

## Research Article

# The Solution of Initial Boundary Value Problem with Time and Space-Fractional Diffusion Equation via a Novel Inner Product

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The main goal of this study is to find the solution of initial boundary value problem for the one-dimensional time and space-fractional diffusion equation which is a very intriguing topic for many researchers. With the aim of newly defined inner product, which is the main contribution of this study, the analytic solution of the boundary value problem is obtained. The time and space-fractional derivatives are defined in the Caputo sense which is more suitable than Riemann-Liouville sense. We apply the separation of variables method to reduce the problem to two separate fractional ODEs. The generalized solution is constructed/formed in the form of a Fourier series with respect to the eigenfunctions of a certain eigenvalue problem. In order to obtain the coefficients of the Fourier series for the solution, we define a new inner product which is the key point of study.

## 1. Introduction

Since PDE of fractional order contributes to modeling for the wide range of processes and systems, including past memories, in various scientific research areas, it has become very intriguing topic for many scientists. Main reason of this trend is that using fractional derivative is global in nature whereas the integer derivative is local in nature. This property makes fractional DEs the best possible choice in modeling physical problems involving past memory and/or delay and attracts growing number of researchers.

The Caputo derivatives are more useful than Riemann-Liouville derivatives since the analysis of the mathematical models involving Caputo derivatives gives closer results to the analysis of ones including integer derivatives. There are various studies in literature supporting this conclusion [1–8]. By making use of Mittag-Leffler function, characteristic equations of fractional ODEs are solved and solutions of them are constructed efficiently. In this sense, the Mittag-Leffler function takes the role of the exponential function which is used in the determination of solutions for ODEs with integer derivatives.

## 2. Preliminary Results

In this section, some basic definitions and known results regarding Caputo fractional derivatives are given.

**Definition 1.** The Caputo fractional derivative of  $u(t)$  of order  $q$  where  $n - 1 < q < n$  is given by equation

$${}^c D^q u(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^t (t-s)^{n-q-1} u^{(n)}(s) ds, \quad (1)$$

$$t \in [t_0, t_0 + T]$$

where  $u^{(n)}(t) = d^n u / dt^n$ . If  $q$  is an integer, then the above definition of fractional derivative coincides with the integer derivative.

**Definition 2.** If  $0 < q < 1$ , the Caputo fractional derivative of order  $q$  can be defined by equation

$${}^c D^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^t (t-s)^{-q} u'(s) ds \quad (2)$$

where  $0 < q < 1$ .

To determine the solution of eigenvalue problem in the following section, we use the two-parameter Mittag-Leffler function defined as

$$E_{\alpha,\beta}(\lambda(t-t_0)^\alpha) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^\alpha)^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0 \quad (3)$$

where  $\lambda$  is a constant. In particular, for  $t_0 = 0$ ,  $\alpha = \beta = q$  we obtain

$$E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk + q)}, \quad q > 0. \quad (4)$$

When  $q = 1$ , Mittag-Leffler function corresponds to usual exponential function, i.e.,  $E_{1,1}(\lambda t) = e^{\lambda t}$ . For further reading see [9, 10].

Let us define the following functions which play an important role in the representation of solution for the problem under consideration:

$$\begin{aligned} \sin_q(\mu t^q) &= \frac{E_{q,1}(i\mu t^q) - E_{q,1}(-i\mu t^q)}{2i} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (\mu t^q)^{2k+1}}{\Gamma((2k+1)q + 1)} \end{aligned} \quad (5)$$

and

$$\begin{aligned} \cos_q(\mu t^q) &= \frac{E_{q,1}(i\mu t^q) + E_{q,1}(-i\mu t^q)}{2} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (\mu t^q)^{2k}}{\Gamma(2kq + 1)}. \end{aligned} \quad (6)$$

Note that for  $q = 1$  these functions are usual trigonometric functions  $\sin(\mu t)$  and  $\cos(\mu t)$ .

### 3. Novel Inner Product

In this section, we define a new inner product to determine the solution which satisfies the initial boundary value problem. The missing of such an inner product makes the space-time fractional initial and boundary value problems open problems. This inner product promotes these kinds of problems to be studied theoretically. Moreover, it provides the best understanding possible to develop the theoretical study of space-time fractional initial boundary value problems. By this inner product, the maximum principle for time-space-fractional PDE will be discussed and investigated for a wide range of problems. In the definition of this inner product, we use the functions which satisfy the initial and boundary value problems with integer orders which make the calculations much more easier.

