\right)\right)\right)\\&\otimes\left(\alpha^{-5}\left(a_{1(0)(0)2(1)}\right)S\left(\alpha^{-5}\left(a_{1(0)(1)11}\right)\right)\right)\\&\cdot\left(\alpha^{-3}\left(a_{1(1)2}\right)\alpha^{-2}\left(h_{12}\right)\right)\otimes\beta^{-2}\left(a_{2(0)(0)}\right)\\&\cdot S\left(\alpha^{-2}\left(a_{2(0)(1)}\right)\right)\otimes\alpha^{-1}\left(a_{2(1)}\right)h_2\\&=\beta^{-5}\left(a_{1(0)(0)1(0)(0)}\right)\\&\cdot S\left(\alpha^{-6}\left(a_{1(0)(0)1(0)(1)}\right)\varepsilon\left(a_{1(0)(1)22}\right)1_H\right)\\&\otimes\left(\alpha^{-5}\left(a_{1(0)(0)1(1)}\right)S\left(\alpha^{-5}\left(a_{1(0)(1)21}\right)\right)\right)\\&\cdot\left(\alpha^{-3}\left(a_{1(1)1}\right)\alpha^{-2}\left(h_{11}\right)\right)\otimes\beta^{-5}\left(a_{1(0)(0)2(0)(0)}\right)\\&\cdot S\left(\alpha^{-6}\left(a_{1(0)(0)2(0)(1)}\right)\varepsilon\left(a_{1(0)(1)12}\right)1_H\right)\\&\otimes\left(\alpha^{-5}\left(a_{1(0)(0)2(1)}\right)S\left(\alpha^{-5}\left(a_{1(0)(1)11}\right)\right)\right)\\&\cdot\left(\alpha^{-3}\left(a_{1(1)2}\right)\alpha^{-2}\left(h_{12}\right)\right)\otimes\beta^{-2}\left(a_{2(0)(0)}\right)\\&\cdot S\left(\alpha^{-2}\left(a_{2(0)(1)}\right)\right)\otimes\alpha^{-1}\left(a_{2(1)}\right)h_2\\&=\beta^{-5}\left(a_{1(0)(0)1(0)(0)}\right)\cdot S\left(\alpha^{-5}\left(a_{1(0)(0)1(0)(1)}\right)\right)\\&\otimes\left(\alpha^{-5}\left(a_{1(0)(0)1(1)}\right)S\left(\alpha^{-4}\left(a_{1(0)(1)2}\right)\right)\right)\\&\cdot\left(\alpha^{-3}\left(a_{1(1)1}\right)\alpha^{-2}\left(h_{11}\right)\right)\otimes\beta^{-5}\left(a_{1(0)(0)2(0)(0)}\right)\\&\cdot S\left(\alpha^{-5}\left(a_{1(0)(0)2(0)(1)}\right)\right)\\&\otimes\left(\alpha^{-5}\left(a_{1(0)(0)2(1)}\right)S\left(\alpha^{-4}\left(a_{1(0)(1)1}\right)\right)\right)\\&\cdot\left(\alpha^{-3}\left(a_{1(1)2}\right)\alpha^{-2}\left(h_{12}\right)\right)\otimes\beta^{-2}\left(a_{2(0)(0)}\right)\\&\cdot S\left(\alpha^{-2}\left(a_{2(0)(1)}\right)\right)\otimes\alpha^{-1}\left(a_{2(1)}\right)h_2\\&=\beta^{-4}\left(a_{1(0)1(0)(0)}\right)\cdot S\left(\alpha^{-4}\left(a_{1(0)1(0)(1)}\right)\right)\\&\otimes\left(\alpha^{-4}\left(a_{1(0)1(1)}\right)S\left(\alpha^{-4}\left(a_{1(1)12}\right)\right)\right)\\&\cdot\left(\alpha^{-4}\left(a_{1(1)21}\right)\alpha^{-2}\left(h_{11}\right)\right)\otimes\beta^{-4}\left(a_{1(0)2(0)(0)}\right)\\&\cdot S\left(\alpha^{-4}\left(a_{1(0)2(0)(1)}\right)\right)\\&\otimes\left(\alpha^{-4}\left(a_{1(0)2(1)}\right)S\left(\alpha^{-4}\left(a_{1(1)11}\right)\right)\right)\\&\cdot\left(\alpha^{-4}\left(a_{1(1)22}\right)\alpha^{-2}\left(h_{12}\right)\right)\otimes\beta^{-2}\left(a_{2(0)(0)}\right)\\&\cdot S\left(\alpha^{-2}\left(a_{2(0)(1)}\right)\right)\otimes\alpha^{-1}\left(a_{2(1)}\right)h_2\end{aligned}$$

