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*Research Article*

# An Accelerated Homotopy Perturbation Method for Solving Nonlinear Two-Dimensional Volterra-Fredholm Integrodifferential Equations

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We propose and apply coupling of the variational iteration method (VIM) and homotopy perturbation method (HPM) to solve nonlinear mixed Volterra-Fredholm integrodifferential equations (VFIDE). In this approach, we use a new formula called variational homotopy perturbation method (VHPM) and variational accelerated homotopy perturbation method (VAHPM). This approach is based on the form of He's polynomials and on a new form of He's polynomials. We discuss the convergence of the technique. Some numerical examples are introduced to verify the efficiency of this technique.

## 1. Introduction

In recent years, there has been a clear interest in integrodifferential equations which are a combination of differential and Volterra-Fredholm integral equations. Integrodifferential equations play an important role in many branches of linear and nonlinear functional analyses and their applications. The mentioned integrodifferential equations are usually difficult to solve analytically, so approximation strategies are required to obtain the solution of the linear and nonlinear integrodifferential equations [1].

Many researchers studied and discussed the linear VFIDE [2]. Al-Jubory [3] introduced some approximation methods to solve Volterra-Fredholm integral and integrodifferential equations. Dadkhah et al. in [4] used a numerical solution of nonlinear VFIDE by using Legendre wavelets. Rabbani and Kiasoltani [5] studied the solving of a nonlinear system of VFIDE by using the discrete collocation method. Gherjalar and Mohammadikia [6] solved integral and integrodifferential equations by using the B-splines function. In this work, we used the HPM and VIM to solve the two-dimensional nonlinear VFIDE as follows:

$$\sum_{j=0}^{k} P_j(x_1,\jmath_1)\,u^j(x_1,\jmath_1) = \acute{f}(x_1,\jmath_1) + \int_a^{x_1}\int_\Omega F(x_1,\jmath_1,y,\tau)\,\gamma\left(u^l(y,\tau)\right)dy\,d\tau, \quad (x_1,\jmath_1)\in \acute{J}=[a,x_1]\times\Omega, \tag{1}$$

with initial conditions

$$u^r(a,\jmath_1) = g_r, \quad r=0,1,\ldots,k-1,\ \Omega=[a,b], \tag{2}$$

where $u^j(x_1,\jmath_1) = d^j u/dx_1^j$. The functions $\acute{f}(x_1,\jmath_1)$, $F(x_1,\jmath_1,y,\tau)$ and $\gamma(u^l(y,\tau))$, $l>0$, are analytic functions on $J'$, and functions $P_j(x_1,\jmath_1)$, $j=0,1,\ldots,k$, $P_k(x_1,\jmath_1)\neq 0$ are given.

For VIM and HPM, which were proposed by He in [7, 8], the solution is considered as the summation of an infinite series which is assumed to be convergent to the exact solution. In recent years, HPM has been applied with great success, so relations and algorithms have been deduced and continuously improved to obtain an accurate solution for a large variety of linear and nonlinear problems. For instance, He in [7] used a strategy to solve some integrodifferential equations where he chose an initial approximate solution in the form of an exact solution with unknown constants.

In this paper, a new approach based on VIM with HPM is introduced to solve the two-dimensional nonlinear VFIDE.

## 2. The HPM

In this section, we will present the HPM. We consider a general integral equation

$$Lu = 0, \tag{3}$$

where $L$ is an integral operator. Define a convex homotopy $\breve{H}(\vartheta, \wp)$ by

$$\breve{H}(\vartheta, \wp) = (1-\wp)\,\mathcal{F}(\vartheta) + \wp L(\vartheta) = 0, \quad \wp \in [0,1], \tag{4}$$

where $\mathcal{F}(\vartheta)$ is a functional operator with solution $\vartheta_0$. Then,

$$\begin{aligned} \breve{H}(\vartheta, 0) &= \mathcal{F}(\vartheta) = 0, \\ \breve{H}(\vartheta, 1) &= L(\vartheta) = 0, \end{aligned} \tag{5}$$

and the process of changing $\wp$ from 0 to 1 is just that of changing $\vartheta$ from $\vartheta_0$ to $u$. In topology, this is called deformation, and $\mathcal{F}(\vartheta)$ and $L(\vartheta)$ are called homotopies.

According to the HPM, we can use the embedding parameter $\wp$ as a "small parameter" and assume that the solution of (4) can be written as a power series in $\wp$:

$$\vartheta = u_0 + \wp u_1 + \wp^2 u_2 + \cdots = \sum_{i=0}^{\infty} \wp^i u_i = u. \tag{6}$$

When $\wp \to 1$, the approximate solution of (3) is obtained with

$$u = \lim_{\wp \to 1} \vartheta = u_0 + u_1 + u_2 + \cdots = \sum_{i=0}^{\infty} u_i. \tag{7}$$

Series (7) is convergent for most cases; however, the rate of convergence depends upon the nonlinear operator $L$ [9].