**Theorem 3.** Let  $V$  be a vector space, made up of all linear combinations of  $\sin_\beta(\mu(x/(b-a))^\beta)$  and  $\cos_\beta(\mu(x/(b-a))^\beta)$  for fixed  $\beta$  where  $0 < \beta \leq 1$ ,  $\mu \in \mathbb{R}$  on the interval  $I = [a, b]$ , i.e.,  $V = \text{span}\{\sin_\beta(\mu(x/(b-a))^\beta), \cos_\beta(\mu(x/(b-a))^\beta)\}$ .

Let  $T : V \rightarrow \text{span}\{\sin(\mu x/(b-a)), \cos(\mu x/(b-a))\}$  be a linear transformation which is one-to-one and onto. Hence it has its inverse transformation  $T^{-1}$ . The mapping  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$  is defined as

$$\begin{aligned} \langle u(x; \beta), v(x; \beta) \rangle &= T^{-1} \left( \int_a^b Tu(x; \beta) \cdot Tv(x; \beta) dx \right) \Big|_{x=a}^b \end{aligned} \quad (7)$$

where  $Tu(x; \beta) = u(x; 1)$  and  $Tv(x; \beta) = v(x; 1)$ . In other words,

$$\langle u(x; \beta), v(x; \beta) \rangle = \int_a^b u(x; 1) \cdot v(x; 1) dx = w(x, 1). \quad (8)$$

At this point we transform the function  $w(x, 1)$  to the corresponding function  $w(x, \beta)$ . Replacing  $\beta$  back and substituting upper and lower values give us the result of the new inner product.

$$\begin{aligned} \langle u(x; \beta), v(x; \beta) \rangle &= w(x, \beta) \Big|_{x=a}^{x=b} \\ &= w(b, \beta) - w(a, \beta). \end{aligned} \quad (9)$$

*Proof.* Let us show that this mapping satisfies the conditions of an inner product:

- (1)  $\langle u(x; \beta), u(x; \beta) \rangle \geq 0$   
 $\langle u(x; \beta), u(x; \beta) \rangle = \int_a^b u^2(x; 1) dx = w(x, 1)$   
 $\langle u(x; \beta), u(x; \beta) \rangle = w(x, \beta) \Big|_{x=a}^{x=b} = w(b, \beta) - w(a, \beta) \geq 0$   
 since  $w(b, 1) - w(a, 1) \geq 0$  and  $-1 \leq \sin_q(x) \leq 1, -1 \leq \cos_q(x) \leq 1, \forall q \in (0, 1)$ .  
 Moreover  $\langle u(x; \beta), u(x; \beta) \rangle = 0 \Leftrightarrow u(x; \beta) = 0$ .
- (2)  $\langle u(x; \beta), v(x; \beta) \rangle = \int_a^b u(x; 1) v(x; 1) dx = \int_a^b v(x; 1) u(x; 1) dx = w(x, 1)$   
 $\langle u(x; \beta), v(x; \beta) \rangle = w(x, \beta) \Big|_{x=a}^{x=b} = w(b, \beta) - w(a, \beta) = \langle v(x; \beta), u(x; \beta) \rangle$ .
- (3)  $\langle u(x; \beta) + v(x; \beta), z(x; \beta) \rangle = \int_a^b (u(x; 1) + v(x; 1)) z(x; 1) dx$   
 $= \int_a^b u(x; 1) z(x; 1) dx + \int_a^b v(x; 1) z(x; 1) dx$   
 $\langle u(x; \beta) + v(x; \beta), z(x; \beta) \rangle = w_1(x, \beta) \Big|_{x=a}^b + w_2(x, \beta) \Big|_{x=a}^b$   
 $= w_1(b, \beta) - w_1(a, \beta) + w_2(b, \beta) - w_2(a, \beta)$   
 $= \langle u(x; \beta), z(x; \beta) \rangle + \langle v(x; \beta), z(x; \beta) \rangle$ .
- (4)  $\langle \alpha u(x; \beta), v(x; \beta) \rangle = \int_a^b \alpha u(x; 1) v(x; 1) dx = \alpha \int_a^b u(x; 1) v(x; 1) dx = \alpha w(x, 1)$   
 $\langle \alpha u(x; \beta), v(x; \beta) \rangle = \alpha w(x, \beta) \Big|_{x=a}^b = \alpha (w(b, \beta) - w(a, \beta)) = \alpha \langle u(x; \beta), v(x; \beta) \rangle$ .  $\square$

