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Research Article

Modeling Anomalous Diffusion by a Subordinated Integrated Brownian Motion

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We consider a particular type of continuous time random walk where the jump lengths between subsequent waiting times are correlated. In a continuum limit, the process can be defined by an integrated Brownian motion subordinated by an inverse α-stable subordinator. We compute the mean square displacement of the proposed process and show that the process exhibits subdiffusion when $0 < \alpha < 1/3$, normal diffusion when $\alpha = 1/3$, and superdiffusion when $1/3 < \alpha < 1$. The time-averaged mean square displacement is also employed to show weak ergodicity breaking occurring in the proposed process. An extension to the fractional case is also considered.

1. Introduction

Anomalous diffusion is found in a wide diversity of systems (see review articles [1–4] and references therein). In one dimension, it is characterized by a mean square displacement (MSD) of the form

$$\left\langle (\Delta x)^2 \right\rangle (t) \propto K_\alpha t^\alpha \tag{1}$$

with $\alpha \neq 1$, which deviates from the linear dependence on time found in normal diffusion. The coefficient K_α is generalized diffusion constant. It is called subdiffusion for $0 < \alpha < 1$ and superdiffusion for $\alpha > 1$ [2].

A fundamental account to anomalous diffusion is provided by a stochastic process called continuous time random walk (CTRW), which was originally introduced by Montroll and Weiss in 1965 [5]. In a continuum limit, the process has been considered by Fogedby [6] via coupled Langevin equations

$$\begin{aligned} \frac{dX}{ds} &= \xi(s), \\ \frac{dt}{ds} &= \zeta(s), \end{aligned} \tag{2}$$

where $\xi(s)$ is a white Gaussian noise with $\langle \xi(s) \rangle = 0$, $\langle \xi(s)\xi(s') \rangle = \delta(s-s')$, and $\zeta(s)$ is a white α-stable Lévy noise, taking positive values only and independent of $\xi(s)$.

In (2), the random walk $x(t)$ is parametrized in terms of a continuous variable s, which is subjected to a random time change. This random time change to the physical time t is described by the second equation. The combined process in the physical time is then given by $x(t) = X(s(t))$, where $s(t)$ is the inverse process to $t(s)$ defined as

$$s(t) = \inf \{ s : t(s) > t \}. \tag{3}$$

Mathematically, the fundamental approach to describe the combined process $x(t) = X(s(t))$ is based on subordination technique, which was first introduced by Bochner [7]. Using the notation of subordination, the process $X(s)$, $t(s)$, and $s(t)$ are named parent process, subordinator, and inverse subordinator, respectively.

In recent years, (2) consisting of Brownian motions with or without external field and inverse α-stable subordinator are becoming a hot topic [8–17]. There are also several other processes considered as parent processes within the subordination framework, for example, Lévy-stable process

[18, 19], arithmetic Brownian motion [20], geometric Brownian motion [21, 22], Ornstein-Uhlenbeck process [23, 24], tempered stable process [25], fractional Brownian motion [26, 27], and fractional Lévy-stable process [28]. Here, we note that, apart from inverse α-stable subordinator, inverse tempered α-stable subordinator and infinitely divisible subordinators are also considered in the literatures [16, 20, 25–31].

In the simplest CTRW process, after each jumps, a new pair of waiting time and jump length is drawn from the associated distributions, independent of the previous values. This independence giving rise to a renewal process is not always justified, for instance, by observations of human motion patterns [32] and active biological movements [33] or in financial market dynamics [34]. Recently, three correlated CTRW models are introduced to model the random walks with some forms of memory [35–37]. Some advances in the field of CTRWs with correlated temporal or/and spatial structure can be also found in [38–45].

In this work, we consider a jump-correlated CTRW model which has the subordination form $x(t) = X(s_\alpha(t))$. Here, the parent process $X(s)$ is an integrated Brownian motion, defined by

$$X(s) = \int_0^s B(s')\,ds' \tag{4}$$

and inverse subordinator $s_\alpha(t)$ is the inverse of one-side α-stable Lévy process $t(s)$, defined by

$$s_\alpha(t) = \inf\{s > 0 : t(s) > t\}. \tag{5}$$

The integrated Brownian motion $X(s)$ is called the random acceleration process in the physical literature and has been studied by many authors. For instance, it appears in the continuum description of the equilibrium Boltzmann weight of a semiflexible polymer chain [46]. It also appears in the description of statistical properties of the Burgers equation with Brownian initial velocity [47]. Some further results of the integrated Brownian motion can be found in the paper [48] reviewing this subject.

