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Research Article

Existence and Uniqueness of Periodic Solutions for a Kind of Second-Order Neutral Functional Differential Equation with Delays

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We consider a kind of second-order neutral functional differential equation. On the basis of Mawhin's coincidence degree, the existence and uniqueness of periodic solutions are proved. It is indicated that the result is related to the deviating arguments. Moreover, we present two simulations to demonstrate the validity of analytical conclusion.

1. Introduction

In this paper, we consider a kind of second-order neutral functional differential equations in the following form:

$$\begin{aligned}(u(t)-ku(t-\tau))'' &= f(u(t))u'(t)+\alpha(t)g(u(t)) \\ &\quad+\sum_{i=1}^{n}\beta_i(t)h(u(t-\tau_i(t))) \\ &\quad+p(t),\end{aligned} \tag{1}$$

where $f,g,h \in C(R,R)$, $p(t)$, $\tau_i(t)$ $(i = 1,2,\dots,n)$ are continuous periodic functions defined on R with period $T > 0$, $\alpha(t)$, $\beta_i(t)$ $(i = 1,2,\dots,n)$ are continuous periodic functions defined on R and have the same sign, and $k,\tau \in R$ are constants such that $|k| \neq 1$.

As we know, neutral differential equations are widely used in physics, biology, medicine, chemistry, economics, ecology, aerospace, and so on. The research of their theory and algorithm is greatly important. Many authors devote themselves to research such kind of NFDE and get some results; see papers [1–15]. These papers were devoted mainly to studying the following types of equations:

$$\begin{aligned}(u(t)-ku(t-\delta))' &= g(t,u(t))+h(t,u(t-\tau(t))) \\ &\quad+p(t), \\ (u(t)-ku(t-\tau))'' &= au'(t)+h(u(t-\tau(t))) \\ &\quad+p(t), \\ (u(t)-ku(t-\tau))'' &= au'(t)+\alpha(t)g(u(t)) \\ &\quad+\sum_{i=1}^{n}\beta_i(t)h(u(t-\tau_i(t))) \\ &\quad+p(t).\end{aligned} \tag{2}$$

By using the coincidence degree theory, the existence and uniqueness of T-periodic solutions for (2) are obtained which are not related to delays. Since the time delay in some equations with practical application background is often very small, it is easier to miss. We know that even a small delay is also likely to have an important impact on the stability of the system. Therefore it is necessary for us to consider the influence of delays. Our work is based on such a background. In this paper, our aim is to establish some criteria to guarantee the existence and uniqueness of periodic solution for (1) by using Mawhin's continuation theorem.

To state our main theorems, we also need some notations as follows:

$$|x|_\infty = \max_{t\in[0,T]} |x(t)|, \qquad |x|_k = \left(\int_0^T |x(t)|^k dt\right)^{1/k}. \tag{3}$$

Throughout this paper we assume that $\tau_i \in C_T^1$, and $\tau_i'(t) < 1$, for all $t \in [0,T]$, $(i = 1, 2, \ldots, n)$.

2. Main Lemmas

Lemma 1 (Gaines and Mawhin [1]). *Let X be a Banach spaces. Suppose that $L : D(L) \subset X \to X$ is a Fredholm operator with index zero and $N : \overline{\Omega} \to X$ is L-compact on $\overline{\Omega}$, where Ω is an open bounded subset of X. Moreover, assume that all the following conditions are satisfied:*

(a) *$Lx \neq \lambda Nx$, for all $\lambda \in (0,1)$ and $x \in D(L) \cap \partial\Omega$.*

(b) *$Nx \notin \operatorname{Im} L$, for all $x \in \operatorname{Ker} L \cap \partial\Omega$.*

(c) $\deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

Lemma 2 (see [5]). *Let $x(t) \in X \cap C^1(R,R)$. Suppose that there exists a constant $D \geq 0$ such that*

$$|x(\tau_0)| \leq D, \quad \tau_0 \in [0,T]. \tag{4}$$

Then

$$|x|_2 \leq \frac{T}{\pi}|x'|_2 + \sqrt{T}D. \tag{5}$$

Lemma 3 (see [10]). *If $|k| \neq 1$, then A has continuous bounded inverse on C_T, and*