*Example 4.* Let us illustrate this inner product by an example in which we take the inner product of Mittag-Leffler functions  $E_\beta(\lambda_1(\beta)x^\beta)$  and  $E_\beta(\lambda_2(\beta)x^\beta)$ :

$$\begin{aligned}
 & \langle E_\beta(\lambda_1(\beta)x^\beta), E_\beta(\lambda_2(\beta)x^\beta) \rangle \\
 &= \int e^{\lambda_1(1)x} e^{\lambda_2(1)x} dx = \int e^{(\lambda_1(1)+\lambda_2(1))x} dx \\
 &= \frac{e^{(\lambda_1(1)+\lambda_2(1))x}}{\lambda_1(1)+\lambda_2(1)} \\
 & \langle E_\beta(\lambda_1(\beta)x^\beta), E_\beta(\lambda_2(\beta)x^\beta) \rangle \quad (10) \\
 &= \frac{E_\beta(\lambda_1(\beta)+\lambda_2(\beta)x^\beta)}{\lambda_1(\beta)+\lambda_2(\beta)} \Big|_{x=a}^b \\
 &= \frac{E_\beta(\lambda_1(\beta)+\lambda_2(\beta)b^\beta) - E_\beta(\lambda_1(\beta)+\lambda_2(\beta)a^\beta)}{\lambda_1(\beta)+\lambda_2(\beta)}
 \end{aligned}$$

Note that this inner product extends orthogonality property between the functions  $u(x; 1)$  and  $v(x; 1)$  to the functions  $u(x; \beta)$  and  $v(x; \beta)$ .

#### 4. Main Results

In this section, we deal with the following initial boundary value problem involving time and space-fractional PDE:

$$D_t^\alpha u(x, t) = a D_x^{2\beta} u, \quad 0 < \alpha < 1, \quad 1 < 2\beta < 2, \quad a > 0 \quad (11)$$

$$u(x, 0) = u_0(x), \quad 0 \leq x \leq l \quad (12)$$

$$u(0, t) = 0,$$

$$u(l, t) = 0 \quad (13)$$

$$0 \leq t \leq T.$$

The generalized solution of this problem can be obtained in an analytical form by using the separation of variables method. Let a particular solution of (11)-(13) be in the following form:

$$u(x, t; \alpha, \beta) = X(x; \beta) T(t; \alpha, \beta) \quad (14)$$

where  $0 \leq x \leq l, 0 \leq t \leq T$ .

Note that the function  $X$  depends on order of fractional derivative with respect to  $x$  and function  $T$  depends on orders of fractional derivatives with respect to  $x$  and  $t$ . Substituting (14) into (11) and applying separation of the variables lead to the equation

$$\frac{(D^\alpha T)(t; \alpha, \beta)}{aT(t; \alpha, \beta)} = \frac{(D^{2\beta} X)(x; \beta)}{X(x; \beta)} = -\lambda(\beta) \quad (15)$$

where  $\lambda$  depends on order of the fractional derivative with respect to  $x$ . The equations in (15) reduce the problem to two separate ODEs including fractional derivatives with

respect to both time and space. The equation on the right with boundary conditions (13) gives the following fractional differential equation:

$$(D^{2\beta} X)(x; \beta) + \lambda(\beta) X(x; \beta) = 0 \quad (16)$$

$$X(0; \beta) = 0, \quad X(l; \beta) = 0 \quad (17)$$

In order to determine the solution of the eigenvalue problem ((16)-(17)) we use the following Mittag-Leffler function:

$$X(x; \beta) = E_{\beta,1}(rx^\beta) \quad (18)$$

which leads to the following characteristic equation for the eigenvalue problem ((16)-(17)):