The structure of the paper is as follows. In Section 2, we introduce the jump-correlated CTRW model. In Section 3, we compute MSD of the proposed process and observe the corresponding anomalous diffusive behaviors. The time-averaged MSD is also employed to show weak ergodicity breaking occurring in the proposed process. In Section 4, we generalize the integrated Brownian motion to the fractional integral of Brownian motion and compute the corresponding MSD. The conclusions are given in Section 5.

2. Model

We begin by recalling the general framework for CTRW theory. Let $\{T_i\}_{i\geq 1}$ be the sequence of nonnegative independent identically distributed (IID) random variable representing waiting times between jumps of a particle. We set $t(0) = 0$ and $t(n) = \sum_{i=1}^{n} T_i$, that is, the time of the nth jump. Let $\{J_i\}_{i\geq 1}$ be the sequence of IID jump lengths of the particle, which are assumed to be independent of waiting times. We set $X(0) = 0$ and $X(n) = \sum_{i=1}^{n} J_i$, that is, the position of the particle after the nth jump. Then, the position of the particle at time t is given by

$$x(t) = X(N(t)) = \sum_{i=1}^{N(t)} J_i, \tag{6}$$

where $N(t) = \max\{n \geq 0 : t(n) \leq t\}$ is the number of jumps up to time t. The process $x(t) = X(N(t))$ is called CTRW.

In what follows, we analyze a particular type of CTRW where the jump lengths are correlated. Assume that each jump is equal to

$$J_i = \xi_1 + \xi_2 + \cdots + \xi_i, \tag{7}$$

where ξ_j are IID random variables with finite second moment (for simplicity we assume that their second moment is equal to 1). Moreover, we assume that each waiting time T_i is nonnegative IID random variable, whose characteristic function $\widehat{\varphi}(k)$ is given by

$$\widehat{\varphi}(k) = \exp\left\{-|k|^\alpha \exp\left(-\frac{i\pi\alpha}{2}\operatorname{sgn}(k)\right)\right\}, \quad 0 < \alpha \leq 1. \tag{8}$$

In the continuous limit, we get the following set of coupled Langevin equations for the position x and time t of the CTRW

$$\begin{aligned}\frac{dX(s)}{ds} &= \int_0^s \xi(s')\,ds' = B(s),\\ \frac{dt(s)}{ds} &= \zeta(s),\end{aligned} \tag{9}$$

where $\xi(s)$ and $\zeta(s)$ are the same as those in (2) and $B(s)$ is the standard Brownian motion with $\langle B(s)\rangle = 0$, $\langle B(s)B(t)\rangle = \min(s,t)$.

An equivalent representation of (9) in the form of subordination is

$$x(t) = X(s_\alpha(t)). \tag{10}$$

Here the parent process $X(s)$ has the form

$$X(s) = \int_0^s B(s')\,ds', \tag{11}$$

and the inverse subordinator $s_\alpha(t)$ is defined by

$$s_\alpha(t) = \inf\{s > 0 : t(s) > t\}, \tag{12}$$

where $t(s) = \int_0^s \zeta(s')ds'$ is an α-stable totally skewed Lévy motion with characteristic function

$$\left\langle e^{-ut(s)}\right\rangle = \exp\{-u^\alpha s\}, \quad 0 < \alpha \leq 1. \tag{13}$$

3. Discussions

At first, let us compute the MSD of subordinated process $x(t) = X(s_\alpha(t))$.