(a) *$\|A^{-1}x\| \leq \|x\| / ||k| - 1|$, for all $x \in C_T$;*

(b) *$\int_0^T |(A^{-1}f)(t)| dt \leq (1/|1 - |k||) \int_0^T |f(s)| ds$, for all $f \in C_T$;*

(c) *$\int_0^T |(A^{-1}f)(t)|^2 dt \leq (1/(1 - |k|)^2) \int_0^T |f(s)|^2 ds$, for all $f \in C_T$.*

Let $X = C_T = \{x \mid x \in C(R,R),\ x(t+T) = x(t)\}$ with the norm $\|x\|_X = |x|_\infty$, and $Y = C_T^1 = \{x \mid x \in C^1(R,R),\ x(t+T) = x(t)\}$ with the norm $\|x\|_Y = \max\{|x|_\infty, |x'|_\infty\}$. Clearly, X and Y are two Banach spaces. We also defined operators A and L in the following, respectively:

$$A : X \longrightarrow X, \qquad (Ax)(t) = u(t) - ku(t-\tau), \tag{6}$$

$$L : D(L) \subset X \longrightarrow X, \qquad Lx = (Ax)'', \tag{7}$$

where $D(L) = \{x \mid x \in C^2(R,R),\ x(t+T) \equiv x(t)\}$.

Next define a nonlinear operator $N : X \to X$ by setting

$$Nx = f(u(t))u'(t) + \alpha(t)g(u(t)) + \sum_{i=1}^n \beta_i(t)h(u(t-\tau_i(t))) + p(t). \tag{8}$$

By Hale's terminology [9], a solution $u(t)$ of (1) is $u \in C'(R,R)$ such that $Au \in C^2(R,R)$ and (1) are satisfied on R. In general, $u \notin C^2(R,R)$. But, under the condition $|k| \neq 1$, we see from Lemma 1 of [4] that $(Ax)'' = Ax''$. So a T-periodic solution $u(t)$ of (1) must be such that $u \in C^2(R,R)$. Meanwhile, according to Lemma 3, we can easily get that $\operatorname{Ker} L = R$, and $\operatorname{Im} L = \{x \mid x \in X,\ \int_0^T x(s)ds = 0\}$. Therefore, the operator L is a Fredholm operator with index zero. Define the continuous projectors $P : Y \to \operatorname{Ker} L$ and $Q : X \to X/\operatorname{Im} L$ by setting

$$Px = x(0), \qquad Qx = \frac{1}{T}\int_0^T x(s)\,ds. \tag{9}$$

Set $L_P = L|_{D(L)\cap \operatorname{Ker} P} : D(L) \cap \operatorname{Ker} P \to \operatorname{Im} L$; then L_P has continuous inverse L_P^{-1} defined by

$$L_P^{-1}y(t) = \left(A^{-1}Fy\right)(t), \tag{10}$$

where

$$(Fy)(t) = -\frac{1}{2}\int_0^T sy(s)\,ds + \int_0^T \frac{t}{T}sy(s)\,ds + \int_0^T (t-s)\,y(s)\,ds - \frac{1}{T}\int_0^T\int_0^u (u-s)\,y(s)\,ds\,du. \tag{11}$$

Therefore, it is easy to see from (7) and (8) that N is L-compact on $\overline{\Omega}$, where Ω is an open bounded set in X.

In view of (7) and (8), the operator equation

$$Lx = \lambda Nx \tag{12}$$

is equivalent to the following equation:

$$(u(t) - ku(t-\tau))'' = \lambda f(u(t))u'(t) + \lambda\alpha(t)g(u(t)) + \lambda\sum_{i=1}^n \beta_i(t)h(u(t-\tau_i(t))) + \lambda p(t). \tag{13}$$

Lemma 4 (see [8]). *Let $g \in C_T$, $\tau \in C_T^1$ with $\tau'(t) < 1$, for all $t \in [0,T]$. Then $g(\mu(t)) \in C_T$, where $\mu(t)$ is inverse function of $t - \tau(t)$.*

The following lemma which we obtained in [16] gives more accurate inequality about the deviating argument τ than Lemma 2 in [4].