$$
\begin{aligned}
&= \beta^{-4}\left(a_{1(0)1(0)(0)}\right)\cdot S\left(\alpha^{-4}\left(a_{1(0)1(0)(1)}\right)\right)\\
&\otimes \alpha^{-3}\left(a_{1(0)1(1)}\right)\\
&\cdot\left(\left(S\left(\alpha^{-5}\left(a_{1(1)12}\right)\right)\alpha^{-5}\left(a_{1(1)21}\right)\right)\alpha^{-2}\left(h_{11}\right)\right)\\
&\otimes \beta^{-4}\left(a_{1(0)2(0)(0)}\right)\cdot S\left(\alpha^{-4}\left(a_{1(0)2(0)(1)}\right)\right)\\
&\otimes \alpha^{-3}\left(a_{1(0)2(1)}\right)\\
&\cdot\left(\left(S\left(\alpha^{-5}\left(a_{1(1)11}\right)\right)\alpha^{-5}\left(a_{1(1)22}\right)\right)\alpha^{-2}\left(h_{12}\right)\right)\\
&\otimes \beta^{-2}\left(a_{2(0)(0)}\right)\cdot S\left(\alpha^{-2}\left(a_{2(0)(1)}\right)\right)\otimes \alpha^{-1}\left(a_{2(1)}\right)h_2\\
&= \beta^{-4}\left(a_{1(0)1(0)(0)}\right)\cdot S\left(\alpha^{-4}\left(a_{1(0)1(0)(1)}\right)\right)\\
&\otimes \alpha^{-3}\left(a_{1(0)1(1)}\right)\varepsilon\left(a_{1(1)21}\right)\alpha^{-1}\left(h_{11}\right)\\
&\otimes \beta^{-4}\left(a_{1(0)2(0)(0)}\right)\cdot S\left(\alpha^{-4}\left(a_{1(0)2(0)(1)}\right)\right)\\
&\otimes \alpha^{-3}\left(a_{1(0)2(1)}\right)\\
&\cdot\left(\left(S\left(\alpha^{-4}\left(a_{1(1)1}\right)\right)\alpha^{-5}\left(a_{1(1)22}\right)\right)\alpha^{-2}\left(h_{12}\right)\right)\\
&\otimes \beta^{-2}\left(a_{2(0)(0)}\right)\cdot S\left(\alpha^{-2}\left(a_{2(0)(1)}\right)\right)\otimes \alpha^{-1}\left(a_{2(1)}\right)h_2\\
&= \beta^{-4}\left(a_{1(0)1(0)(0)}\right)\cdot S\left(\alpha^{-4}\left(a_{1(0)1(0)(1)}\right)\right)\\
&\otimes \alpha^{-3}\left(a_{1(0)1(1)}\right)\alpha^{-1}\left(h_{11}\right)\otimes \beta^{-4}\left(a_{1(0)2(0)(0)}\right)\\
&\cdot S\left(\alpha^{-4}\left(a_{1(0)2(0)(1)}\right)\right)\otimes \alpha^{-3}\left(a_{1(0)2(1)}\right)\\
&\cdot\left(\left(S\left(\alpha^{-4}\left(a_{1(1)1}\right)\right)\alpha^{-4}\left(a_{1(1)2}\right)\right)\alpha^{-2}\left(h_{12}\right)\right)\\
&\otimes \beta^{-2}\left(a_{2(0)(0)}\right)\cdot S\left(\alpha^{-2}\left(a_{2(0)(1)}\right)\right)\otimes \alpha^{-1}\left(a_{2(1)}\right)h_2\\
&= \beta^{-4}\left(a_{1(0)1(0)(0)}\right)\cdot S\left(\alpha^{-4}\left(a_{1(0)1(0)(1)}\right)\right)\\
&\otimes \alpha^{-3}\left(a_{1(0)1(1)}\right)\alpha^{-1}\left(h_{11}\right)\otimes \beta^{-4}\left(a_{1(0)2(0)(0)}\right)\\
&\cdot S\left(\alpha^{-4}\left(a_{1(0)2(0)(1)}\right)\right)\otimes \alpha^{-3}\left(a_{1(0)2(1)}\right)\varepsilon\left(a_{1(1)}\right)\\
&\cdot \alpha^{-1}\left(h_{12}\right)\otimes \beta^{-2}\left(a_{2(0)(0)}\right)\cdot S\left(\alpha^{-2}\left(a_{2(0)(1)}\right)\right)\\
&\otimes \alpha^{-1}\left(a_{2(1)}\right)h_2 = \beta^{-3}\left(a_{11(0)(0)}\right)\cdot S\left(\alpha^{-3}\left(a_{11(0)(1)}\right)\right)\\
&\otimes \alpha^{-2}\left(a_{11(1)}\right)\alpha^{-1}\left(h_{11}\right)\otimes \beta^{-3}\left(a_{12(0)(0)}\right)\\
&\cdot S\left(\alpha^{-3}\left(a_{12(0)(1)}\right)\right)\otimes \alpha^{-2}\left(a_{12(1)}\right)\alpha^{-1}\left(h_{12}\right)\\
&\otimes \beta^{-2}\left(a_{2(0)(0)}\right)\cdot S\left(\alpha^{-2}\left(a_{2(0)(1)}\right)\right)\otimes \alpha^{-1}\left(a_{2(1)}\right)h_2.
\end{aligned}
\tag{74}
$$