Assume that $p(x,t)$, $f(x,s)$, and $g(s,t)$ are PDFs of subordinated process $x(t)$, parent process $X(s)$, and inverse subordinator $s_\alpha(t)$, respectively. In terms of subordination, we have

$$p(x,t) = \int_0^\infty f(x,s)\,g(s,t)\,ds. \tag{14}$$

Since the first moment of parent process $X(s)$

$$\langle X(s)\rangle = \left\langle \int_0^s B(s')\,ds' \right\rangle = \int_0^s \langle B(s')\rangle\,ds' = 0 \tag{15}$$

and the second moment

$$\begin{aligned}\left\langle X^2(s)\right\rangle &= \left\langle \int_0^s B(s')\,ds' \cdot \int_0^s B(s'')\,ds'' \right\rangle \\ &= \int_0^s ds' \int_0^s \langle B(s')\,B(s'')\rangle\,ds'' \\ &= \int_0^s ds' \int_0^s \min\{s',s''\}\,ds'' = \frac{s^3}{3},\end{aligned} \tag{16}$$

we obtain

$$\begin{aligned}\langle x(t)\rangle &= \int_0^\infty xp(x,t)\,dx \\ &= \int_0^\infty dx \int_0^\infty xf(x,s)\,g(s,t)\,ds \\ &= \int_0^\infty \langle X(s)\rangle\,g(s,t)\,ds = 0,\end{aligned} \tag{17}$$

$$\begin{aligned}\left\langle x^2(t)\right\rangle &= \int_{-\infty}^\infty x^2 p(x,t)\,dx \\ &= \int_{-\infty}^\infty dx \int_0^\infty x^2 f(x,s)\,g(s,t)\,ds \\ &= \int_0^\infty \left\langle X^2(s)\right\rangle g(s,t)\,ds \\ &= \frac{1}{3}\int_0^\infty s^3 g(s,t)\,ds.\end{aligned} \tag{18}$$

Thus, the MSD of the subordinated process $x(t)$ is

$$\begin{aligned}\left\langle (\Delta x)^2\right\rangle(t) &= \left\langle x^2(t)\right\rangle - \langle x(t)\rangle^2 \\ &= \frac{1}{3}\int_0^\infty s^3 g(s,t)\,ds.\end{aligned} \tag{19}$$

Let us turn to the inverse subordinator $s_\alpha(t)$. Observing the equivalence from (12)

$$s_\alpha \le s \Longleftrightarrow t(s) > t, \tag{20}$$

we obtain the relation

$$P(s_\alpha \le s) = P(t(s) > t) = 1 - P(t(s) \le t), \tag{21}$$

which gives the formula for the PDF $g(s,t)$ in terms of the PDF $h(t,s)$:

$$g(s,t) = -\frac{\partial}{\partial s}\int_0^t h(t',s)\,dt'. \tag{22}$$

Taking the Laplace transform for (22) about variable t, we get

$$\tilde{g}(s,u) = -\frac{\partial}{\partial s}\frac{1}{u}\tilde{h}(u,s) = u^{\alpha-1}\exp\{-u^\alpha s\}, \quad 0 < \alpha \le 1. \tag{23}$$

Thus, the MSD of the subordinated process $x(t)$ in Laplace space is

$$\begin{aligned}\left\langle \left(\widetilde{\Delta x}\right)^2\right\rangle(u) &= \frac{1}{3}\int_0^\infty s^3 \tilde{g}(s,u)\,ds \\ &= \frac{1}{3}\int_0^\infty s^3 u^{\alpha-1}\exp\{-u^\alpha s\}\,ds = \frac{2}{u^{3\alpha+1}},\end{aligned} \tag{24}$$

which implies that the MSD of $x(t)$ is

$$\left\langle (\Delta x)^2\right\rangle(t) = \frac{2}{\Gamma(3\alpha+1)}t^{3\alpha}, \quad 0 < \alpha \le 1. \tag{25}$$

It is easy to observe from (25) that the process is subdiffusive when $0 < \alpha < 1/3$, normally diffusive when $\alpha = 1/3$, and superdiffusive when $1/3 < \alpha \le 1$.

It is well-known that the MSD of the process given by (2) is of the form

$$\left\langle (\Delta x)^2\right\rangle(t) = \frac{t^\alpha}{\Gamma(1+\alpha)} \quad 0 < \alpha < 1. \tag{26}$$

Comparing (25) with (26), we see that Fogedby's model can only represent anomalous subdiffusion, but our model can represent subdiffusion, normal diffusion, and superdiffusion.

Next, we study weak ergodicity breaking of the subordinated process $x(t)$.