Lemma 5 (see [16]). *Let $\tau \in (-T/2, 0)\cup(0, T/2)$ be a constant. Then for for all $x \in C^1(R, R) \cap C_T$, we have*

$$\int_0^T |x(t) - x(t-\tau)|^2 dt \le \tau^2 \left(1 + \frac{|\tau|}{2T}\right) \int_0^T |x'(t)|^2 dt. \quad (14)$$

Lemma 6. *Assume that the following conditions are satisfied:*

(H_1) *One of the following conditions holds:*

(a) $(g(x_1)-g(x_2))(x_1-x_2) > 0$, $(h(x_1)-h(x_2))(x_1-x_2) > 0$, *for all* $x_1, x_2 \in R$, $x_1 \ne x_2$,

(b) $(g(x_1)-g(x_2))(x_1-x_2) < 0$, $(h(x_1)-h(x_2))(x_1-x_2) < 0$, *for all* $x_1, x_2 \in R$, $x_1 \ne x_2$.

(H_2) *One of the following conditions holds:*

(a) *There exists two positive constants L_i $(i = 1, 2)$ such that* $|k|+(\alpha^* T^2/\pi^2)L_1+(nT^2\beta^*/\pi^2)L_2 < 1$;

$$\begin{aligned} &|g(x_1) - g(x_2)| \le L_1 |x_1 - x_2|, \quad \forall x_1, x_2 \in R; \\ &|h(x_1) - h(x_2)| \le L_2 |x_1 - x_2|, \quad \forall x_1, x_2 \in R. \end{aligned} \quad (15)$$

(b) *There exists three positive constants L_i $(i = 2)$ such that* $1+(\alpha^* T^2/\pi^2)L_1+(nT^2\beta^*/\pi^2)L_2 < |k|$;

$$\begin{aligned} &|g(x_1) - g(x_2)| \le L_1 |x_1 - x_2|, \quad \forall x_1, x_2 \in R; \\ &|h(x_1) - h(x_2)| \le L_2 |x_1 - x_2|, \quad \forall v_1, v_2 \in R, \end{aligned} \quad (16)$$

where $\alpha^* = \max_{t\in[0,T]}|\alpha(t)|$ *and* $\beta^* = \max_{t\in[0,T]}|\beta_i(t)|$.

Then (1) has at most one T-periodic solution.

Proof. Suppose that $u_1(t)$ and $u_2(t)$ are two T-periodic solutions of (1). Then we have

$$\begin{aligned} &[(u_1(t) - u_2(t)) - k(u_1(t-\tau) - u_2(t-\tau))]'' \\ &\quad - [f(u_1(t))u_1'(t) - f(u_2(t))u_2'(t)] \\ &\quad - \alpha(t)[g(u_1(t)) - g(u_2(t))] \\ &\quad - \sum_{i=1}^n \beta_i(t)[h(u_1(t-\tau_i(t))) - h(u_2(t-\tau_i(t)))] \\ &= 0. \end{aligned} \quad (17)$$

Set $\psi(t) = u_1(t) - u_2(t)$; we obtain

$$\begin{aligned} &\psi''(t) - k\psi''(t-\tau) \\ &\quad - [f(u_1(t))u_1'(t) - f(u_2(t))u_2'(t)] \\ &\quad - \alpha(t)[g(u_1(t)) - g(u_2(t))] \\ &\quad - \sum_{i=1}^n \beta_i(t)[h(u_1(t-\tau_i(t))) - h(u_2(t-\tau_i(t)))]\, dt \\ &= 0. \end{aligned} \quad (18)$$

Integrating (18) from 0 to T, we have

$$\begin{aligned} &\int_0^T \alpha(t)[g(u_1(t)) - g(u_2(t))]\, dt - \int_0^T \sum_{i=1}^n \beta_i(t) \\ &\quad \cdot [h(u_1(t-\tau_i(t))) - h(u_2(t-\tau_i(t)))]\, dt = 0. \end{aligned} \quad (19)$$

By using the integral mean value theorem, we find that there is $\xi \in [0, T]$ such that

$$\begin{aligned} &\alpha(\xi)[g(u_1(\xi)) - g(u_2(\xi))] \\ &\quad + \sum_{i=1}^n \beta_i(\xi)[h(u_1(t-\tau_i(\xi))) - h(u_2(t-\tau_i(\xi)))] \\ &= 0. \end{aligned} \quad (20)$$