$$r^2 + \lambda(\beta) = 0. \quad (19)$$

*Case 1.* If  $\lambda < 0$ , we have two real and distinct solutions  $r = \pm \sqrt{-\lambda(\beta)}$ , and the general solution of the eigenvalue problem ((16)-(17)) is represented by

$$X(x; \beta) = c_1 E_{\beta,1}(\sqrt{-\lambda}x^\beta) + c_2 E_{\beta,1}(-\sqrt{-\lambda}x^\beta)$$

$$X(0; \beta) = 0 \implies c_1 E_{\beta,1}(0) + c_2 E_{\beta,1}(0) = 0 \implies c_2 = -c_1 \quad (20)$$

$$= -c_1$$

which leads to the following solution:

$$X(x; \beta) = c_1 (E_{\beta,1}(\sqrt{-\lambda}x^\beta) - E_{\beta,1}(-\sqrt{-\lambda}x^\beta)). \quad (21)$$

From the second boundary condition we have

$$\begin{aligned}
 X(l; \beta) = 0 & \implies X(l; \beta) \\
 &= c_1 (E_{\beta,1}(\sqrt{-\lambda}l^\beta) - E_{\beta,1}(-\sqrt{-\lambda}l^\beta)) = 0 \quad (22)
 \end{aligned}$$

which implies that

$$E_{\beta,1}(\sqrt{-\lambda}l^\beta) = E_{\beta,1}(-\sqrt{-\lambda}l^\beta). \quad (23)$$

Thus

$$\begin{aligned}
 \sqrt{-\lambda}l^\beta &= 0 \implies \\
 \lambda &= 0 \quad (24)
 \end{aligned}$$

$$X(x; \beta) = c_1 (E_{\beta,1}(0) - E_{\beta,1}(0)) = 0$$

which means that there is no solution for the case  $\lambda < 0$ .

*Case 2.* If  $\lambda = 0$ , the characteristic equation becomes

$$\begin{aligned}
 r^2 &= 0 \implies \\
 r_{1,2} &= 0. \quad (25)
 \end{aligned}$$

Hence the general solution of the eigenvalue problem ((16)-(17)) is represented by

$$X(x; \beta) = c_1 + c_2 \frac{x^\beta}{\beta} \quad (26)$$

From the first boundary condition we have

$$\begin{aligned} X(0; \beta) &= 0 \implies \\ c_1 &= 0 \end{aligned} \quad (27)$$

which leads to the following solution:

$$X(x; \beta) = c_2 \frac{x^\beta}{\beta}. \quad (28)$$

From the last boundary condition we have

$$\begin{aligned} X(l; \beta) &= 0 \implies \\ c_2 \frac{l^\beta}{\beta} &= 0 \implies \\ c_2 &= 0 \end{aligned} \quad (29)$$

which leads to  $X(x; \beta) = 0$  which means that there is no solution for  $\lambda = 0$ .

*Case 3.* If  $\lambda > 0$ , we obtain two complex roots  $r = \mp i\sqrt{\lambda}$ . Then the general solution of the eigenvalue problem ((16)-(17)) is represented by

$$X(x; \beta) = c_1 \sin_\beta(\sqrt{\lambda} x^\beta) + c_2 \cos_\beta(\sqrt{\lambda} x^\beta). \quad (30)$$

Let us check if the boundary conditions are satisfied by the solution. From the first boundary condition we have

$$\begin{aligned} X(0; \beta) &= 0 \implies \\ c_1 \sin_\beta(0) + c_2 \cos_\beta(0) &= 0 \implies \\ c_2 &= 0 \end{aligned} \quad (31)$$

which gives the following solution:

$$X(x; \beta) = c_1 \sin_\beta(\sqrt{\lambda} x^\beta) \quad (32)$$

From the second boundary condition we have

$$\begin{aligned} X(l; \beta) &= 0 \implies \\ X(l) &= c_1 \sin_\beta(\sqrt{\lambda} l^\beta) = 0 \end{aligned} \quad (33)$$

which implies that

$$\sin_\beta(\sqrt{\lambda} l^\beta) = 0. \quad (34)$$

Let  $w_n(\beta) = \sqrt{\lambda_n(\beta)} l^\beta$ . Hence the eigenvalues can be represented in terms of  $w_n(\beta)$  as follows:

$$\lambda_n(\beta) = \frac{w_n^2(\beta)}{l^{2\beta}}, \quad 0 < w_1(\beta) < w_2(\beta) < \dots \quad (35)$$