## 3. The HPM for Solving Nonlinear Mixed VFIDE

In what follows, we display an outline for utilizing the HPM for solving the nonlinear VFIDE. Equation (1) can be written as follows:

$$\begin{aligned} &u(x_1, \jmath_1) \\ &\quad = L^{-1}\left(\frac{\dot{f}(x_1, \jmath_1)}{P_k(x_1, \jmath_1)}\right) + \sum_{r=0}^{k-1} \frac{1}{(r!)} (x_1 - \jmath_1)^r g_r \\ &\quad + L^{-1}\left(\int_a^{x_1}\int_\Omega \frac{F(x_1, \jmath_1, y, \tau)\,\gamma\left(u^l(y,\tau)\right)}{P_k(x_1, \jmath_1)}\,dy\,d\tau\right) \\ &\quad - L^{-1}\left(\sum_{j=0}^{k-1} \frac{P_j(x_1, \jmath_1)}{P_k(x_1, \jmath_1)} u^j(x_1, \jmath_1)\right), \end{aligned} \tag{8}$$

where $L^{-1}$ is the multiple integration operator given as follows:

$$L^{-1}(\cdot) = \int_a^{x_1}\int_a^{x_1}\cdots\int_a^{x_1} (\cdot)\,dx_1 dx_1 \cdots dx_1, \qquad (n \text{ times}). \tag{9}$$

So, (8) takes the form

$$\begin{aligned} u(x_1, \jmath_1) &= L^{-1}\left(\frac{\dot{f}(x_1, \jmath_1)}{P_k(x_1, \jmath_1)}\right) + \sum_{r=0}^{k-1} \frac{1}{(r!)} (x_1 - \jmath_1)^r g_r \\ &+ \int_a^{x_1}\int_\Omega \frac{(x_1 - \jmath_1)^k}{(k)!} \frac{F(x_1, \jmath_1, y, \tau)\,\gamma\left(u^l(y,\tau)\right)}{P_k(x_1, \jmath_1)}\,dy\,d\tau \\ &- \sum_{j=0}^{k-1} \int_a^{x_1} \frac{(x_1 - \jmath_1)^{k-1}}{(k-1)!} \frac{P_j(x_1, \jmath_1)}{P_k(x_1, \jmath_1)} u^j(x_1, \jmath_1)\,d\jmath_1 \end{aligned} \tag{10}$$

since

$$\begin{aligned} &\sum_{j=0}^{k-1} L^{-1}\left(\frac{P_j(x_1, \jmath_1)}{P_k(x_1, \jmath_1)}\right) u^j(x_1, \jmath_1) \\ &\quad = \sum_{j=0}^{k-1} \int_a^{x_1} \frac{(x_1 - \jmath_1)^{k-1}}{(k-1)!} \frac{P_j(x_1, \jmath_1)}{P_k(x_1, \jmath_1)} u^j(x_1, \jmath_1)\,d\jmath_1. \end{aligned} \tag{11}$$

To illustrate the HPM, for nonlinear mixed VFIDE, let us consider (8):

$$\begin{aligned} \breve{H}(\vartheta, \wp) &= \vartheta(x_1, \jmath_1) - L^{-1}\left(\frac{\dot{f}(x_1, \jmath_1)}{P_k(x_1, \jmath_1)}\right) - \sum_{r=0}^{k-1} \frac{1}{(r!)} (x_1 \\ &- \jmath_1)^r g_r \\ &- \wp\left[L^{-1}\left(\int_a^{x_1}\int_\Omega \frac{F(x_1, \jmath_1, y, \tau)\,\gamma\left(\vartheta^l(y,\tau)\right)}{P_k(x_1, \jmath_1)}\,dy\,d\tau\right)\right. \\ &\left. - L^{-1}\left(\sum_{j=0}^{k-1} \frac{P_j(x_1, \jmath_1)}{P_k(x_1, \jmath_1)} \vartheta^j(x_1, \jmath_1)\right)\right] = 0. \end{aligned} \tag{12}$$

By the HPM, we can expand $\vartheta(x_1, \jmath_1)$ into the form

$$\begin{aligned} \vartheta(x_1, \jmath_1) &= u_0(x_1, \jmath_1) + \wp u_1(x_1, \jmath_1) + \wp^2 u_2(x_1, \jmath_1) \\ &+ \cdots = \sum_{i=0}^{\infty} \wp^i u_i(x_1, \jmath_1) = u(x_1, \jmath_1). \end{aligned} \tag{13}$$

When $\wp \to 1$, the approximate solution is obtained with

$$\begin{aligned} u(x_1, \jmath_1) &= \lim_{\wp \to 1} \vartheta(x_1, \jmath_1) \\ &= u_0(x_1, \jmath_1) + u_1(x_1, \jmath_1) + u_2(x_1, \jmath_1) + \cdots \end{aligned} \tag{14}$$

and in sum, according to [10], He's HPM considers the nonlinear term $\gamma(u)$ as

$$\gamma(u)=\sum_{i=0}^{\infty}\wp^{i}\breve{H}_{i}=\breve{H}_{0}+\wp\breve{H}_{1}+\wp^{2}\breve{H}_{2}+\cdots, \tag{15}$$

where $H_n$'s are the so-called He's polynomials [10], which can be calculated by using the formula

$$\breve{H}_{n}=\frac{1}{n}\frac{\partial^{n}}{\partial\wp^{n}}\left[\gamma\left(\sum_{i=0}^{\infty}\wp^{i}u_{i}\right)\right]_{\wp=0},\quad n=0,1,2,\ldots. \tag{16}$$

Substituting (13) and (15) into (12) and equating the terms with identical powers of $\wp$, we have

$$\begin{aligned}
\wp^{0}&: u_{0}(x_{1},\jmath_{1})\\
&=L^{-1}\left(\frac{\dot{f}(x_{1},\jmath_{1})}{P_{k}(x_{1},\jmath_{1})}\right)+\sum_{r=0}^{k-1}\frac{1}{(r!)}(x_{1}-\jmath_{1})^{r}g_{r},\\
\wp^{i+1}&: u_{i+1}(x_{1},\jmath_{1})\\
&=L^{-1}\left(\int_{a}^{x_{1}}\int_{\Omega}\frac{F(x_{1},\jmath_{1},y,\tau)}{P_{k}(x_{1},\jmath_{1})}\breve{H}_{i}dy\,d\tau\right)\\
&\quad-L^{-1}\left(\sum_{j=0}^{k-1}\frac{P_{j}(x_{1},\jmath_{1})}{P_{k}(x_{1},\jmath_{1})}L_{i_{j}}\right),\quad i\geq 0.
\end{aligned} \tag{17}$$