From (H_1), (20) implies that

$$(u_1(\xi) - u_2(\xi))(u_1(\xi-\tau(\xi)) - u_2(\xi-\tau(\xi))) \le 0. \quad (21)$$

Since $\psi(t) = u_1(t) - u_2(t)$ is a continuous function on R, it follows that there exists a constant $\eta \in R$ such that

$$\psi(\eta) = 0. \quad (22)$$

Let $\eta = nT+\eta^*$, where $\eta^* \in [0, T]$ and n is an integer. Then (22) implies that there exists a constants $\eta^* \in [0, T]$ such that

$$\psi(\eta) = \psi(\eta^*) = 0. \quad (23)$$

From Lemma 2, using Schwarz inequality and the following relation:

$$|\psi(t)| = \left|\psi(\eta^*) + \int_{\eta^*}^t \psi'(s)\, ds\right| \le \int_0^T |\psi'(s)|\, ds, \quad t \in [0, T], \quad (24)$$

we obtain

$$|\psi|_\infty \le \sqrt{T}\, |\psi'|_2, \quad (25)$$

$$|\psi|_2 \le \frac{T}{\pi} |\psi'|_2. \quad (26)$$

Case 1. If (H_2)(a) holds, multiplying both sides of (18) by $\psi''(t)$ and then integrating them from 0 to T, using (26), we have

$$\begin{aligned}\left|\psi''\right|_2^2 &\le |k|\left|\psi''\right|_2^2 + |\alpha(t)| \\ &\cdot \int_0^T L_1\left|u_1(t)-u_2(t)\right|\left|\psi''(t)\right|dt + \sum_{i=1}^n\left|\beta_i(t)\right| \\ &\cdot \int_0^T L_2\left|u_1(t-\tau_i(t))-u_2(t-\tau_i(t))\right|\left|\psi''(t)\right|dt \\ &\le |k|\left|\psi''\right|_2^2 + \frac{\alpha^* T}{\pi}L_1\left|\psi'\right|_2\left|\psi''\right|_2 + \frac{nT\beta^*}{\pi}L_2\left|\psi'\right|_2 \\ &\cdot \left|\psi''\right|_2.\end{aligned} \tag{27}$$

Since $\psi(0) = \psi(T)$, it follows that there exists a constant $t^* \in [0,T]$ such that $\psi'(t^*) = 0$. From Lemma 2, using Schwarz inequality and the following relation:

$$\left|\psi'(t)\right| = \left|\psi'(t^*) + \int_{t^*}^t \psi''(s)\,ds\right| \le \int_0^T\left|\psi''(s)\right|ds, \quad t \in [0,T], \tag{28}$$

we obtain

$$\left|\psi'\right|_\infty \le \sqrt{T}\left|\psi''\right|_2, \tag{29}$$

$$\left|\psi'\right|_2 \le \frac{T}{\pi}\left|\psi''\right|_2. \tag{30}$$

Substituting (30) into (27), we get

$$\left|\psi''\right|_2^2 \le |k|\left|\psi''\right|_2^2 + \frac{\alpha^* T^2}{\pi^2}L_1\left|\psi''\right|_2^2 + \frac{nT^2\beta^*}{\pi^2}L_2\left|\psi''\right|_2^2. \tag{31}$$

Since $|k| + (\alpha^* T^2/\pi^2)L_1 + (nT^2\beta^*/\pi^2)L_2 < 1$, thus (31) implies that

$$\psi(t) \equiv \psi'(t) \equiv 0, \quad \forall t \in R. \tag{32}$$

Hence, $u_1(t) = u_2(t)$ for all $t \in R$. Therefore, (1) has at most one T-periodic solution.

