# Quasi-Exact Coulomb Dynamics of $n$ Charges $n-1$ of Which Are Equal 

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#### Abstract

For $n \geq 3$ point charges $n-1$ of which are negative and equal quasi-exact periodic solutions of their Coulomb equation of motion are found. These solutions describe a motion of the negative charges around a coordinate axis in such a way that their coordinates coincide with vertices of a regular polygon in planes perpendicular to the axis along which the positive charge moves. The Weinstein and center Lyapunov theorems are utilized.


## 1. Introduction

In this paper we consider the Coulomb nonplanar dynamics of $n-1 \geq 2$ negative identical charges $-e_{0}$ and a positive charge $e_{n}$. In our paper [1] we found planar exact solutions of the Coulomb equation of motion for these charges such that the immobile positive charge and the equal negative charges occupy the origin and all vertices of a regular polygon centered at the origin, respectively. The charges coordinates in the complex representation were written as

$$
\begin{align*}
x_{j}=\left(x_{j}^{1}+i x_{j}^{2} ; 0\right) & =\left(z_{j} q(t) ; 0\right) \in \mathbb{R}^{3}, \\
& q(t)=e^{i u(t)} r(t), j=1, \ldots, n-1, x_{n}=0, \tag{1}
\end{align*}
$$

where $z_{j}$ is one of the $n-1$ vertices of a regular polygon in the complex plane. As a result the $2 n$-dimensional Coulomb equation of motion is reduced to equations for the radial and angular variables which determine a Keplerian orbit.

In our paper [2] we generalized this result for the system with two additional nonplanar equal charges located at the same distance from the origin where the positive charge $e_{n}$ is immobile. The charges coordinates were written as

$$
\begin{align*}
& x_{j}=\left(x_{j}^{1}+i x_{j}^{2} ; 0\right)=\left(z_{j} q(t) ; 0\right) \in \mathbb{R}^{3}, \\
& q(t)=e^{i u(t)} r_{1}(t), j=1, \ldots, n-1, x_{n}=0,  \tag{2}\\
& -x_{n+2}=x_{n+1}=\left(0,0, r_{2}(t)\right) \in \mathbb{R}^{3}, \\
& \quad\left|z_{j}\right|=a,
\end{align*}
$$

where $x_{n}, x_{n+1}, x_{n+2}$ are the coordinates of the positive charge $e_{n}$ and the two additional nonplanar charges, the coordinates $x_{j}, j<n$ of the negative planar charges have the complex representation, and $z_{j}$ is one of the $n-1$ vertices of a regular polygon in the complex plane. As a result the Coulomb equation of motion for $n+2$ charges is reduced to equations for two radial $r_{1}$ and $r_{2}$ and angular variables. The equations for $r_{1}, r_{2}$ determine a two-dimensional mechanical system, that is, a four-dimensional Hamiltonian one, and have an equilibrium which generates the time-dependent quasi-exact solutions of the Coulomb equation of motion for $n+2$ charges.

In this paper we find quasi-exact nonplanar solutions of the Coulomb equation of motion for the $n$ mentioned charges such that the positive charge moves along the vertical coordinate axis and the equal $n-1$ negative charges occupy all the vertices of regular polygons centered at the origin located
at different horizontal planes. The charges coordinates in the complex representation are written in this case as

$$
\begin{align*}
& x_{j}=\left(x_{j}^{1}+i x_{j}^{2} ; q_{2}(t)\right)=\left(z_{j} q(t) ; q_{2}(t)\right) \in \mathbb{R}^{3},  \tag{3}\\
& q(t)=e^{i u(t)} q_{1}(t), j=1, \ldots, n-1, x_{n}=\left(0,0, q_{3}(t)\right),
\end{align*}
$$

where $z_{j}$ is one of the $n-1$ vertices of a regular polygon in the complex plane. We derive the equation of motion for the vector $q_{j}, j=1,2,3$, which determines a three-dimensional mechanical system with an equilibrium. This equation is translation invariant in $q_{2}, q_{3}$ and this allows us to derive an equation of motion for $q_{1}$ and $q_{2}-q_{3}$, which determines a two-dimensional mechanical system with an equilibrium, and establish the existence of its periodic solutions with the help of the Lyapunov center theorem [3-7] and Weinstein theorem [8, 9]. The existence of periodic orbits for the Coulomb equation of motion in systems of charges other than considered by us was announced in [10].

Our main result relies on the following theorem and proposition proved in [1,2].