In a similar way

$$
\begin{aligned}
&\left(\left(\beta\otimes\alpha\right)\otimes\Delta\right)\Delta\left(a\otimes h\right)\\
&\quad= \beta^{-2}\left(a_{1(0)(0)}\right)\cdot S\left(\alpha^{-2}\left(a_{1(0)(0)}\right)\right)\otimes \alpha^{-1}\left(a_{1(1)}\right)h_1\\
&\qquad\otimes \beta^{-3}\left(a_{21(0)(0)}\right)\cdot S\left(\alpha^{-3}\left(a_{21(0)(1)}\right)\right)\\
&\qquad\otimes \alpha^{-2}\left(a_{21(1)}\right)\alpha^{-1}\left(h_{21}\right)\otimes \beta^{-3}\left(a_{22(0)(0)}\right)\\
&\qquad\cdot S\left(\alpha^{-3}\left(a_{22(0)(1)}\right)\right)\otimes \alpha^{-2}\left(a_{22(1)}\right)\alpha^{-1}\left(h_{22}\right).
\end{aligned}
\tag{75}
$$

Thus, $(A\otimes H,\Delta,\varepsilon,\beta\otimes\alpha)$ is a Hom-coalgebra.

Next we show that $\varphi$ is a Hom-coalgebra morphism.