Case 2. If (H_2)(b) holds, multiplying both sides of (18) by $\psi''(t-\tau)$ and then integrating them from 0 to T, using (26) and (30), we have

$$|k|\left|\psi''\right|_2^2 \le \left|\psi''\right|_2^2 + \frac{\alpha^* T^2}{\pi^2}L_1\left|\psi''\right|_2^2 + \frac{nT^2\beta^*}{\pi^2}L_2\left|\psi''\right|_2^2. \tag{33}$$

Since $1+(\alpha^* T^2/\pi^2)L_1+(nT^2\beta^*/\pi^2)L_2 < |k|$, thus (33) implies that

$$\psi(t) \equiv \psi'(t) \equiv 0, \quad \forall t \in R. \tag{34}$$

Hence, $u_1(t) = u_2(t)$ for all $t \in R$. Therefore, (1) has at most one T-periodic solution. □

Lemma 7 (see [17]). *Assume that (H_1) holds and the following conditions are satisfied:*

> (H_3) *There exists a constant $\rho > 0$ such that one of the following conditions holds:*
>
> (a) $u[\alpha(t)g(u) + \sum_{i=1}^n \beta_i(t)h(u) + p(t)] > 0$, *for all* $t \in R$, $|u| \ge \rho$;
>
> (b) $u[\alpha(t)g(u) + \sum_{i=1}^n \beta_i(t)h(u) + p(t)] < 0$, *for all* $t \in R$, $|u| \ge \rho$.

If $u(t)$ is a T-periodic solution of (13), then

$$|u|_\infty \le \rho + \sqrt{T}\left|u'\right|_2. \tag{35}$$

3. Main Result

Theorem 1. *Let (H_1) and (H_2) hold. Assume that either the condition (H_3)(a) or the condition (H_3)(b) is satisfied and the following inequality holds:*

$$\frac{T^2|k+1|}{(1-|k|)^2}\left(\alpha^* L_1 + \sum_{i=1}^n\left|\frac{1}{1-\tau_i'}\right|_\infty^{1/2}\beta^* L_2\right) < 1. \tag{36}$$

Then (1) has a unique T-periodic solution.

Proof. From Lemma 5, together with Lemma 6, it is easy to see that (1) has at most one T-periodic solution. Thus, to prove Theorem 1, it suffices to show that (1) has at least one T-periodic solution.

Let $u(t)$ be a T-periodic solution of (13). If (H_2) holds, multiplying both sides of (13) by $u(t) - ku(t-\tau)$ and then integrating them from 0 to T, we have

$$\begin{aligned}\int_0^T\left[\left(Au'\right)(t)\right]^2 dt &\le \alpha^* T L_1|k+1||u|_\infty^2 \\ &+ \alpha^*\sqrt{T}\left|g(0)\right|\left(|\tau|^2 + \frac{|\tau|^3}{2T}\right)^{1/2}\left|u'\right|_2 + \beta^* L_2|k \\ &+ 1|\sum_{i=1}^n\int_0^T\left|u(t-\tau_i(t))\right||u(t)|\,dt \\ &+ \left(n\beta^* T|k+1||h(0)| + T|k+1|\left|p\right|_\infty\right. \\ &\left.+ \alpha^* T\left|g(0)\right||k-1|\right)|u|_\infty.\end{aligned} \tag{37}$$

Since

$$\begin{aligned}\int_0^T|u(t-\tau(t))|^2 dt &= \int_{-\tau(0)}^{T-\tau(T)}\frac{|u(s)|^2}{1-\tau'(\mu(s))}ds \\ &\le \left|\frac{1}{1-\tau'}\right|_\infty\int_0^T|u(s)|^2 ds,\end{aligned} \tag{38}$$

following from (37), we have

$$\int_0^T \left|(Au')(t)\right|^2 dt \le |k+1|\,T\left[\alpha^* L_1 + \beta^* L_2\left(\sum_{i=1}^{n}\left|\frac{1}{1-\tau_i'}\right|_\infty^{1/2}\right)\right]|u|_\infty^2 + \left(n\beta^* T|k+1||h(0)| + T|k+1|\left|p\right|_\infty + \alpha^* T\left|g(0)\right||k-1|\right)|u|_\infty + \alpha^*\sqrt{T}\left|g(0)\right|\left(|\tau|^2 + \frac{|\tau|^3}{2T}\right)^{1/2}\left|u'\right|_2. \quad (39)$$