The nonlinear terms $\gamma(u^{l}(x_1,\jmath_1))$ and $D^{j}(u(x_1,\jmath_1))$ ($D^{j}=\partial^{j}u(x_1,\jmath_1)/\partial x_1^{j}$ is a derivative operator) are usually represented by an infinite series of the so-called He's polynomials as follows:

$$\begin{aligned}
\gamma\left(u^{l}(x_{1},\jmath_{1})\right)&=\sum_{i=0}^{\infty}\breve{H},\\
D^{j}\left(u(x_{1},\jmath_{1})\right)&=\sum_{i=0}^{\infty}L_{i_{j}}.
\end{aligned} \tag{18}$$

The components $u_i(x_1,\jmath_1)$, $i\geq 0$, can be computed by using the recursive relations (17).

## 4. A New Formula to He's Polynomials

He's polynomials are not unique; another formula of He's polynomials ($\breve{H}_n$), called accelerated He's polynomials, is represented by ($\overline{H}_n$); in [11], the author proved that

$$\gamma(u)=\sum_{n=0}^{\infty}\breve{H}_{n}=\sum_{n=0}^{\infty}\overline{H}_{n} \tag{19}$$

in which $\breve{H}_n$ can be written in the new mathematical form

$$\overline{H}_{n}=\gamma(\delta_{n})-\sum_{i=0}^{n-1}\overline{H}_{i}, \tag{20}$$

where the partial sum $\delta_n=\sum_{i=0}^{n}u_i(x_1,\jmath_1)$ and $\overline{H}_0=\gamma(u_0)$. Substituting (13) and $\gamma(u)=\sum_{n=0}^{\infty}\wp^{n}\overline{H}_n$ into (12) and equating the terms with identical powers of $\wp$, we obtain the following accelerated recursive formula:

$$\begin{aligned}
\wp^{0}&: u_{0}(x_{1},\jmath_{1})\\
&=L^{-1}\left(\frac{\dot{f}(x_{1},\jmath_{1})}{P_{k}(x_{1},\jmath_{1})}\right)+\sum_{r=0}^{k-1}\frac{1}{(r!)}(x_{1}-\jmath_{1})^{r}g_{r},\\
\wp^{i+1}&: u_{i+1}(x_{1},\jmath_{1})\\
&=L^{-1}\left(\int_{a}^{x_{1}}\int_{\Omega}\frac{F(x_{1},\jmath_{1},y,\tau)}{P_{k}(x_{1},\jmath_{1})}\overline{H}_{i}dy\,d\tau\right)\\
&\quad-L^{-1}\left(\sum_{j=0}^{k-1}\frac{P_{j}(x_{1},\jmath_{1})}{P_{k}(x_{1},\jmath_{1})}L_{i_{j}}\right),\quad i\geq 0.
\end{aligned} \tag{21}$$

For example, if $\gamma(u)=u^3$, the first four polynomials using formulas (16) and (20) are computed to be as follows.

Using formula (16),

$$\begin{aligned}
\breve{H}_{0}&=u_{0}^{3}.\\
\breve{H}_{1}&=3u_{0}^{2}u_{1}.\\
\breve{H}_{2}&=3u_{0}u_{1}^{2}+3u_{0}^{2}u_{2}.\\
\breve{H}_{3}&=u_{1}^{3}+6u_{0}u_{1}u_{2}+3u_{0}^{2}u_{3}.\\
\breve{H}_{4}&=3u_{1}^{2}u_{2}+3u_{0}u_{2}^{2}+6u_{0}u_{1}u_{3}+3u_{0}^{2}u_{4}.
\end{aligned} \tag{22}$$

Using formula (20),

$$\begin{aligned}
\overline{H}_{0}&=u_{0}^{3}.\\
\overline{H}_{1}&=3u_{0}^{2}u_{1}+3u_{0}u_{1}^{2}+u_{1}^{3}.\\
\overline{H}_{2}&=3u_{0}^{2}u_{2}+3u_{0}u_{2}^{2}+3u_{1}^{2}u_{2}+3u_{1}u_{2}^{2}+6u_{0}u_{1}u_{2}\\
&\quad+u_{2}^{3}.\\
\overline{H}_{3}&=3u_{0}^{2}u_{3}+3u_{0}u_{3}^{2}+3u_{1}^{2}u_{3}+3u_{1}u_{3}^{2}+3u_{2}^{2}u_{3}\\
&\quad+3u_{2}u_{3}^{2}+6u_{0}u_{1}u_{3}+6u_{0}u_{2}u_{3}+6u_{1}u_{2}u_{3}\\
&\quad+u_{3}^{3}.\\
\overline{H}_{4}&=3u_{0}^{2}u_{4}+3u_{0}u_{4}^{2}+3u_{1}^{2}u_{4}+3u_{1}u_{4}^{2}+3u_{2}^{2}u_{4}\\
&\quad+3u_{2}u_{4}^{2}+3u_{3}^{2}u_{4}+3u_{3}u_{4}^{2}+6u_{0}u_{1}u_{4}\\
&\quad+6u_{0}u_{2}u_{4}+6u_{0}u_{3}u_{4}+6u_{1}u_{2}u_{4}+6u_{1}u_{3}u_{4}\\
&\quad+6u_{2}u_{3}u_{4}+u_{4}^{3}.
\end{aligned} \tag{23}$$

Clearly, the first four polynomials computed using the suggested formula (20) include the first four polynomials computed using formula (16) in addition to other terms that should appear in $\breve{H}_5,\breve{H}_6,\breve{H}_7,\ldots$ using formula (16). Thus, the solution that was obtained using formula (20) enforces many terms to the calculation processes earlier, yielding faster convergence.