Theorem 1. Let the positive charge $e_{n}=e_{n}^{0}>0$ have the coordinate $z_{n}=0$ and the negative planar charges $e_{j}=e_{j}^{0}=-e_{0}<0$, $j=1, \ldots, n-1 \geq 2$, have the coordinates $z_{k}$ which determine all vertices of a regular polygon centered at the origin for $n>3$ and $z_{1}+z_{2}=0$ for $n=3$. Let also $\left|z_{n}-z_{j}\right|=a$. Then such the configuration is an equilibrium of the Coulomb system of these charges if

$$
\begin{align*}
e_{n}^{0} & =2^{-3 / 2} e_{0} \sum_{k=1}^{n-2}\left(1-\cos \frac{2 \pi k}{n-1}\right)^{-1 / 2} \\
& =2^{-2} e_{0} \sum_{k=1}^{n-2} \sin ^{-1} \frac{\pi k}{n-1} \tag{4}
\end{align*}
$$

Proposition 2. Let $z_{j}$ and charges $e_{j}, j=1, \ldots, n-1$, with the equal mass $m$ be the same as in Theorem 1. Then

$$
\begin{equation*}
-w^{2} m_{j} z_{j}=\sum_{k=1, k \neq j}^{n} e_{j} e_{k} \frac{z_{j}-z_{k}}{\left|z_{j}-z_{k}\right|^{3}}, \tag{5}
\end{equation*}
$$

where

$$
\begin{equation*}
w^{2} a^{3} \frac{m}{e_{0}}=e_{n}-e_{n}^{0} \tag{6}
\end{equation*}
$$

A crucial role in the proofs is played by the equality for the coordinates of the regular polygon vertices

$$
\begin{equation*}
\sum_{k=1}^{n-1} z_{k}=0, \quad z_{k}=z^{k}, z=e^{i(2 \pi /(n-1))} \tag{7}
\end{equation*}
$$

which in its turn is a consequence of the equality

$$
\begin{equation*}
z \sum_{k=1}^{n-1} z_{k}=\sum_{k=1}^{n-1} z_{k} \tag{8}
\end{equation*}
$$

In the formula for $e_{n}^{0}$ we used the equality

$$
\begin{equation*}
\cos ^{2} x=\frac{1}{2}(1+\cos 2 x) . \tag{9}
\end{equation*}
$$

We believe that classical Coulomb dynamics and our results may help to understand better quantum systems [11]. The results of this paper and [1,2] show the existence of the Coulomb classical models of atoms and molecules with three nuclei if a number of equal negative charges (electrons) are not large. A comment concerning classical models of molecules can be found in [2].

Our paper is organized as follows. In the second section the equation of motion is derived for $q_{j}, j=1,2,3$, and eigenvalues of its linear vector field at an equilibrium are found. In the third section the equation of motion is derived for $q_{1}, q_{2}-q_{3}$ and our main result is formulated in two theorems which follows from the Weinstein and center Lyapunov theorems.

## 2. Reduced Equation of Motion

The Coulomb potential energy of our system is given by

$$
\begin{equation*}
U\left(x_{(n)}\right)=\sum_{1 \leq k<j \leq n} \frac{e_{j} e_{k}}{\left|x_{j}-x_{k}\right|} \tag{10}
\end{equation*}
$$

The equation of motion is given by

$$
\begin{array}{r}
m_{j} \frac{d^{2} x_{j}}{d t^{2}}=-\frac{\partial U\left(x_{(n)}\right)}{\partial x_{j}}=\sum_{k=1, k \neq j}^{n} e_{j} e_{k} \frac{x_{j}-x_{k}}{\left|x_{j}-x_{k}\right|^{3}}  \tag{11}\\
j=1, \ldots, n
\end{array}
$$

where $|x|$ is the Euclidean norm,

$$
\begin{aligned}
& e_{j}=-e_{0}<0, \\
& m_{j}=m, \\
& \qquad j=1, \ldots, n-1, e_{n}>0, m_{n}=m^{\prime}, \\
& x_{(n)}=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{3 n}, \\
& x_{j}=\left(x_{j}^{1}, x_{j}^{2}, x_{j}^{3}\right) \in \mathbb{R}^{3} .
\end{aligned}
$$

If (3) is true and $\left|z_{j}\right|=a$ then

$$
\begin{align*}
\left|x_{j}-x_{n}\right|^{2} & =q_{1}^{2}(t)\left|z_{j}\right|^{2}+\left(q_{2}-q_{3}\right)^{2}  \tag{13}\\
& =a^{2} q_{1}^{2}(t)+\left(q_{2}-q_{3}\right)^{2}, \quad j<n .
\end{align*}
$$