As ${u'}^2(t) = |A^{-1}Au'(t)|^2 \le (1/(1-|k|)^2)|(Au')(t)|^2$, using (35), from (39) we get

$$\left|u'\right|_2^2 \le \frac{1}{(1-|k|)^2}\left\{|k+1|\,T\left[\alpha^* L_1 + \beta^* L_2\left(\sum_{i=1}^{n}\left|\frac{1}{1-\tau_i'}\right|_\infty^{1/2}\right)\right]\left(\rho^2 + 2\rho\sqrt{T}\left|u'\right|_2 + T\left|u'\right|_2^2\right) + \left(n\beta^* T|k+1||h(0)| + T|k+1|\left|p\right|_\infty + \alpha^* T\left|g(0)\right||k-1|\right)\left(\rho + \sqrt{T}\left|u'\right|_2\right)\right\} + \alpha^*\sqrt{T}\left|g(0)\right|\left(|\tau|^2 + \frac{|\tau|^3}{2T}\right)^{1/2}\left|u'\right|_2. \quad (40)$$

Since $(T^2|k+1|/(1-|k|)^2)(\alpha^* L_1 + \sum_{i=1}^{n}|1/(1-\tau_i')|_\infty^{1/2}\beta^* L_2) < 1$, thus (40) implies that there exists a positive constant M_2^*, such that

$$\left|u'\right|_2^2 \le M_2^*, \qquad |u|_\infty \le \rho + \sqrt{TM_2^*} := M_2. \quad (41)$$

If $u \in \Omega_1 = \{u : u \in \operatorname{Ker} L \cap X \text{ and } Nu \in \operatorname{Im} L\}$, then there exists a constant M_1 such that

$$u(t) \equiv M_1, \qquad \int_0^T\left[\alpha(t)\,g(M_1) + \sum_{i=1}^{n}\beta_i(t)\,h(M_1) + p(t)\right]dt = 0. \quad (42)$$

Thus

$$|u(t)| \equiv \left|M_1\right| < \rho, \quad \forall x(t) \in \Omega_1. \quad (43)$$

Let $M = |M_1| + M_2^* + M_2 + 1$, and take $\Omega = \{u \in X : \|u\|_X \le M\}$. It is easy to see that N is L-compact on $\overline{\Omega}$. We have from (42), (43) that the conditions (a) and (b) in Lemma 1 hold.

Furthermore, define continuous functions $H_1(u,\mu)$ and $H_2(u,\mu)$ by setting

$$H_1(u,\mu) = (1-\mu)\,u + \frac{\mu}{T}\int_0^T\left[\alpha(t)\,g(u) + \sum_{i=1}^{n}\beta_i(t)\,h(u) + p(t)\right]dt; \quad \mu \in [0,1],$$
$$H_2(u,\mu) = -(1-\mu)\,u + \frac{\mu}{T}\int_0^T\left[\alpha(t)\,g(u) + \sum_{i=1}^{n}\beta_i(t)\,h(u) + p(t)\right]dt; \quad \mu \in [0,1]. \quad (44)$$

If the condition (H_3)(a) holds, then

$$uH_1(u,\mu) \ne 0, \quad \forall u \in \partial\Omega \cap \operatorname{Ker} L. \quad (45)$$

Hence, using the homotopy invariance theorem, we have

$$\deg\{QN, \Omega\cap\operatorname{Ker} L, 0\} = \deg\left\{\frac{1}{T}\cdot\int_0^T\left[\alpha(t)\,g(u) + \sum_{i=1}^{n}\beta_i(t)\,h(u) + p(t)\right]dt, \Omega \cap \operatorname{Ker} L, 0\right\} = \deg\{x, \Omega\cap\operatorname{Ker} L, 0\} \ne 0. \quad (46)$$

If the condition (H_3)(b) holds, then

$$uH_2(u,\mu) \ne 0, \quad \forall u \in \partial\Omega \cap \operatorname{Ker} L. \quad (47)$$

Hence, using the homotopic invariance theorem, we have

$$\deg\{QN, \Omega\cap\operatorname{Ker} L, 0\} = \deg\left\{\frac{1}{T}\cdot\int_0^T\left[\alpha(t)\,g(u) + \sum_{i=1}^{n}\beta_i(t)\,h(u) + p(t)\right]dt, \Omega \cap \operatorname{Ker} L, 0\right\} = \deg\{-x, \Omega\cap\operatorname{Ker} L, 0\} \ne 0. \quad (48)$$

Which completes the condition (c) in Lemma 1.