## 5. The VIM

Consider the differential equation

$$\imath u + \aleph u = \dot{f}(x_1), \tag{24}$$

where $\imath$ is a linear operator, $\aleph$ is a nonlinear operator, and $\dot{f}(x_1)$ is a given continuous function. The VIM presents a correction functional for (24) in the form

$$u_{i+1}(x_1) = u_i(x_1) + \int_0^{x_1} \lambda(\varsigma)\left(\imath u_i(\varsigma) + \aleph \tilde{u}_i(\varsigma) - \dot{f}(\varsigma)\right) d\varsigma, \tag{25}$$

where $\lambda$ is a Lagrange multiplier [8, 9] which can be identified optimally via variational theory, $u_i$ is the $n$th approximate solution, and $\tilde{u}_i$ denotes a restricted variation (i.e., $\varrho u_i = 0$).

## 6. Adapting VIM with HPM for Solving (1) and (2)

This modified version of VHPM is obtained by the coupling of VIM with HPM. First, by using formula (16), we obtain

$$\begin{aligned}\sum_{i=0}^{\infty}\wp^i u_{i+1}(x_1, \jmath_1) &= u_i(x_1, \jmath_1) + \wp\int_0^{\jmath_1}\lambda(\varsigma) \\ &\cdot\left[\sum_{i=0}^{\infty}\wp^i u_i^k(x_1,\varsigma) - \frac{\dot{f}(x_1,\varsigma)}{P_k(x_1,\varsigma)}\right. \\ &- \int_a^{x_1}\int_\Omega \frac{F(x_1,\varsigma,y,\tau)}{P_k(x_1,\varsigma)}\sum_{i=0}^{\infty}\wp^i \breve{H}_i dy\, d\tau \\ &\left.+ \sum_{j=0}^{k-1}\frac{P_j(x_1,\varsigma)}{P_k(x_1,\varsigma)}\left(\sum_{i=0}^{\infty}\wp^i u_i^j(x_1,\varsigma)\right)\right] d\varsigma\end{aligned} \tag{26}$$

which is called VHPM.

Second, by using formula (20),

$$\begin{aligned}\sum_{i=0}^{\infty}\wp^i u_{i+1}(x_1, \jmath_1) &= u_i(x_1, \jmath_1) + \wp\int_0^{\jmath_1}\lambda(\varsigma) \\ &\cdot\left[\sum_{i=0}^{\infty}\wp^i u_i^k(x_1,\varsigma) - \frac{\dot{f}(x_1,\varsigma)}{P_k(x_1,\varsigma)}\right. \\ &- \int_a^{x_1}\int_\Omega \frac{F(x_1,\varsigma,y,\tau)}{P_k(x_1,\varsigma)}\sum_{i=0}^{\infty}\wp^i \overline{H}_i dy\, d\tau \\ &\left.+ \sum_{j=0}^{k-1}\frac{P_j(x_1,\varsigma)}{P_k(x_1,\varsigma)}\left(\sum_{i=0}^{\infty}\wp^i u_i^j(x_1,\varsigma)\right)\right] d\varsigma\end{aligned} \tag{27}$$

which is called VAHPM.

The following is the algorithm for calculating $u_0, u_1, u_2, \ldots, u_{n-1}, u_n$:

Step 1: input nonlinear term $\gamma(u^l)$ and $n$ that is the order of He's polynomials, endpoint $a, b$, initial conditions $g_1, g_2, \ldots, g_r$, free term, and $F(x_1, \jmath_1, y, \tau)$.

Step 2: set $u = u_0 + \wp u_1 + \wp^2 u_2 + \cdots + \wp^n u_n$.

Step 3: let $\sum_{k=0}^{n}\wp^k \breve{H}_k = \gamma(u_0 + \wp u_1 + \wp^2 u_2 + \cdots + \wp^n u_n)$.

Step 4: For $i = 0, 1, \ldots, n$, do

(a) $i$th-order derivative of both sides of the equality with respect to $\wp$:

$$\frac{\partial^i\left(\sum_{k=0}^{n}\wp^k\breve{H}_k\right)}{\partial\wp^i} = \frac{\partial\gamma\left(u_0 + \wp u_1 + \wp^2 u_2 + \cdots + \wp^n u_n\right)}{\partial\wp^i}; \tag{28}$$

(b) let $\wp = 0$ of the above equality and determine $\breve{H}_i$ by solving the equation with respect to $\breve{H}_i$.

End do.

Step 5: put $u_0$ = initial conditions.

Step 6: for $i = 1, \ldots, n$, do

Step 7: calculate $u_i$ by applying (26),

end do.

Step 8: set $u = \lim_{i\to\infty} u_i$ as the approximate of the exact solution.

## 7. Convergence Analysis

In this section, the sufficient condition that guarantees the existence of a unique solution is introduced in Theorem 1, convergence of the methods is proved in Theorems 2 and 3, and finally the maximum absolute error of the truncated series ($u_i(x_1, \jmath_1) = \sum_{i=0}^{\infty} u_i(x_1, \jmath_1)$) is estimated in Theorem 4.