Thus the solution of the eigenvalue problem is represented in the following form:

$$X_n(x; \beta) = c_n \sin_\beta\left(w_n(\beta) \left(\frac{x}{l}\right)^\beta\right). \quad (36)$$

The equation on the left of (15) for each eigenvalue  $\lambda_n$  gives the following fractional differential equation:

$$(D^\alpha T_n)(t; \alpha, \beta) + \left(\frac{aw_n^2(\beta)}{l^{2\beta}}\right) T_n(t; \alpha, \beta) = 0 \quad (37)$$

By using the similar calculations the solution of (37) is determined in the following form:

$$T_n(t; \alpha, \beta) = c_n E_\alpha\left(-\left(\frac{aw_n^2(\beta)}{l^{2\beta}}\right) t^\alpha\right). \quad (38)$$

For each eigenvalue  $\lambda_n$  we obtain the following function:

$$\begin{aligned} u_n(x, t; \alpha, \beta) &= X_n(x; \beta) T_n(t; \alpha, \beta) \\ &= c_n \sin_\beta\left(w_n(\beta) \left(\frac{x}{l}\right)^\beta\right) E_\alpha\left(-\left(\frac{aw_n^2(\beta)}{l^{2\beta}}\right) t^\alpha\right) \end{aligned} \quad (39)$$

and hence we have the following sum:

$$\begin{aligned} u(x, t; \alpha, \beta) &= \sum_{n=1}^{\infty} u_n(x, t; \alpha, \beta) \\ &= \sum_{n=1}^{\infty} c_n \sin_\beta\left(w_n(\beta) \left(\frac{x}{l}\right)^\beta\right) E_\alpha\left(-\left(\frac{aw_n^2(\beta)}{l^{2\beta}}\right) t^\alpha\right) \end{aligned} \quad (40)$$

which satisfy both the fractional equation (11) and boundary condition (13).

In order to determine the solution which satisfies the initial condition (12) we use the inner product defined in (7).

In (38), replacing  $t$  by 0 and using the initial condition (12) we have

$$\begin{aligned} u(x, 0; \alpha, \beta) &= u_0(x; \alpha, \beta) = \sum_{n=1}^{\infty} u_n(x, 0; \alpha, \beta) \\ &= \sum_{n=1}^{\infty} c_n \sin_\beta\left(w_n(\beta) \left(\frac{x}{l}\right)^\beta\right). \end{aligned} \quad (41)$$

Via the inner product (7) we will obtain the coefficients  $c_n$  for  $n = 1, 2, 3, \dots$  as follows:

$$\begin{aligned} c_n &= \left\langle u_0(x; \alpha, \beta), \sin_\beta\left(w_n(\beta) \left(\frac{x}{l}\right)^\beta\right) \right\rangle, \\ &n = 1, 2, 3, \dots \end{aligned} \quad (42)$$

## 5. Illustrative Example

Before solving fractional heat-like problem we first solve the following problem for  $\alpha = \beta = 1$  in order to compare the solutions of these two problems:

$$\begin{aligned} D_t u(x, t) &= \frac{1}{4} D_x^2 u(x, t), \quad 0 \leq x \leq 2, \quad t > 0 \\ u(x, 0) &= -\sin(\pi x), \quad 0 \leq x \leq 2 \\ u(0, t) &= 0, \\ u(2, t) &= 0 \\ 0 &\leq t \leq T \end{aligned} \quad (43)$$

The solution of this problem is obtained as follows:

$$u(x, t) = -\sin(\pi x) \exp\left(\frac{-\pi^2}{4} t\right). \quad (44)$$

Let us consider the following fractional heat-like problem:

$$D_t^\alpha u(x, t) = \frac{1}{4} D_x^{2\beta} u(x, t), \quad 0 < \alpha < 1, \quad 1 < 2\beta < 2, \quad (45)$$

$$u(x, 0) = -\sin(\pi x), \quad 0 \leq x \leq 2 \quad (46)$$

$$u(0, t) = 0,$$

$$u(2, t) = 0 \quad (47)$$

$$0 \leq t \leq T$$

Applying separation of the variables to (45) leads to the equation

$$\frac{4(D^\alpha T)(t; \alpha, \beta)}{T(t; \alpha, \beta)} = \frac{(D^{2\beta} X)(x; \beta)}{X(x; \beta)} = -\lambda(\beta) \quad (48)$$