$$
\begin{aligned}
&\varphi\left(\left(a\otimes h\right)_1\right)\otimes\varphi\left(\left(a\otimes h\right)_2\right)\\
&\quad= \left(\beta^{-3}\left(a_{1(0)(0)}\right)\cdot S\left(\alpha^{-3}\left(a_{1(0)(1)}\right)\right)\right)\\
&\qquad\cdot \alpha^{-2}\left(a_{1(1)}\right)\alpha^{-1}\left(h_1\right)\\
&\qquad\otimes\left(\beta^{-3}\left(a_{2(0)(0)}\right)\cdot S\left(\alpha^{-3}\left(a_{2(0)(1)}\right)\right)\right)\\
&\qquad\cdot \alpha^{-2}\left(a_{2(1)}\right)\alpha^{-1}\left(h_2\right)\\
&\quad= \beta^{-1}\left(a_{1(0)}\right)\\
&\qquad\cdot\left(\left(S\left(\alpha^{-4}\left(a_{1(1)1}\right)\right)\alpha^{-4}\left(a_{1(1)2}\right)\right)\alpha^{-1}\left(h_1\right)\right)\\
&\qquad\otimes \beta^{-1}\left(a_{2(0)}\right)\\
&\qquad\cdot\left(\left(S\left(\alpha^{-4}\left(a_{2(1)1}\right)\right)\alpha^{-4}\left(a_{2(1)2}\right)\right)\alpha^{-1}\left(h_2\right)\right)\\
&\quad= a_1\cdot h_1\otimes a_2\cdot h_2 = \varphi\left(a_1\otimes h_1\right)\otimes\varphi\left(a_2\otimes h_2\right).
\end{aligned}
\tag{76}
$$

Therefore, $B\simeq A\,\dot{\square}\,H$ as Hom-coalgebras. □

By the above results, we obtain the following result.

**Theorem 15.** *Let $(H,\alpha)$ be a Hom-Hopf algebra and $(B,\cdot,\phi,\beta)$ a cleft coextension, then*

$$B\simeq C\rtimes_\lambda H\simeq A\,\dot{\square}\,H \tag{77}$$

*as Hom-coalgebras, where $C=B/BH^+$, and*

$$
\begin{aligned}
A = \Big\{ b\in B \mid\ & \beta^{-2}\left(b_{11}\right)\cdot\alpha^{-3}\left(\phi^{-1}\left(b_{12}\right)\phi\left(b_2\right)_1\right)\\
&\otimes\alpha^{-1}\left(\phi\left(b_2\right)_2\right)=\beta\left(b\right)\otimes 1\Big\}.
\end{aligned}
\tag{78}
$$

*Proof.* By Theorems 10 and 14, we have $B\simeq C\rtimes_\lambda H$ and $B\simeq A\,\dot{\square}\,H$ as Hom-coalgebras, respectively. Since $(B,\cdot,\phi,\beta)$ is a cleft coextension, we get $(B,\cdot,\rho,\beta)$ is a right $H$-Hom-Hopf module by Proposition 12, where the comodule structure is defined by

$$
\begin{aligned}
\rho\left(b\right) = {}& \beta^{-2}\left(b_{11}\right)\cdot\alpha^{-3}\left(\phi^{-1}\left(b_{12}\right)\phi\left(b_2\right)_1\right)\\
&\otimes\alpha^{-1}\left(\phi\left(b_2\right)_2\right).
\end{aligned}
\tag{79}
$$

We know that

$$
\begin{aligned}
\psi : C &\longrightarrow B^{coH},\\
\overline{b} &\longmapsto b_{(0)}\cdot S\left(b_{(1)}\right)
\end{aligned}
\tag{80}
$$

is an isomorphism of Hom-coalgebras by Lemma 11. Set

$$
\begin{aligned}
\overline{\psi} &= \psi\otimes id,\\
\overline{\psi} : C\rtimes_\lambda H &\longrightarrow A\,\dot{\square}\,H.
\end{aligned}
\tag{81}
$$

We only need to verify that $\overline{\psi}$ is a Hom-coalgebra morphism. First we prove that

$$\psi\left(\overline{b}\right) = b_1 \cdot \phi^{-1}\left(b_2\right). \tag{82}$$