In an ergodic system, one can find the equivalence

$$\left\langle (\Delta x)^2\right\rangle(\Delta) = \left\langle \overline{\delta^2(\Delta)}\right\rangle. \tag{27}$$

Here, $\overline{\delta^2(\Delta)}$ is the time-averaged MSD of the process $x(t)$, defined as

$$\overline{\delta^2(\Delta)} = \frac{1}{T-\Delta}\int_0^{T-\Delta}[x(t+\Delta) - x(t)]^2\,dt, \tag{28}$$

where Δ is the lag time and T is the overall measure time.

For anomalous diffusion, the behavior of the ensemble MSD $\langle(\Delta x)^2\rangle(\Delta)$ and the time-averaged MSD (28) may be fundamentally different. The disparity $\langle(\Delta x)^2\rangle(\Delta) \ne \langle\overline{\delta^2(\Delta)}\rangle$ is usually called weak ergodicity breaking (or weak nonergodicity) [49]. In recent years, weak nonergodicity of anomalous diffusion process attracts more and more attentions [49–55].

Since, for any $a > 0$, parent process $X(s)$ satisfies

$$X(as) = \int_0^{as} B(s')\,ds' = a\int_0^s B(a\tau)\,d\tau \stackrel{d}{=} a^{3/2}X(s), \tag{29}$$

where $=^d$ means an equality in distribution, we have

$$x(t) = X(s_\alpha(t)) =^d X(t^\alpha s_\alpha(1)) =^d \left(\frac{t^{3\alpha}}{2}\right) X(s_\alpha(1)) = t^{3\alpha/2} x(1). \quad (30)$$

Thus,

$$\begin{aligned}\left\langle\overline{\delta^2(\Delta)}\right\rangle &= \frac{\langle x^2(1)\rangle}{T-\Delta}\int_0^{T-\Delta}\left[(t+\Delta)^{3\alpha/2} - t^{3\alpha/2}\right]^2 dt \\ &= \frac{\langle x^2(1)\rangle}{T-\Delta}\int_0^{T-\Delta}\left[(t+\Delta)^{3\alpha} + t^{3\alpha} - 2t^{3\alpha/2}(t+\Delta)^{3\alpha/2}\right] dt \\ &= \frac{\langle x^2(1)\rangle}{T-\Delta}\left\{\frac{1}{(3\alpha+1)}\left[T^{3\alpha+1} - \Delta^{3\alpha+1} + (T-\Delta)^{3\alpha+1}\right] - 2I_1\right\},\end{aligned} \quad (31)$$

where $I_1 = \int_0^{T-\Delta} t^{3\alpha/2}(t+\Delta)^{3\alpha/2} dt$.

In the limit $\Delta \ll T$,

$$\begin{aligned}I_1 &= \int_0^{T-\Delta} t^{3\alpha/2}(t+\Delta)^{3\alpha/2} dt \\ &= T^{3\alpha+1}\int_0^{1-\Delta/T}\left(\tau + \frac{\Delta}{T}\right)^{3\alpha/2}\tau^{3\alpha/2} d\tau \\ &\simeq T^{3\alpha+1}\int_0^{1-\Delta/T}\tau^{3\alpha} d\tau = \frac{1}{3\alpha+1}(T-\Delta)^{3\alpha+1}.\end{aligned} \quad (32)$$

Hence,

$$\begin{aligned}\left\langle\overline{\delta^2(\Delta)}\right\rangle &\simeq \frac{\langle x^2(1)\rangle}{T-\Delta} \cdot \frac{1}{3\alpha+1}\left[T^{3\alpha+1} - \Delta^{3\alpha+1} - (T-\Delta)^{3\alpha+1}\right] \\ &= \frac{\langle x^2(1)\rangle}{T-\Delta}\frac{1}{3\alpha+1} \cdot T^{3\alpha+1}\left[1 - \left(\frac{\Delta}{T}\right)^{3\alpha+1} - \left(1 - \frac{\Delta}{T}\right)^{3\alpha+1}\right] \simeq \langle x^2(1)\rangle \cdot \frac{\Delta}{T^{1-3\alpha}}.\end{aligned} \quad (33)$$

Since

$$\langle(\Delta x)^2\rangle(\Delta) = \frac{2}{\Gamma(3\alpha+1)}\Delta^{3\alpha}, \quad 0 < \alpha \le 1, \quad (34)$$

comparing (33) with (34), we see that the linear lag time dependence of $\langle\overline{\delta^2(\Delta)}\rangle$ is different from the power-law form $\Delta^{3\alpha}$ of $\langle(\Delta x)^2\rangle(\Delta)$, which implies that subordinated process $x(t)$ is weakly nonergodic.