Considering (10), we set

$$\begin{aligned} L^{-1}\left(\frac{\dot{f}(x_1,\jmath_1)}{P_k(x_1,\jmath_1)}\right) + \sum_{r=0}^{k-1}\frac{1}{(r!)}(x_1-\jmath_1)^r g_r &= f_1(x_1,\jmath_1), \\ k_1(x_1,\jmath_1) &= \int_\Omega \frac{(x_1-\jmath_1)^k}{(k)!}\frac{F(x_1,\jmath_1,y,\tau)}{P_k(x_1,\jmath_1)}dy \\ k_2(x_1,\jmath_1) &= \frac{(x_1-\jmath_1)^{k-1}}{(k-1)!}\frac{P_j(x_1,\jmath_1)}{P_k(x_1,\jmath_1)}. \end{aligned} \tag{29}$$

We can write (10) as

$$u(x_1, \jmath_1) = f_1(x_1, \jmath_1) + \int_a^{x_1} k_1(x_1, \jmath_1)\gamma(u^l(y,\tau))\,d\tau - \cdot \sum_{j=0}^{k-1}\int_a^{x_1} k_2(x_1, \jmath_1) u^j(x_1, \jmath_1)\,d\jmath_1. \quad (30)$$

We assume $f_1(x_1, \jmath_1)$ is bounded for all $x_1, \jmath_1$ in $\acute{\jmath}$ and

$$\left|k_1(x_1, \jmath_1)\right| \le N_1$$
$$\left|k_2(x_1, \jmath_1)\right| \le N_{1_j}, \quad j = 0, 1, \ldots, k-1, \ \forall x_1, \jmath_1 \in \acute{\jmath}. \quad (31)$$

Also, we suppose the nonlinear terms $\gamma(u^l(x_1, \jmath_1))$ and $D^j(u(x_1, \jmath_1))$ are Lipschitz continuous with

$$\left|\gamma\left(u^{(l)}(x_1, \jmath_1)\right) - \gamma\left(u^{(l)^*}(x_1, \jmath_1)\right)\right| \le d\left|u(x_1, \jmath_1) - u^*(x_1, \jmath_1)\right|$$
$$\left|D^j(u(x_1, \jmath_1)) - D^j(u^*(x_1, \jmath_1))\right| \le Z_j\left|u(x_1, \jmath_1) - u^*(x_1, \jmath_1)\right|, \quad j = 0, 1, \ldots, k-1. \quad (32)$$

Hence, we set

$$\Gamma = (b-a)(dN_1 + kZN),$$
$$Z = \max\left|Z_j\right|,$$
$$N = \max\left|N_{1_j}\right|, \quad j = 0, 1, \ldots, k-1. \quad (33)$$

**Theorem 1.** *Two-dimensional nonlinear VFIDE has a unique solution whenever* $0 < \Gamma < 1$.

*Proof.* Let $u$ and $u^*$ be two different solutions of (30). Then,

$$\begin{aligned}
\left|u(x_1, \jmath_1) - u^*(x_1, \jmath_1)\right| &= \left|\int_a^{x_1} k_1(x_1, \jmath_1) \cdot \left[\gamma\left(u^{(l)}(x_1, \jmath_1)\right) - \gamma\left(u^{(l)^*}(x_1, \jmath_1)\right)\right]d\tau - \sum_{j=0}^{k-1}\int_a^{x_1} k_2(x_1, \jmath_1) \cdot \left[D^j(u(x_1, \jmath_1)) - D^j(u^*(x_1, \jmath_1))\right]d\jmath_1\right| \\
&\le \int_a^{x_1}\left|k_1(x_1, \jmath_1)\right|\left|\gamma\left(u^{(l)}(x_1, \jmath_1)\right) - \gamma\left(u^{(l)^*}(x_1, \jmath_1)\right)\right|d\tau + \sum_{j=0}^{k-1}\int_a^{x_1}\left|k_2(x_1, \jmath_1)\right| \cdot \left|D^j(u(x_1, \jmath_1)) - D^j(u^*(x_1, \jmath_1))\right|d\jmath_1 \\
&\le (b-a)(dN_1 + kZN)\left|u(x_1, \jmath_1) - u^*(x_1, \jmath_1)\right| \\
&= \Gamma\left|u(x_1, \jmath_1) - u^*(x_1, \jmath_1)\right|,
\end{aligned} \quad (34)$$

from which we get $(1-\Gamma)|u - u^*| \le 0$. Since $0 < \Gamma < 1$, therefore $|u - u^*| = 0$. Therefore, $u = u^*$ and this completes the proof. □

**Theorem 2.** *The series solution* $u(x_1, \jmath_1) = \sum_{i=0}^{\infty} u_i(x_1, \jmath_1)$ *of (1) using HPM convergence when* $0 < \Gamma < 1$ *and* $\|u_1(x_1, \jmath_1)\| < \infty$.

*Proof.* Denote with $(C[\acute{\jmath}], \|\cdot\|)$ the Banach space of all continuous functions on $\acute{\jmath}$ with the norm $\|\dot{f}(x_1, \jmath_1)\| = \max|\dot{f}(x_1, \jmath_1)|$ for all $x_1, \jmath_1$ in $\acute{\jmath}$. Define the sequence of partial sums $\delta_n$, and let $\delta_n$ and $\delta_m$ be arbitrary partial sums with $n \ge m$. We are going to prove that $\delta_n = \sum_{i=0}^{n} u_i(x_1, \jmath_1)$ is a Cauchy sequence in this Banach space:

$$\begin{aligned}
\left\|\delta_n - \delta_m\right\| &= \max_{\forall x_1, \jmath_1 \in \acute{\jmath}}\left|\delta_n - \delta_m\right| \\
&= \max_{\forall x_1, \jmath_1 \in \acute{\jmath}}\left|\sum_{i=m+1}^{n} u_i(x_1, \jmath_1)\right| \\
&= \max_{\forall x_1, \jmath_1 \in \acute{\jmath}}\left|\sum_{i=m+1}^{n}\left[\int_a^{x_1} k_1(x_1, \jmath_1)\overline{H}_i\,d\tau - \sum_{j=0}^{k-1}\int_a^{x_1} k_2(x_1, \jmath_1) L_{i_j}\,d\jmath_1\right]\right| \\
&= \max_{\forall x_1, \jmath_1 \in \acute{\jmath}}\left|\int_a^{x_1} k_1(x_1, \jmath_1)\left(\sum_{i=m}^{n-1}\overline{H}_i\right)d\tau - \sum_{j=0}^{k-1}\int_a^{x_1} k_2(x, \jmath_1)\left(\sum_{i=m}^{n-1} L_{i_j}\right)d\jmath_1\right|.
\end{aligned} \quad (35)$$