In fact

$$\begin{aligned}\psi\left(\overline{b}\right) &= b_{(0)} \cdot S\left(b_{(1)}\right)\\ &= \beta^{-2}\left(b_{11}\right) \cdot \alpha^{-3}\left(\phi^{-1}\left(b_{12}\right)\phi\left(b_2\right)_1\right)\\ &\quad \cdot S\left(\alpha^{-1}\left(\phi\left(b_2\right)_2\right)\right)\\ &= \beta^{-1}\left(b_{11}\right) \cdot \alpha^{-1}\left(\phi^{-1}\left(b_{12}\right)\right)\varepsilon\left(\phi\left(b_2\right)\right)\\ &= b_1 \cdot \phi^{-1}\left(b_2\right).\end{aligned} \tag{83}$$

On one hand, we can calculate

$$\begin{aligned}\Delta\overline{\psi}\left(\overline{b}\otimes h\right) &\overset{(83)}{=} \Delta\left(b_1 \cdot \phi^{-1}\left(b_2\right)\otimes h\right)\\ &= \beta^{-3}\left(\left(b_1 \cdot \phi^{-1}\left(b_2\right)\right)_{1(0)(0)}\right)\\ &\quad \cdot S\left(\alpha^{-3}\left(\left(b_1 \cdot \phi^{-1}\left(b_2\right)\right)_{1(0)(1)}\right)\right)\\ &\quad \otimes \alpha^{-2}\left(\left(b_1 \cdot \phi^{-1}\left(b_2\right)\right)_{1(1)}\right)\alpha^{-1}\left(h_1\right)\\ &\quad \otimes \beta^{-3}\left(\left(b_1 \cdot \phi^{-1}\left(b_2\right)\right)_{2(0)(0)}\right)\\ &\quad \cdot S\left(\alpha^{-3}\left(\left(b_1 \cdot \phi^{-1}\left(b_2\right)\right)_{2(0)(1)}\right)\right)\\ &\quad \otimes \alpha^{-2}\left(\left(b_1 \cdot \phi^{-1}\left(b_2\right)\right)_{2(1)}\right)\alpha^{-1}\left(h_2\right)\\ &= \beta^{-3}\left(b_{11(0)(0)} \cdot \phi^{-1}\left(b_2\right)_{111}\right)\\ &\quad \cdot S\left(\alpha^{-3}\left(b_{11(0)(1)}\phi^{-1}\left(b_2\right)_{112}\right)\right)\\ &\quad \otimes \alpha^{-2}\left(b_{11(1)}\phi^{-1}\left(b_2\right)_{12}\right)\alpha^{-1}\left(h_1\right)\\ &\quad \otimes \beta^{-3}\left(b_{12(0)(0)} \cdot \phi^{-1}\left(b_2\right)_{211}\right)\\ &\quad \cdot S\left(\alpha^{-3}\left(b_{12(0)(1)}\phi^{-1}\left(b_2\right)_{212}\right)\right)\\ &\quad \otimes \alpha^{-2}\left(b_{12(1)}\phi^{-1}\left(b_2\right)_{22}\right)\alpha^{-1}\left(h_2\right) = \beta^{-2}\left(b_{11(0)(0)}\right)\\ &\quad \cdot \left(\alpha^{-4}\left(\phi^{-1}\left(b_2\right)_{111}\right)S\left(\alpha^{-4}\left(\phi^{-1}\left(b_2\right)_{112}\right)\right)\right)\\ &\quad \cdot S\left(\alpha^{-3}\left(b_{11(0)(1)}\right)\right)\otimes \alpha^{-2}\left(b_{11(1)}\phi^{-1}\left(b_2\right)_{12}\right)\alpha^{-1}\left(h_1\right)\\ &\quad \otimes \beta^{-2}\left(b_{12(0)(0)}\right)\\ &\quad \cdot \left(\alpha^{-4}\left(\phi^{-1}\left(b_2\right)_{211}\right)S\left(\alpha^{-4}\left(\phi^{-1}\left(b_2\right)_{212}\right)\right)\right)\\ &\quad \cdot S\left(\alpha^{-3}\left(b_{12(0)(1)}\right)\right)\otimes \alpha^{-2}\left(b_{12(1)}\phi^{-1}\left(b_2\right)_{22}\right)\alpha^{-1}\left(h_2\right)\\ &= \beta^{-2}\left(b_{11(0)(0)}\right) \cdot S\left(\alpha^{-2}\left(b_{11(0)(1)}\right)\right)\\ &\quad \otimes \left(\alpha^{-2}\left(b_{11(1)}\right)\alpha^{-1}\left(\phi^{-1}\left(b_2\right)_1\right)\right)\alpha^{-1}\left(h_1\right)\\ &\quad \otimes \beta^{-2}\left(b_{12(0)(0)}\right) \cdot S\left(\alpha^{-2}\left(b_{12(0)(1)}\right)\right)\\ &\quad \otimes \left(\alpha^{-2}\left(b_{12(1)}\right)\alpha^{-1}\left(\phi^{-1}\left(b_2\right)_2\right)\right)\alpha^{-1}\left(h_2\right)\\ &= \beta^{-1}\left(b_{11(0)}\right) \cdot S\left(\alpha^{-2}\left(b_{11(1)1}\right)\right)\\ &\quad \otimes \left(\alpha^{-3}\left(b_{11(1)2}\right)\alpha^{-1}\left(\phi^{-1}\left(b_2\right)_1\right)\right)\alpha^{-1}\left(h_1\right)\\ &\quad \otimes \beta^{-1}\left(b_{12(0)}\right) \cdot S\left(\alpha^{-2}\left(b_{12(1)1}\right)\right)\\ &\quad \otimes \left(\alpha^{-3}\left(b_{12(1)2}\right)\alpha^{-1}\left(\phi^{-1}\left(b_2\right)_2\right)\right)\alpha^{-1}\left(h_2\right)\\ &= \left(\beta^{-3}\left(b_{1111}\right) \cdot \alpha^{-4}\left(\phi^{-1}\left(b_{1112}\right)\phi\left(b_{112}\right)_1\right)\right)\\ &\quad \cdot S\left(\alpha^{-3}\left(\phi\left(b_{112}\right)_{21}\right)\right)\\ &\quad \otimes \left(\alpha^{-4}\left(\phi\left(b_{112}\right)_{22}\right)\alpha^{-1}\left(\phi^{-1}\left(b_2\right)_1\right)\right)\alpha^{-1}\left(h_1\right)\\ &\quad \otimes \left(\beta^{-3}\left(b_{1211}\right) \cdot \alpha^{-4}\left(\phi^{-1}\left(b_{1212}\right)\phi\left(b_{122}\right)_1\right)\right)\\ &\quad \cdot S\left(\alpha^{-3}\left(\phi\left(b_{122}\right)_{21}\right)\right)\\ &\quad \otimes \left(\alpha^{-4}\left(\phi\left(b_{122}\right)_{22}\right)\alpha^{-1}\left(\phi^{-1}\left(b_2\right)_2\right)\right)\alpha^{-1}\left(h_2\right)\\ &= \beta^{-2}\left(b_{1111}\right) \cdot \alpha^{-3}\left(\phi^{-1}\left(b_{1112}\right)\right)\\ &\quad \cdot \left(\alpha^{-4}\left(\phi\left(b_{112}\right)_1\right)S\left(\alpha^{-5}\left(\phi\left(b_{112}\right)_{21}\right)\right)\right)\\ &\quad \otimes \left(\alpha^{-4}\left(\phi\left(b_{112}\right)_{22}\right)\alpha^{-1}\left(\phi^{-1}\left(b_2\right)_1\right)\right)\alpha^{-1}\left(h_1\right)\\ &\quad \otimes \beta^{-2}\left(b_{1211}\right) \cdot \alpha^{-3}\left(\phi^{-1}\left(b_{1212}\right)\right)\\ &\quad \cdot \left(\alpha^{-4}\left(\phi\left(b_{122}\right)_1\right)S\left(\alpha^{-5}\left(\phi\left(b_{122}\right)_{21}\right)\right)\right)\\ &\quad \otimes \left(\alpha^{-4}\left(\phi\left(b_{122}\right)_{22}\right)\alpha^{-1}\left(\phi^{-1}\left(b_2\right)_2\right)\right)\alpha^{-1}\left(h_2\right)\\ &= \beta^{-2}\left(b_{1111}\right) \cdot \alpha^{-2}\left(\phi^{-1}\left(b_{1112}\right)\right)\\ &\quad \otimes \left(\alpha^{-2}\left(\phi\left(b_{112}\right)\right)\alpha^{-1}\left(\phi^{-1}\left(b_2\right)_1\right)\right)\alpha^{-1}\left(h_1\right)\\ &\quad \otimes \beta^{-2}\left(b_{1211}\right) \cdot \alpha^{-2}\left(\phi^{-1}\left(b_{1212}\right)\right)\\ &\quad \otimes \left(\alpha^{-2}\left(\phi\left(b_{122}\right)\right)\alpha^{-1}\left(\phi^{-1}\left(b_2\right)_2\right)\right)\alpha^{-1}\left(h_2\right).\end{aligned} \tag{84}$$