Up to now, all conditions in Lemma 1 are satisfied, and hence (1) has a unique T-periodic solution. □

Theorem 2. *Let (H_1) and (H_2) hold. Assume that either the condition (H_3)(a) or the condition (H_3)(b) is satisfied and the following inequality holds:*

$$\alpha^* TL_1 + nT\beta^* L_2 < 1. \quad (49)$$

Then (1) has a unique T-periodic solution.

Proof.

Case 1. If (H_2)(a) holds, multiplying the two sides of (13) by $u(t)$ and integrating them on $[0, T]$, we have

$$\begin{aligned}|u'|_2^2 &= \int_0^T |u'(t)|^2\, dt \le \alpha^* T \left(L_1 |u|_\infty^2 + |g(0)|\,|u|_\infty\right) \\ &\quad + nT\beta^* \left(L_2 |u|_\infty^2 + |h(0)|\,|u|_\infty\right) + T\,|p|_\infty\, |u|_\infty \\ &\le (\alpha^* T L_1 + nT\beta^* L_2)\,|u'|_2^2 \\ &\quad + \left[2\rho\sqrt{T}\,(\alpha^* T L_1 + nT\beta^* L_2)\right. \\ &\quad \left. + \sqrt{T}\,(T\,|p|_\infty + \alpha^* T\,|g(0)| + nT\beta^*\,|h(0)|)\right]|u'|_2 \\ &\quad + \sigma_1,\end{aligned} \tag{50}$$

where $\sigma_1 = \rho^2(\alpha^* T L_1 + nT\beta^* L_2) + \rho(T|p|_\infty + \alpha^* T|g(0)| + nT\beta^*|h(0)|)$.

As $\alpha^* T L_1 + nT\beta^* L_2 < 1$, there is a constant $M_3 > 0$ such that

$$\begin{aligned}|u'|_2^2 &\le M_3, \\ |u|_\infty &\le \rho + \sqrt{TM_3}.\end{aligned} \tag{51}$$

Case 2. If (H_2)(b) holds, multiplying the two sides of (13) by $u(t-\tau)$ and integrating them on $[0, T]$, using the methods similar to those used in Case 1, we can show that (51) holds.

The rest of the proof is similar to that of Theorem 1 and is omitted. □

4. Application

Finally, we give two examples to illustrate our main results.

Example 1. Consider the following equation:

$$\begin{aligned}(u(t) - 771u(t-\tau))'' &= -\sin(u(t))\,u'(t) - \frac{1}{8071}\left(u(t) - \frac{1}{2}\right) \\ &\quad - \frac{1}{19}\sin(u(t-1))\,u(t-1) - \frac{1}{199}\sin(t).\end{aligned} \tag{52}$$

In this example $k = 771$, $f(u(t))u'(t) = \sin(u(t))u'(t)$, $\alpha(t) = 1/8071$, $g(u(t)) = u(t) - 1/2$, $\beta_1(t) = 1/19$, $h(u(t-\tau_1(t))) = \sin(u(t-1))u(t-1)$, $\tau_1(t) = 1$, $p(t) = 1/199\sin(t)$; then $\alpha^* = 1/8071$, $\beta^* = 1/19$, $L_1 = L_2 = 1$, $T = 34\pi$, $\tau_i' = 0$ which implies that $1 + (\alpha^* T^2/\pi^2)L_1 + (nT^2\beta^*/\pi^2)L_2 < |k|$ and

$$\frac{T^2\,|k+1|}{(1-|k|)^2}\left(\alpha^* L_1 + \sum_{i=1}^{n}\left|\frac{1}{1-\tau_i'}\right|_\infty^{1/2}\beta^* L_2\right) < 1. \tag{53}$$

Thus by applying Theorem 1, we have that (52) has a unique T-periodic solution (see Figures 1 and 2).

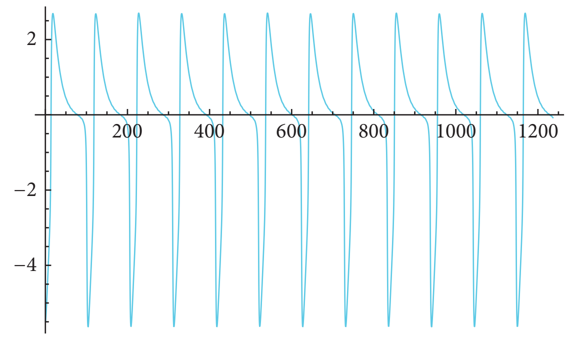

FIGURE 1: Periodic solution of (52) with delayed condition $u(x) = -5.5$ $(x \le 0)$ and initial condition $u'(0) = 0$.