This equality and (7) imply that both sides of (11) for the first two components of $x_{n}$ contain zero. For $x_{j}^{1}+i x_{j}^{2}, j<n$, we
derive from the equation of motion and Proposition 2 putting $z_{n}=0$

$$
\begin{align*}
& m z_{j} \frac{d^{2} q}{d t^{2}}=q\left(|q|^{-3} \sum_{k=1, k \neq j}^{n} e_{j} e_{k} \frac{z_{j}-z_{k}}{\left|z_{j}-z_{k}\right|^{3}}\right. \\
& \left.\quad+e_{0} e_{n}\left(\frac{z_{j}}{q^{3}\left|z_{j}\right|^{3}}-\frac{z_{j}}{\left|x_{j}-x_{n}\right|^{3}}\right)\right) \\
& \quad=z_{j} q\left\{\left[-w^{2}+\left(m a^{3}\right)^{-1} e_{0} e_{n}\right] m|q|^{-3}\right.  \tag{14}\\
& \left.\quad-\frac{e_{0} e_{n}}{\left(a^{2} q_{1}^{2}+\left(q_{2}-q_{3}\right)^{2}\right)^{3 / 2}}\right\}=z_{j} q\left(m w_{0}^{2}|q|^{-3}\right. \\
& \\
& \left.-\frac{e_{0} e_{n}}{\left(a^{2} q_{1}^{2}+\left(q_{2}-q_{3}\right)^{2}\right)^{3 / 2}}\right)
\end{align*}
$$

where

$$
\begin{equation*}
w_{0}^{2} a^{3} \frac{m}{e_{0}}=e_{n}^{0} . \tag{15}
\end{equation*}
$$

From the equalities

$$
\begin{align*}
e^{-i u(t)} \frac{d^{2} q}{d t^{2}}= & \frac{d^{2} q_{1}}{d t^{2}}+i\left(2 \frac{d q_{1}}{d t} \frac{d u}{d t}+\frac{d^{2} u}{d t^{2}} q_{1}\right) \\
& -\left(\frac{d u}{d t}\right)^{2} q_{1},  \tag{16}\\
2 \frac{d q_{1}}{d t} \frac{d u}{d t}+\frac{d^{2} u}{d t^{2}} q_{1}= & q_{1}^{-1} \frac{d}{d t}\left(q_{1}^{2} \frac{d u}{d t}\right)=0, \quad q_{1}^{2} \frac{d u}{d t}=\eta
\end{align*}
$$

one derives for $\eta \in \mathbb{R}$

$$
\begin{align*}
\frac{d u}{d t}= & q_{1}^{-2} \eta \\
m \frac{d^{2} q_{1}}{d t^{2}}= & \eta^{2} m q_{1}^{-3}+w_{0}^{2} m q_{1}^{-2}  \tag{17}\\
& -\frac{e_{0} e_{n} q_{1}}{\left(a^{2} q_{1}^{2}+\left(q_{2}-q_{3}\right)^{2}\right)^{3 / 2}}
\end{align*}
$$

From (3) and (11) for $x_{j}^{3}$ and

$$
\begin{align*}
x_{j}^{3} & =x_{k}^{3} \\
\left|x_{j}-x_{k}\right|^{2} & =q_{1}^{2}(t)\left|z_{j}-z_{k}\right|^{2}>0 \tag{18}
\end{align*}
$$

$$
j \neq k<n
$$

it follows that

$$
\begin{align*}
m \frac{d^{2} q_{2}}{d t^{2}} & =-e_{0} e_{n} \frac{q_{2}-q_{3}}{\left(a^{2} q_{1}^{2}+\left(q_{2}-q_{3}\right)^{2}\right)^{3 / 2}}  \tag{19}\\
m^{\prime} \frac{d^{2} q_{3}}{d t^{2}} & =-e_{0} e_{n}(n-1) \frac{q_{3}-q_{2}}{\left(a^{2} q_{1}^{2}+\left(q_{2}-q_{3}\right)^{2}\right)^{3 / 2}} \tag{20}
\end{align*}
$$

where $\partial_{j}$ is the partial derivative in $q_{j}$.
Equations (17), (19), and (20) determine the equation of motion for the three-dimensional mechanical system