At last, we consider the propagator $p(x,t)$ associated with the subordinated process $x(t)$. By the total probability formula, we obtain an integral representation of $p(x,t)$:

$$p(x,t) = \int_0^\infty f(x,s)\, g(s,t)\, ds. \quad (35)$$

For fixed $s > 0$, the random variable $X(s) = \int_0^s B(s')ds'$ is normally distributed. From (15) and (16), we have

$$f(x,s) = \frac{\sqrt{3}}{\sqrt{2\pi s^3}}\exp\left(-\frac{3x^2}{2s^3}\right). \quad (36)$$

It follows from

$$\tilde{g}(s,u) = u^{\alpha-1}\exp\{-u^\alpha s\}, \quad 0 < \alpha \le 1, \quad (37)$$

and the Laplace transform $s \mapsto q$ for $\tilde{g}(s,u)$ that we obtain

$$\tilde{\tilde{g}}(q,u) = \frac{u^{\alpha-1}}{u^\alpha + q}. \quad (38)$$

After taking the inverse Laplace transform $u \mapsto t$ for $\tilde{\tilde{g}}(q,u)$, we get

$$\tilde{g}(q,t) = E_\alpha(-qt^\alpha), \quad (39)$$

where

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+1)} \quad (40)$$

is the Mittag-Leffler function with parameter α [56].

4. An Extension to the Fractional Case

In this section, we introduce the dependent sequence of jump lengths J_i in the following manner:

$$J_i = \sum_{j=1}^{i} M(i-j+1)\,\xi_j, \quad (41)$$

where $M(\cdot)$ is a memory function. The continuous limit is of the form

$$\frac{dX(s)}{ds} = \int_0^s M(s-s')\,\xi(s')\,ds' = \int_0^s M(s-s')\,dB(s'). \quad (42)$$

Integrating (42) we get

$$X(s) = \int_0^s ds' \int_0^{s'} M(s'-s'')\,dB(s''). \quad (43)$$

After taking $M(t) = t^{-\mu}/\Gamma(1-\mu)$ $(0 < \mu < 1)$ and using the integration by parts, (43) can be written as

$$X(s) = \frac{1}{\Gamma(1-\mu)}\int_0^s \frac{B(s')}{(s-s')^\mu} ds' = {}_0I_t^p B(s), \quad (44)$$

where ${}_0I_t^p$ is the Riemann-Liouville fractional integration operator of order p, defined by [56]

$$ {}_0I_t^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau)\, d\tau, \quad (p>0). \tag{45} $$

As a result, the jump-correlated CTRW has the subordination form $x(t) = X(s_\alpha(t))$, where parent process $X(s)$ is of the form (44), and inverse subordinator $s_\alpha(t)$ is defined by (12).

Here, we are interested in the competition between the memory parameter μ and stability index α. In what follows, we will not discuss any properties of motion other than the MSD.

In terms of (44), we get

$$ \begin{aligned} \left\langle X^2(s) \right\rangle &= \frac{1}{\Gamma^2(1-\mu)} \\ &\quad \cdot \int_0^s \frac{ds'}{(s-s')^\mu} \int_0^s \frac{\left\langle B(s') B(s'') \right\rangle}{(s-s'')^\mu} ds'' \\ &= \frac{1}{\Gamma^2(1-\mu)} \int_0^s \frac{ds'}{(s-s')^\mu} \int_0^s \frac{\min\{s', s''\}}{(s-s'')^\mu} ds'' \\ &= \frac{1}{\Gamma^2(1-\mu)} \int_0^s \frac{ds'}{(s-s')^\mu} \left[\int_0^{s'} \frac{s''}{(s-s'')^\mu} ds'' \right. \\ &\quad \left. + \int_{s'}^s \frac{s'}{(s-s'')^\mu} ds'' \right]. \end{aligned} \tag{46} $$