From (20), we have

$$\sum_{i=m}^{n-1}\overline{H}_i = \gamma(\delta_{n-1}) - \gamma(\delta_{m-1})$$
$$\sum_{i=m}^{n-1} L_{i_j} = D^j(\delta_{n-1}) - D^j(\delta_{m-1}). \quad (36)$$

So,

$$\begin{aligned}
&\|\delta_n - \delta_m\| \\
&= \max_{\forall x_1, \jmath_1 \in \acute{\jmath}} \left( \left| \int_a^{x_1} k_1(x_1, \jmath_1)\left[\gamma(\delta_{n-1}) - \gamma(\delta_{m-1})\right] d\tau \right.\right. \\
&\quad \left.\left. - \sum_{j=0}^{k-1} \int_a^{x_1} k_2(x_1, \jmath_1)\left[D^j(\delta_{n-1}) - D^j(\delta_{m-1})\right] d\jmath_1 \right| \right) \\
&\leq \max_{\forall x_1, \jmath_1 \in \acute{\jmath}} \left( \int_a^{x_1} |k_1(x_1, \jmath_1)| \left|\gamma(\delta_{n-1}) - \gamma(\delta_{m-1})\right| d\tau \right. \\
&\quad \left. + \sum_{j=0}^{k-1} \left( \int_a^{x_1} |k_2(x_1, \jmath_1)| \left|D^j(\delta_{n-1}) - D^j(\delta_{m-1})\right| d\jmath_1 \right) \right) \\
&\leq \Gamma \|\delta_{n-1} - \delta_{m-1}\|.
\end{aligned} \tag{37}$$

Let $n = m + 1$; then,

$$\begin{aligned}
\|\delta_{m+1} - \delta_m\| &\leq \Gamma \|\delta_m - \delta_{m-1}\| \leq \Gamma^2 \|\delta_{m-1} - \delta_{m-2}\| \\
&\leq \cdots \leq \Gamma^m \|\delta_1 - \delta_0\|.
\end{aligned} \tag{38}$$

So,

$$\begin{aligned}
\|\delta_n - \delta_m\| &\leq \|\delta_{m+1} - \delta_m\| + \|\delta_{m+2} - \delta_{m+1}\| + \cdots \\
&\quad + \|\delta_n - \delta_{n-1}\| \\
&\leq \left[\Gamma^m + \Gamma^{m+1} + \cdots + \Gamma^{n-1}\right] \|\delta_1 - \delta_0\| \\
&\leq \Gamma^m \left[1 + \Gamma + \Gamma^2 + \cdots + \Gamma^{n-m-1}\right] \|\delta_1 - \delta_0\| \\
&\leq \Gamma^m \left[\frac{1 - \Gamma^{n-m}}{1 - \Gamma}\right] \|u_1(x_1, \jmath_1)\|.
\end{aligned} \tag{39}$$

Since $0 < \Gamma < 1$, we have $(1 - \Gamma^{n-m}) < 1$; then,

$$\|\delta_n - \delta_m\| \leq \frac{\Gamma^m}{1 - \Gamma} \max_{\forall x_1, \jmath_1 \in \acute{\jmath}} \|u_1(x_1, \jmath_1)\|. \tag{40}$$

But $|u_1(x_1, \jmath_1)| < \infty$ (since $\dot{f}_1(x_1, \jmath_1)$ is bounded), so, as $m \to \infty$, then $\|\delta_n - \delta_m\| \to 0$. We conclude that $\delta_n$ is a Cauchy sequence in $C[\acute{J}]$, and therefore the series is convergent and the proof is complete. □

**Theorem 3.** *When using VIM for solving two-dimensional nonlinear VFIDE where $0 < \Gamma < 1$ and $P_k(x_1, \jmath_1) = 1$, then $u(x_1, \jmath_1) = \lim_{n \to \infty} u_n(x_1, \jmath_1)$ converges.*

*Proof.* One has

$$\begin{aligned}
u_{n+1}(x_1, \jmath_1) &= u_n(x_1, \jmath_1) - \int_0^{\jmath_1} \left[ u_n(x_1, \varsigma) \right. \\
&\quad - f_1(x_1, \varsigma) - \int_a^{x_1} k_1(x_1, \varsigma)\, \gamma\left(u_n^l(x_1, \varsigma)\right) d\tau \\
&\quad \left. + \sum_{j=0}^{k-1} \int_a^{x_1} k_2(x_1, \varsigma) (u_n)^j (x_1, \varsigma)\, d\jmath_1 \right] d\varsigma
\end{aligned} \tag{41}$$

$$\begin{aligned}
u(x_1, \jmath_1) &= u(x_1, \jmath_1) - \int_0^{\jmath_1} \left[ u(x_1, \varsigma) - f_1(x_1, \varsigma) \right. \\
&\quad - \int_a^{x_1} k_1(x_1, \varsigma)\, \gamma\left(u^l(x_1, \varsigma)\right) d\tau \\
&\quad \left. + \sum_{j=0}^{k-1} \int_a^{x_1} k_2(x_1, \varsigma) (u)^j (x_1, \varsigma)\, d\jmath_1 \right] d\varsigma.
\end{aligned} \tag{42}$$