$$
\begin{equation*}
m_{j}^{\prime} \frac{d^{2} q_{j}}{d t^{2}}=-\partial_{j} U, \quad j=1,2,3 \tag{21}
\end{equation*}
$$

where

$$
\begin{align*}
U\left(q_{(3)}\right)= & \frac{1}{2} m(a \eta)^{2} q_{1}^{-2}+m\left(a w_{0}\right)^{2} q_{1}^{-1} \\
& -\frac{e_{0} e_{n}}{\left(a^{2} q_{1}^{2}+\left(q_{2}-q_{3}\right)^{2}\right)^{1 / 2}},  \tag{22}\\
a^{-2} m_{1}^{\prime}= & m_{2}^{\prime}=m \\
(n-1) m_{3}^{\prime}= & m^{\prime}
\end{align*}
$$

The simplest equilibrium $q_{2}^{0}=q_{3}^{0}=0, q_{1}^{0}=a_{1}=\eta^{2} w^{-2}>0$, of (21) is easily found for $e_{n}>e_{n}^{0}$ since (17) yields the following equilibrium relation:

$$
\begin{align*}
\eta^{2} q_{1}^{-3}+w_{0}^{2} q_{1}^{-2}-\frac{e_{0} e_{n}}{m a^{3} q_{1}^{2}} & =0  \tag{23}\\
\eta^{2} q_{1}^{-3}-w^{2} q_{1}^{-2} & =0
\end{align*}
$$

In order to solve (21) we have to find the matrix of the linear part of its right-hand side at the equilibrium and determine its eigenvalues. It coincides with $-M^{-1} U^{0}$, where

$$
M=\left(\begin{array}{ccc}
m_{1}^{\prime} & 0 & 0  \tag{24}\\
0 & m_{2}^{\prime} & 0 \\
0 & 0 & m_{3}^{\prime}
\end{array}\right)
$$

The matrix elements of $U^{0}$ are given by

$$
\begin{align*}
U_{j, k}^{0} & =U_{j, k}\left(q_{(3)}^{0}\right) \\
U_{j, k}\left(q_{(3)}\right) & =\frac{\partial^{2} U\left(q_{(3)}\right)}{\partial q_{j} \partial q_{k}} \tag{25}
\end{align*}
$$

$$
k, j=1,2,3
$$

It is not difficult to calculate them:

$$
\begin{align*}
\left(a^{2} m\right)^{-1} U_{1,1}^{0} & =3 \eta^{2} a_{1}^{-4}+2 w_{0}^{2} a_{1}^{-3}-2 \frac{e_{0} e_{n}}{m a^{3} a_{1}^{3}} \\
& =3 \eta^{2} a_{1}^{-4}-2 w^{2} a_{1}^{-3}=w^{2} a_{1}^{-3}=w^{8} \eta^{-6} \\
U_{1, k}^{0} & =U_{k, 1}^{0}=0, \quad k=2,3 ;  \tag{26}\\
U_{3,3}^{0} & =U_{2,2}^{0}=\frac{e_{0} e_{n}}{a^{3} a_{1}^{2}}=g \\
U_{2,3}^{0} & =-\frac{e_{0} e_{n}}{a^{3} a_{1}^{2}}=U_{3,2}^{0} .
\end{align*}
$$

That is,

$$
\begin{align*}
& M^{-1} U^{0} \\
& \quad=\left(\begin{array}{ccc}
w^{8} \eta^{-6} & 0 & 0 \\
0 & m^{-1} g & -m^{-1} g \\
0 & -(n-1) m^{\prime-1} g & (n-1) m^{\prime-1} g
\end{array}\right) . \tag{27}
\end{align*}
$$

The eigenvalues $\sigma_{j}, j=1,2,3$ of this matrix look like

$$
\begin{align*}
& \sigma_{3}=0 \\
& \sigma_{1}=w^{8} \eta^{-6}  \tag{28}\\
& \sigma_{2}=g\left(m^{-1}+(n-1) m^{\prime-1}\right)
\end{align*}
$$

It is well-known that the linear part of the first-order ordinary form of (21)

$$
\begin{align*}
& \frac{d x_{j}}{d t}=v_{j} \\
& \frac{d v_{j}}{d t}=-m_{j}^{\prime-1} \partial_{j} U \tag{29}
\end{align*}
$$

$$
j=1,2,3
$$

is determined by the six-dimensional matrix with imaginary eigenvalues $\pm \sqrt{-\sigma_{j}}$ [12]. Its zero eigenvalue does not allow finding periodic solutions of (21) with the help of the Weinstein and center Lyapunov theorems. An introduction of the difference variable $q_{2}-q_{3}$ permits applying the theorems.