By denoting $I(s) = \int_0^s (ds'/(s-s')^\mu) \int_0^{s'} (s''/(s-s'')^\mu) ds''$ and exchanging the order of quadratic integral $I(s)$, we obtain

$$ I(s) = \int_0^s \frac{ds''}{(s-s'')^\mu} \int_{s''}^s \frac{s''}{(s-s')^\mu} ds'. \tag{47} $$

Thus,

$$ \begin{aligned} &\left\langle X^2(s) \right\rangle \\ &\quad = \frac{2}{\Gamma^2(1-\mu)} \int_0^s \frac{s'}{(s-s')^\mu} ds' \int_{s'}^s \frac{1}{(s-s'')^\mu} ds'' \\ &\quad = \frac{2}{(1-\mu)\Gamma^2(1-\mu)} \int_0^s s' \left(s-s'\right)^{1-2\mu} ds' \\ &\quad = K_\mu s^{3-2\mu}, \quad (0<\mu<1), \end{aligned} \tag{48} $$

where $K_\mu = 2B(2, 2-2\mu)/(1-\mu)\Gamma^2(1-\mu)$ and $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$ is Beta function.

We observe from (48) that, in the limiting case $\mu \to 0$, memory function $M(t) = 1$, the parent process $X(s)$ defined by (43) reduces to the form defined by (11), and the second moment of $X(s)$ computed by (48) reduces to $K_0 s^3$, where $K_0 = 2B(2,2) = 1/3$, the same form as (16). We are also interested in the limiting case $\mu \to 1$. At the moment, the memory function $M(t)$ is a Dirac δ-function; $X(s)$ defined by (43) reduces to the standard Brownian motion.

Let us turn to the MSD of the subordinated process $x(t)$. In terms of (18) and (48), we obtain

$$ \left\langle (\Delta x)^2 \right\rangle (t) = \left\langle x^2(t) \right\rangle = K_\mu \int_0^\infty s^{3-2\mu} g(s,t)\, ds. \tag{49} $$

In the Laplace space, the MSD is of the form

$$ \begin{aligned} \left\langle \left(\widetilde{\Delta x}\right)^2 \right\rangle (u) &= K_\mu u^{\alpha-1} \int_0^\infty s^{3-2\mu} e^{-u^\alpha s} ds \\ &= K_\mu \frac{\Gamma(4-2\mu)}{u^{\alpha(3-2\mu)+1}}. \end{aligned} \tag{50} $$

Taking the inverse Laplace transform for $\langle (\widetilde{\Delta x})^2 \rangle (u)$, we have

$$ \left\langle (\Delta x)^2 \right\rangle (t) = K_{\mu,\alpha} t^{\alpha(3-2\mu)}, \quad 0<\mu<1,\ 0<\alpha\le 1, \tag{51} $$

where $K_{\mu,\alpha} = K_\mu \Gamma(4-2\mu)/\Gamma(\alpha(3-2\mu)+1)$. In the limiting case $\mu \to 0$, the parameter $K_{\mu,\alpha}$ reduces to

$$ K_{0,\alpha} = \frac{K_0 \Gamma(4)}{\Gamma(3\alpha+1)} = \frac{2}{\Gamma(3\alpha+1)}. \tag{52} $$

Thus, (51) reduces to (25).

It is easy to observe from (51) that there exists a competition between the memory parameter μ and stability index α. For the case $\alpha \le 1/3$, the subordinated process exhibits subdiffusive behaviors. For the case $1/3 < \alpha < 1$, the process is subdiffusive when $1 < 3-2\mu < 1/\alpha$, normal diffusive when $3-2\mu = 1/\alpha$, and superdiffusive when $1/\alpha < 3-2\mu < 3$.

5. Conclusions

We introduce an integrated Brownian motion subordinated by inverse α-stable one-sided Lévy motion, which is a continuous limit of a jump-correlated CTRW. In terms of the ensemble MSD of the proposed process, we conclude that the process is subdiffusive when $0 < \alpha < 1/3$, normal diffusive when $\alpha = 1/3$, and superdiffusive when $1/3 < \alpha \le 1$. The time-averaged MSD is also employed to show weak ergodicity breaking occurring in the proposed process.