By subtracting relation (41) from (42),

$$\begin{aligned}
u_{n+1}(x_1, \jmath_1) - u(x_1, \jmath_1) &= u_n(x_1, \jmath_1) - u(x_1, \jmath_1) - \int_0^{\jmath_1} \left[ u_n(x_1, \varsigma) - u(x_1, \varsigma) \right. \\
&\quad \left. - \int_a^{x_1} k_1(x_1, \varsigma)\left[\gamma\left(u_n^l(x_1, \varsigma)\right) - \gamma\left(u^l(x_1, \varsigma)\right)\right] d\tau + \sum_{j=0}^{k-1} \int_a^{x_1} k_2(x_1, \varsigma)\left[D^j(u_n(x_1, \varsigma)) - D^j(u(x_1, \varsigma))\, d\jmath_1\right] \right] d\varsigma.
\end{aligned} \tag{43}$$

Hence, we set

$$\begin{aligned}
e_{n+1}(x_1, \jmath_1) &= u_{n+1}(x_1, \jmath_1) - u(x_1, \jmath_1), \\
e_n(x_1, \jmath_1) &= u_n(x_1, \jmath_1) - u(x_1, \jmath_1).
\end{aligned} \tag{44}$$

Then,

$$\begin{aligned}
e_{n+1}(x_1, \jmath_1) &= e_n(x_1, \jmath_1) - \int_0^{\jmath_1} \left[ u_n(x_1, \varsigma) - u(x_1, \varsigma) \right. \\
&\quad - \int_a^{x_1} k_1(x_1, \varsigma)\left[\gamma\left(u_n^l(x_1, \varsigma)\right) - \gamma\left(u^l(x_1, \varsigma)\right)\right] d\tau \\
&\quad + \sum_{j=0}^{k-1} \int_a^{x_1} k_2(x_1, \varsigma) \\
&\quad \left. \cdot \left[D^j(u_n(x_1, \varsigma)) - D^j(u(x_1, \varsigma))\right] d\jmath_1 \right] d\varsigma \\
&\leq e_n(x_1, \jmath_1)\left(1 - (b - a)(dN_1 + kZN)\right) = (1 - \Gamma) \\
&\quad \cdot e_n(x_1, \jmath_1).
\end{aligned} \tag{45}$$

Therefore,

$$\|e_{n+1}\| = \max_{\forall x_1, \jmath_1 \in \acute{J}} |e_{n+1}| \le (1-\Gamma) \max_{\forall x_1, \jmath_1 \in \acute{J}} |e_n| = \|e_n\|. \quad (46)$$

Since $0 < \Gamma < 1$, then $\|e_n\| \to 0$. So, the series converges and the proof is complete. □

**Theorem 4.** *The maximum absolute truncation error of the series $u(x_1, \jmath_1) = \sum_{i=0}^{\infty} u_i(x_1, \jmath_1)$ to (1) is estimated to be*

$$\begin{aligned} &\max_{\forall x_1, \jmath_1 \in \acute{J}} \left| u(x_1, \jmath_1) - \sum_{i=0}^{m} u_i(x_1, \jmath_1) \right| \\ &\quad \le \frac{\Gamma^{m+1}}{1-\Gamma} \max_{\forall x_1, \jmath_1 \in \acute{J}} |\gamma(u_0)|. \end{aligned} \quad (47)$$

*Proof.* From Theorem 2 and inequality (40), we have

$$\|\delta_n - \delta_m\| \le \frac{\Gamma^m}{1-\Gamma} \max_{\forall x_1, \jmath_1 \in \acute{J}} \|u_1(x_1, \jmath_1)\|. \quad (48)$$

As $n \to \infty$, then $\delta_n \to u(x_1, \jmath_1)$ and we have

$$\begin{aligned} &\max_{\forall x, t \in \acute{J}} |u_1(x_1, \jmath_1)| \\ &\quad \le (b-a)(dN_1 + kZN) \max_{\forall x_1, \jmath_1 \in \acute{J}} |\gamma(u_0)| \\ &\quad \le \Gamma \max_{\forall x_1, \jmath_1 \in \acute{J}} |\gamma(u_0)|. \end{aligned} \quad (49)$$

So,

$$\|u(x_1, \jmath_1) - \delta_m\| \le \frac{\Gamma^{m+1}}{1-\Gamma} \max_{\forall x_1, \jmath_1 \in \acute{J}} |\gamma(u_0)|. \quad (50)$$

Finally, the maximum absolute truncation error in the interval $\acute{J}$ is

$$\begin{aligned} &\max_{\forall x_1, \jmath_1 \in \acute{J}} \left| u(x_1, \jmath_1) - \sum_{i=0}^{m} u_i(x_1, \jmath_1) \right| \\ &\quad \le \frac{\Gamma^{m+1}}{1-\Gamma} \max_{\forall x_1, \jmath_1 \in \acute{J}} |\gamma(u_0)|. \end{aligned} \quad (51)$$