## 3. Main Result

Let us introduce the new variables $q_{j}^{\prime}$ :

$$
\begin{aligned}
& q_{3}^{\prime}=m(n-1) q_{2}+m^{\prime} q_{3} \\
& q_{2}^{\prime}=q_{2}-q_{3} \\
& q_{1}^{\prime}=q_{1}
\end{aligned}
$$

Then equations (17), (19), and (20) are transformed into

$$
\begin{align*}
m \frac{d^{2} q_{1}^{\prime}}{d t^{2}} & =m \eta^{2} q_{1}^{\prime-3}+m w_{0}^{2} q_{1}^{\prime-2}-\frac{e_{0} e_{n} q_{1}^{\prime}}{\left(a^{2} q_{1}^{\prime 2}+q_{2}^{\prime 2}\right)^{3 / 2}}  \tag{31}\\
\mu_{2} \frac{d^{2} q_{2}^{\prime}}{d t^{2}} & =-e_{0} e_{n} \frac{q_{2}^{\prime}}{\left(a^{2} q_{1}^{\prime 2}+q_{2}^{\prime 2}\right)^{3 / 2}}  \tag{32}\\
\frac{d^{2} q_{3}^{\prime}}{d t^{2}} & =0, \quad q_{3}^{\prime}=c_{0}+t c_{1} \tag{33}
\end{align*}
$$

where

$$
\begin{equation*}
\mu_{2}^{-1}=m^{\prime-1}(n-1)+m^{-1} . \tag{34}
\end{equation*}
$$

The second equation is derived by subtracting (20) from (19) after multiplying the former and latter by $m^{\prime-1}$ and $m^{-1}$, respectively. The third equation is derived by adding (19) and (20) after multiplying the former by $n-1$.

Equations (31)-(32) are rewritten as

$$
\begin{equation*}
\mu_{j} \frac{d^{2} q_{j}^{\prime}}{d t^{2}}=-\partial_{j} \widetilde{U}, \quad j=1,2 \tag{35}
\end{equation*}
$$

where $\mu_{1}=m a^{2}$,

$$
\begin{align*}
\widetilde{U}\left(q_{(2)}^{\prime}\right)= & \frac{1}{2} m(a \eta)^{2} q_{1}^{\prime-2}+m\left(a w_{0}\right)^{2} q_{1}^{\prime-1} \\
& -\frac{e_{0} e_{n}}{\left(a^{2} q_{1}^{\prime 2}+q_{2}^{\prime 2}\right)^{1 / 2}} \tag{36}
\end{align*}
$$

The equilibrium of (31)-(32) is given by $q_{2}^{\prime 0}=0, q_{1}^{\prime 0}=a_{1}=$ $\eta^{2} w^{-2}>0$. The linear part of the vector field of (31)-(32) is determined by the matrix $-\widetilde{M}^{-1} \widetilde{U}^{0}$, where

$$
\begin{align*}
\widetilde{M} & =\left(\begin{array}{cc}
\mu_{1} & 0 \\
0 & \mu_{2}
\end{array}\right),  \tag{37}\\
\widetilde{U}_{j, k}^{0} & =\widetilde{U}_{j, k}\left(q_{(2)}^{\prime 0}\right), \\
\widetilde{U}_{j, k}\left(q_{(2)}^{\prime}\right) & =\frac{\partial^{2} \widetilde{U}\left(q_{(2)}^{\prime}\right)}{\partial q_{j}^{\prime} \partial q_{k}^{\prime}}, \tag{38}
\end{align*}
$$

$$
k, j=1,2
$$

These matrix elements are easily calculated as

$$
\begin{align*}
& \widetilde{U}_{2,1}^{0}=\widetilde{U}_{1,2}^{0}=0 \\
& \widetilde{U}_{2,2}^{0}=\frac{e_{0} e_{n}}{a^{3} a_{1}^{2}}=g  \tag{39}\\
& \widetilde{U}_{1,1}^{0}=U_{1,1}^{0}=m a^{2} w^{8} \eta^{-6}
\end{align*}
$$

That is,

$$
\widetilde{M}^{-1} \widetilde{U}^{0}=\left(\begin{array}{cc}
w^{8} \eta^{-6} & 0  \tag{40}\\
0 & \mu_{2}^{-1} g
\end{array}\right)
$$