For all $b \in B$, $h \in H$, we define $\rho' : C \to H \otimes C$ by

$$\rho'\left(\overline{b}\right) = \overline{b}_{-1} \otimes \overline{b}_0 = \phi\left(b_{11}\right)\alpha\left(\phi^{-1}\left(b_2\right)\right) \otimes \beta^{-1}\left(\overline{b}_{12}\right), \tag{85}$$

and $\lambda : C \to H \otimes H$ is defined by

$$\begin{aligned}\lambda\left(\overline{b}\right) &= \lambda_1\left(\overline{b}\right) \otimes \lambda_2\left(\overline{b}\right)\\ &= \phi\left(b_1\right)\alpha^{-2}\left(\phi^{-1}\left(b_{22}\right)_1\right)\\ &\quad \otimes \alpha^{-1}\left(\phi\left(b_{21}\right)\right)\alpha^{-2}\left(\phi^{-1}\left(b_{22}\right)_2\right).\end{aligned} \tag{86}$$

On the other hand, we have

$$\begin{aligned}\left(\overline{\psi}\otimes\overline{\psi}\right)\Delta_\lambda\left(\overline{b}\otimes h\right) &= \left(\overline{\psi}\otimes\overline{\psi}\right)\left(\overline{b}_1 \otimes \left(\alpha^{-4}\left(\overline{b}_{21-1}\right)\right.\right.\\ &\quad \left.\cdot \alpha^{-3}\left(\lambda_1\left(\overline{b}_{22}\right)\right)\right)\alpha^{-1}\left(h_1\right) \otimes \beta^{-2}\left(\overline{b}_{210}\right)\\ &\quad \otimes \alpha^{-2}\left(\lambda_2\left(\overline{b}_{22}\right)\right)\alpha^{-1}\left(h_2\right)\Big) = \left(\overline{\psi}\otimes\overline{\psi}\right)\left(\overline{b}_1\right.\end{aligned}$$