□

## 8. Numerical Examples

*Example 1.* Consider the nonlinear integrodifferential equation

$$\begin{aligned} &\frac{\partial^2 u(x_1, \jmath_1)}{\partial \jmath_1^2} + \frac{\partial^2 u(x_1, \jmath_1)}{\partial x_1 \partial \jmath_1} - x_1 u^3(x_1, \jmath_1) \\ &\quad + \int_0^{x_1} \int_0^1 u^2(y, \tau)\, dy\, d\tau = \acute{f}(x_1, \jmath_1), \\ &\qquad x_1 \in [0, 1], \end{aligned} \quad (52)$$

where

$$\acute{f}(x_1, \jmath_1) = 2\jmath_1 + \frac{1}{15} x_1^3 \jmath_1^5 - x_1^4 \jmath_1^5 + 2x_1, \quad (53)$$

with the initial conditions

$$\text{I.Cs}: \begin{cases} u(x_1, 0) = 0, \\ \dfrac{\partial u(x_1, 0)}{\partial \jmath_1} = 0, \end{cases} \quad (54)$$

which has exact solution $u(x_1, t_1) = x_1 \jmath_1^2$. This example is solved by using the variational iteration method with He's polynomials VHPM (see (26)) and VAHPM (see (27)) expressing the nonlinear terms of $\breve{H}$ and $\overline{H}$, respectively, in Table 1.

*Example 2.* Consider the nonlinear integrodifferential equation

$$\begin{aligned} &u(x_1, \jmath_1) \frac{\partial^2 u(x_1, \jmath_1)}{\partial \jmath_1^2} - 4u(x_1, \jmath_1) \frac{\partial^2 u(x_1, \jmath_1)}{\partial x_1^2} \\ &\quad + 4 \int_0^{x_1} \int_0^1 u^2(y, \tau)\, dy\, d\tau = \acute{f}(x_1, \jmath_1), \\ &\qquad x_1 \in [0, 1], \end{aligned} \quad (55)$$

where

$$\begin{aligned} &\acute{f}(x_1, \jmath_1) \\ &\quad = \left( x_1 - \frac{1}{2\pi} \sin(2\pi x_1) \right) \left( \jmath_1 - \frac{1}{4\pi} \sin(4\pi \jmath_1) \right), \end{aligned} \quad (56)$$

with the boundary conditions

$$\text{B.Cs}: u(0, \jmath_1) = u(1, \jmath_1) = 0 \quad (57)$$

and the initial conditions

$$\text{I.Cs}: \begin{cases} u(x_1, 0) = \sin(\pi x_1), & 0 \le x_1 \le 1 \\ \dfrac{\partial u(x_1, 0)}{\partial \jmath_1} = 0, & 0 \le x_1 \le 1, \end{cases} \quad (58)$$

which has exact solution $u(x_1, \jmath_1) = \sin(\pi x_1) \cos(2\pi \jmath_1)$. This example is solved by using VHPM (see (26)) and VAHPM (see (27)) expressing the nonlinear terms of $\breve{H}$ and $\overline{H}$, respectively, in Table 2.

## 9. Conclusion

In this paper, we applied VHPM and VAHPM to solve nonlinear mixed VFIDE. The proposed VAHPM converges faster than the VHPM. Based on the proposed formula (27) with accelerated He's polynomials formula (20), the convergence of the technique is proved. The presented technique is very easy to implement and it reduces the computation size.

Table 1: Exact solution, approximate solution, and error by using VHPM and VAHPM.

| $x_1$ | $y_1$ | Exact | $\text{Appr.}_{(\text{VHPM})}$ | $\text{Err.}_{(\text{VHPM})}$ | $\text{Appr.}_{(\text{VAHPM})}$ | $\text{Err.}_{(\text{VAHPM})}$ |
|---|---|---|---|---|---|---|
| $0.00E+00$ | $0.00E+00$ | $0.00000E+00$ | $0.00000E+00$ | $0.00000E+00$ | $0.00000E+00$ | $0.00000E+00$ |
| $3.00E-03$ | $3.00E-03$ | $2.70000E-08$ | $9.90000E-08$ | $2.70000E-08$ | $3.60000E-08$ | $9.00000E-09$ |
| $5.00E-03$ | $5.00E-03$ | $1.25000E-07$ | $4.58330E-07$ | $3.33333E-07$ | $1.66667E-07$ | $4.16670E-08$ |
| $7.00E-03$ | $7.00E-03$ | $3.43000E-07$ | $1.25767E-06$ | $9.14670E-07$ | $4.57333E-07$ | $1.14333E-07$ |
| $9.00E-03$ | $9.00E-02$ | $7.290000E-07$ | $2.67300E-06$ | $1.94400E-06$ | $9.72000E-07$ | $2.43000E-07$ |

Table 2: Exact solution, approximate solution, and error by using VHPM and VAHPM.

| $x_1$ | $y_1$ | Exact | $\text{Appr.}_{(\text{VHPM})}$ | $\text{Err.}_{(\text{VHPM})}$ | $\text{Appr.}_{(\text{VAHPM})}$ | $\text{Err.}_{(\text{VAHPM})}$ |
|---|---|---|---|---|---|---|
| $0.00E+00$ | $0.00E+00$ | $0.00000E+00$ | $0.00000E+00$ | $0.00000E+00$ | $0.00000E+00$ | $0.00000E+00$ |
| $3.00E-03$ | $3.00E-03$ | $9.42295E-03$ | $9.41128E-03$ | $1.16700E-05$ | $9.42443E-03$ | $1.48000E-06$ |
| $5.00E-03$ | $5.00E-03$ | $1.56997E-02$ | $1.56384E-02$ | $6.13000E-05$ | $1.57054E-02$ | $5.70000E-06$ |
| $7.00E-03$ | $7.00E-03$ | $2.19680E-02$ | $2.17975E-02$ | $1.70500E-04$ | $2.19817E-02$ | $1.37000E-05$ |
| $9.00E-03$ | $9.00E-02$ | $2.82253E-02$ | $2.78610E-02$ | $3.64300E-04$ | $2.82534E-02$ | $2.81000E-05$ |

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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