Eigenvalues $\sigma_{j}, j=1,2$, of this matrix are obvious:

$$
\begin{align*}
& \sigma_{1}=w^{8} \eta^{-6} \\
& \sigma_{2}=g\left(m^{-1}+(n-1) m^{\prime-1}\right) \tag{41}
\end{align*}
$$

In order to obtain periodic solutions of (17), (19), and (20) we have to put $c_{1}=0$ and $q_{3}^{\prime}=c_{0}$. As a result $q_{j}, j=1,2,3$, will be periodic in time if $q_{j}^{\prime}, j=1,2$, are periodic in time and

$$
\begin{align*}
q_{2}(t) & =\left(c_{0}+m^{\prime} q_{2}^{\prime}(t)\right)\left(m(n-1)+m^{\prime}\right)^{-1} \\
q_{3} & =\left(c_{0}-m(n-1) q_{2}^{\prime}(t)\right)\left(m^{\prime}+m(n-1)\right)^{-1}  \tag{42}\\
c_{0} & =m(n-1) q_{2}(0)+m^{\prime} q_{3}(0)
\end{align*}
$$

If a coordinate $q_{1}^{\prime}$ is translated by $a_{1}$ then (35) determines the four-dimensional Hamiltonian system with the Hamiltonian $H^{\prime}$ as follows:

$$
\begin{equation*}
H^{\prime}\left(p_{(2)}, q_{(2)}\right)=\frac{p_{1}^{2}}{2 \mu_{1}}+\frac{p_{2}^{2}}{2 \mu_{2}}+\widetilde{U}\left(q_{1}+a_{1}, q_{2}\right) \tag{43}
\end{equation*}
$$

The origin is its equilibrium and its Hessian; that is, the matrix of second partial derivatives is positive definite at the origin. Moreover the Hamiltonian is a holomorphic function at a neighborhood of the origin(equilibrium); the linear part of the first-order ordinary form of (35) coincides with

$$
\left(\begin{array}{cc}
0 & 1  \tag{44}\\
-\sigma_{1} & 0
\end{array}\right) \oplus\left(\begin{array}{cc}
0 & 1 \\
-\sigma_{2} & 0
\end{array}\right)
$$

and its eigenvalues look like $\lambda_{j}= \pm i \sqrt{\sigma_{j}}, j=1,2$. Hence the Weinstein theorem implies validity of the following theorem.

Theorem 3. If $e_{n}>e_{n}^{0}$ then equation of motion (35) possesses 2 periodic solutions whose periods are close to those of the linearized at the origin equation of motion, that is, $2 \pi / \sqrt{\sigma_{j}}, j=$ 1,2. These periodic solutions $q_{j}^{\prime}(t)$ of (35) generate the periodic solutions $q_{j}(t)$ of (21) through (42).

The center Lyapunov theorem implies the following theorem.

Theorem 4. Let $e_{n}>e_{n}^{0}, \sigma_{1} \neq \sigma_{2}$ and $\sigma_{1}=k^{2} \sigma_{2}$, or $\sigma_{2}=$ $k^{2} \sigma_{1}$, where $k \neq 1$ is an integer; then equation of motion (35) possesses a periodic solution such that it depends on a real parameter $c$. This solution and its period $\tau(c)$ are real analytical functions in $c$ at the origin and $\tau(0)=2 \pi / \sqrt{\sigma_{1}}$ or $\tau(0)=$ $2 \pi / \sqrt{\sigma_{2}}$. If neither of the two resonance conditions hold then (35) possesses two periodic solutions such that each of them depends on one of the real parameters $c_{1}, c_{2}$. These solutions and their periods $\tau_{1}\left(c_{1}\right), \tau_{2}\left(c_{2}\right)$ are real analytical functions at the origin in the parameters and $\tau_{j}(0)=2 \pi / \sqrt{\sigma_{j}}, j=1,2$. These periodic solutions $q_{j}^{\prime}(t)$ of (35) generate the periodic solutions $q_{j}(t)$ of (21) through (42).

## 4. Conclusion

The Coulomb equation of motion of $n-1$ equal negative charges with the same mass and a positive charge is reduced by us to an equation of motion of a two-dimensional mechanical system with an equilibrium assuming that coordinates of the negative charges coincide with vertices of a regular polygon in planes perpendicular to a coordinate axis along which the positive charge moves. We found the periodic solutions of this equation with the help of the Weinstein and center Lyapunov theorems. This result poses the following question: what is the quantum analog of such a highly synchronized motion of point charges?

## Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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