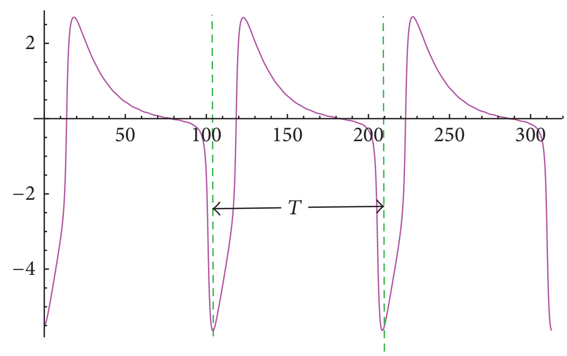

FIGURE 2: Time series of periodic solution of (52).

Example 2. Consider the following equation:

$$\begin{aligned}\left(u(t) + \frac{1}{4}u(t-\tau)\right)'' &= \frac{1}{20}(u(t)+1)\,u'(t) - \frac{t}{512\pi^2}\left|u(t) - \frac{1}{2}\right| \\ &\quad + \left(|\mathrm{sgn}\, t|^2 - \frac{191}{192}\right)\left[u\left(t - \frac{1}{15}\sin t\right) + 2\right] \\ &\quad - \frac{t}{70}u\left(t - \frac{\pi}{25}\cos t\right) + e^{|\cos t|}.\end{aligned} \tag{54}$$

In this example $k = -1/4$, $f(u(t))u'(t) = (1/20)(u(t)+1)u'(t)$, $\alpha(t) = -t/512\pi^2$, $g(u(t)) = |u(t) - 1/2|$, $\beta_1(t) = |\mathrm{sgn}\, t|^2 - 191/192$, $h(u(t-\tau_1(t))) = u(t - (1/15)\sin t) + 2$, $\beta_2(t) = t/70$, $h(u(t-\tau_2(t))) = u(t - (\pi/25)\cos t)$, $p(t) = e^{|\cos t|}$; then $\alpha^* = 1/256\pi$, $\beta^* = 1/192$, $L_1 = L_2 = 1$, $T = 2\pi$ which implies that $|k| + (\alpha^* T^2/\pi^2)L_1 + (nT^2\beta^*/\pi^2)L_2 < 1$ and

$$\alpha^* T L_1 + nT\beta^* L_2 < 1. \tag{55}$$

Thus by applying Theorem 2, we have that (54) has a unique T-periodic solution.

5. Result and Discussions

Remark 1. Obviously, (1) which we study in this paper is more general. Even if for the case of $f(x) \equiv a$, the conditions imposed on g and approaches to estimate a priori bound of solutions to (1) are different from the corresponding ones of the past work [1–3, 6].

Remark 2. In the past works, many scholars studied some kinds of second-order neutral functional differential equations and obtained some good results of existence and uniqueness of periodic solutions. However, these results are not related to delays. As we know that even a small delay is also likely to have an important impact on the system. Therefore, we focus on the relationship between the existence of periodic solutions and the delays. By using Mawhin's coincidence degree, the existence and uniqueness of periodic solutions of (1) are obtained. *The interesting is that we get some existence results which are related to the delays* $\tau_i(t)$ $(i = 1, 2, \dots, n)$. *That is different from the past results (Theorem 4 [3], Theorem 3.1 [5], and Theorem 3.1 [17]).*

Remark 3. From the above examples, we see the results are related to the deviating argument $\tau_i(t)$ $(i = 1, 2, \dots, n)$, which are different from the theorems in papers [1–6, 15, 17–19] and the references therein. The studies indicate this kind of system with time delays can exhibit periodic solutions, which shows that second-order neutral functional delayed differential equation has the potential to reproduce the complex dynamics of real applied background in physics, economics, ecology, mechanics, and so forth.

Disclosure

The author carried out the main part of this article and the main theorem.

Competing Interests

The author declares that no conflict of interests exists.

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