We also generalize the process to the case, where the parent process is fractional integral of Brownian motion. In terms of the MSD, we observe a competition between the memory parameter μ and stability index α. Other types of inverse subordinators may be also candidates.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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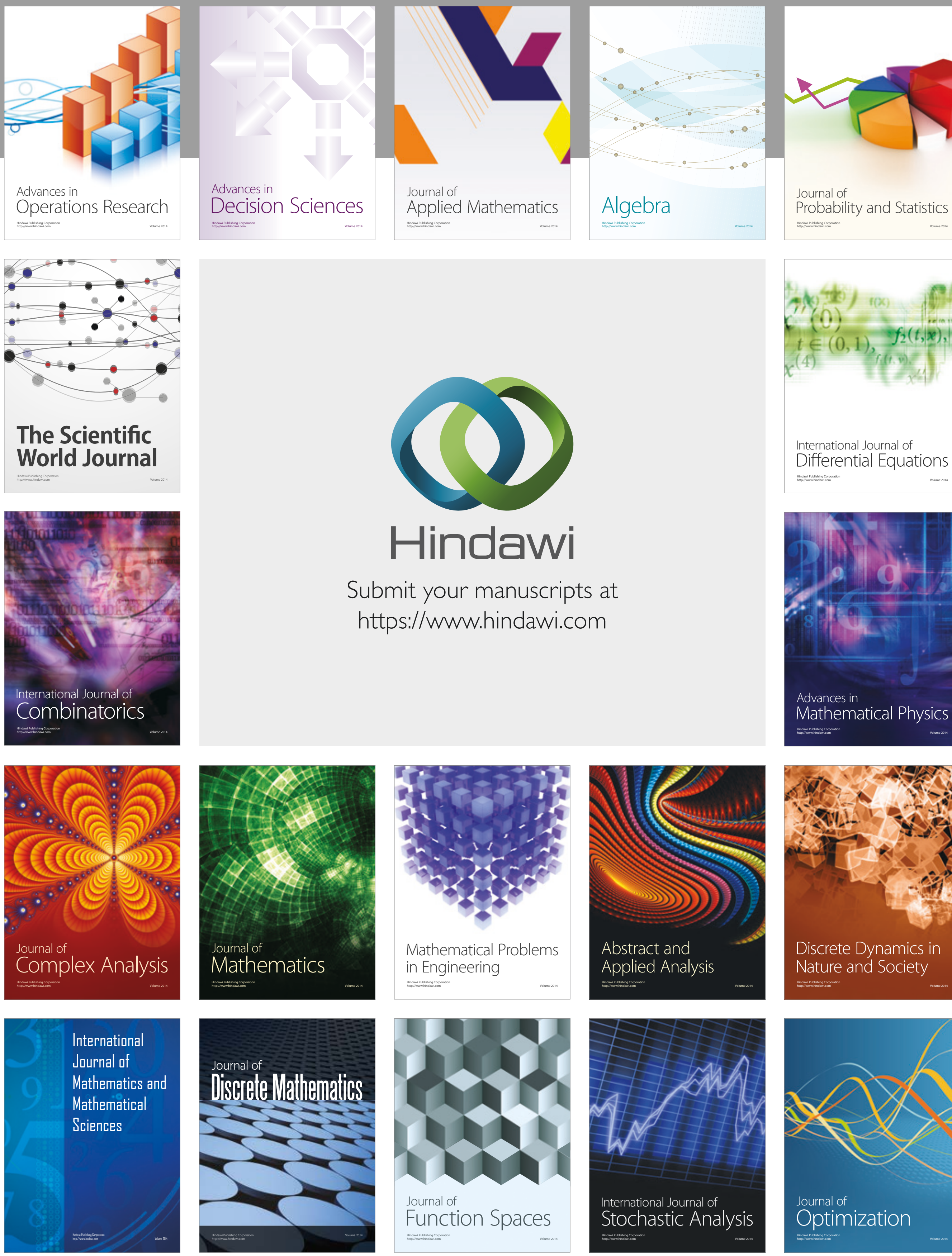
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