$$
\begin{aligned}
&\otimes\left(\alpha^{-4}\left(\phi\left(b_{2111}\right)\alpha\left(\phi^{-1}\left(b_{212}\right)\right)\right)\alpha^{-3}\left(\phi\left(b_{221}\right)\right.\right.\\
&\left.\left.\cdot\alpha^{-2}\left(\phi^{-1}\left(b_{2222}\right)_1\right)\right)\right)\alpha^{-1}\left(h_1\right)\otimes\beta^{-3}\left(\overline{b}_{2112}\right)\\
&\otimes\alpha^{-2}\left(\alpha^{-1}\left(\phi\left(b_{2221}\right)\right)\alpha^{-2}\left(\phi^{-1}\left(b_{2222}\right)_2\right)\right)\alpha^{-1}\left(h_2\right)\Big)\\
&=\left(\overline{\psi}\otimes\overline{\psi}\right)\left(\overline{b}_1\otimes\left(\alpha^{-3}\left(\phi\left(b_{2111}\right)\right)\right.\right.\\
&\cdot\left(\left(\alpha^{-4}\left(\phi^{-1}\left(b_{212}\right)\right)\alpha^{-4}\left(\phi\left(b_{221}\right)\right)\right)\right.\\
&\left.\left.\cdot\alpha^{-5}\left(\phi^{-1}\left(b_{2222}\right)_1\right)\right)\right)\alpha^{-1}\left(h_1\right)\otimes\beta^{-3}\left(\overline{b}_{2112}\right)\\
&\otimes\alpha^{-2}\left(\alpha^{-1}\left(\phi\left(b_{2221}\right)\right)\alpha^{-2}\left(\phi^{-1}\left(b_{2222}\right)_2\right)\right)\alpha^{-1}\left(h_2\right)\\
&=\left(\overline{\psi}\otimes\overline{\psi}\right)\left(\overline{b}_1\otimes\left(\alpha^{-2}\left(\phi\left(b_{211}\right)\right)\varepsilon\left(b_{221}\right)\right.\right.\\
&\left.\cdot\alpha^{-4}\left(\phi^{-1}\left(b_{2222}\right)_1\right)\right)\alpha^{-1}\left(h_1\right)\otimes\beta^{-2}\left(\overline{b}_{212}\right)\\
&\otimes\alpha^{-2}\left(\alpha^{-1}\left(\phi\left(b_{2221}\right)\right)\alpha^{-2}\left(\phi^{-1}\left(b_{2222}\right)_2\right)\right)\alpha^{-1}\left(h_2\right)\\
&=\left(\overline{\psi}\otimes\overline{\psi}\right)\left(\overline{b}_1\otimes\left(\alpha^{-2}\left(\phi\left(b_{211}\right)\right)\alpha^{-3}\left(\phi^{-1}\left(b_{222}\right)_1\right)\right)\right)\\
&\cdot\alpha^{-1}\left(h_1\right)\otimes\beta^{-2}\left(\overline{b}_{212}\right)\otimes\alpha^{-2}\left(\phi\left(b_{221}\right)\right.\\
&\left.\cdot\alpha^{-1}\left(\phi^{-1}\left(b_{222}\right)_2\right)\right)\alpha^{-1}\left(h_2\right)\overset{(83)}{=}b_{11}\cdot\phi^{-1}\left(b_{12}\right)\\
&\otimes\left(\alpha^{-2}\left(\phi\left(b_{211}\right)\right)\alpha^{-3}\left(\phi^{-1}\left(b_{222}\right)_1\right)\right)\alpha^{-1}\left(h_1\right)\\
&\otimes\beta^{-2}\left(b_{2121}\cdot\phi^{-1}\left(b_{2122}\right)\right)\otimes\alpha^{-2}\left(\phi\left(b_{221}\right)\right.\\
&\left.\cdot\alpha^{-1}\left(\phi^{-1}\left(b_{222}\right)_2\right)\right)\alpha^{-1}\left(h_2\right)=\beta^{-2}\left(b_{1111}\right)\\
&\cdot\alpha^{-2}\left(\phi^{-1}\left(b_{1112}\right)\right)\otimes\left(\alpha^{-2}\left(\phi\left(b_{112}\right)\right)\right.\\
&\left.\cdot\alpha^{-1}\left(\phi^{-1}\left(b_2\right)_1\right)\right)\alpha^{-1}\left(h_1\right)\otimes\beta^{-2}\left(b_{1211}\right)\\
&\cdot\alpha^{-2}\left(\phi^{-1}\left(b_{1212}\right)\right)\otimes\left(\alpha^{-2}\left(\phi\left(b_{122}\right)\right)\right.\\
&\left.\cdot\alpha^{-1}\left(\phi^{-1}\left(b_2\right)_2\right)\right)\alpha^{-1}\left(h_2\right).
\end{aligned}
\tag{87}
$$

Therefore, $B\simeq C\rtimes_\lambda H\simeq A\,\square^{\cdot}\,H$ as Hom-coalgebras. □

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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