where  $\lambda$  depends on  $\beta$ . Hence we get two ODEs having fractional derivatives with respect to time and space separately. The equation on the right of (48) with boundary conditions (47) gives the fractional differential equation

$$(D^{2\beta} X)(x; \beta) + \lambda X(x; \beta) = 0 \quad (49)$$

$$X(0; \beta) = 0, \quad (50)$$

$$X(2; \beta) = 0.$$

Using the Mittag-Leffler function  $X(x; \beta) = E_{\beta,1}(rx^\beta)$  we obtain the following characteristic equation  $r^2 + \mu^2(\beta) = 0$  where we take  $\lambda = \mu^2$ . Hence the solution becomes

$$X(x; \beta) = c_1 \sin_\beta(\mu x^\beta) + c_2 \cos_\beta(\mu x^\beta). \quad (51)$$

Using boundary conditions

$$X(0; \beta) = 0 \implies$$

$$c_1 \sin_\beta(0) + c_2 \cos_\beta(0) = 0 \implies \quad (52)$$

$$c_2 = 0$$

which gives the following solution:

$$X(x; \beta) = c_1 \sin_\beta(\mu x^\beta). \quad (53)$$

From the second boundary condition we have

$$X(2; \beta) = 0 \implies \quad (54)$$

$$X(2; \beta) = c_1 \sin_\beta(\mu 2^\beta) = 0$$

which implies that

$$\sin_\beta(\mu 2^\beta) = 0. \quad (55)$$

Let  $w_n(\beta) = \mu_n 2^\beta$ , i.e.,  $\sin_\beta(w_n) = 0$ . Hence the eigenvalues can be represented in terms of  $w_n(\beta)$  as follows:

$$\lambda_n(\beta) = \frac{w_n^2(\beta)}{2^{2\beta}}, \quad 0 < w_1(\beta) < w_2(\beta) < \dots \quad (56)$$

Thus the solution of the eigenvalue problem is represented in the following form:

$$X_n(x; \beta) = c_n \sin_\beta\left(w_n(\beta) \left(\frac{x}{2}\right)^\beta\right). \quad (57)$$

The equation on the left of (48) for each eigenvalue  $\lambda_n(\beta)$  gives the following fractional differential equation:

$$(D^\alpha T_n)(t; \alpha, \beta) + \left(\frac{w_n^2(\beta)}{2^{2\beta+2}}\right) T_n(t; \alpha, \beta) = 0. \quad (58)$$

By using the similar calculations the solution of (58) is determined in the following form:

$$T_n(t; \alpha, \beta) = E_\alpha\left(-\left(\frac{w_n(\beta)}{2^{\beta+1}}\right)^2 t^\alpha\right). \quad (59)$$

For each eigenvalue  $\lambda_n$  we obtain the following function:

$$\begin{aligned} u_n(x, t; \alpha, \beta) &= X_n(x; \beta) T_n(t; \alpha, \beta) \\ &= c_n \sin_\beta\left(w_n(\beta) \left(\frac{x}{2}\right)^\beta\right) E_\alpha\left(-\left(\frac{w_n(\beta)}{2^{\beta+1}}\right)^2 t^\alpha\right) \end{aligned} \quad (60)$$

and thus we have the following sum:

$$\begin{aligned} u(x, t; \alpha, \beta) &= \sum_{n=1}^{\infty} u_n(x, t; \alpha, \beta) \\ &= \sum_{n=1}^{\infty} c_n \sin_\beta\left(w_n(\beta) \left(\frac{x}{2}\right)^\beta\right) E_\alpha\left(-\left(\frac{w_n(\beta)}{2^{\beta+1}}\right)^2 t^\alpha\right) \end{aligned} \quad (61)$$

which satisfy both (45) and boundary condition (47). In order to determine the solution which satisfies the initial condition (46),

$$\begin{aligned} u(x, 0; \alpha, \beta) &= \sum_{n=1}^{\infty} u_n(x, 0; \alpha, \beta) \\ &= \sum_{n=1}^{\infty} c_n \sin_\beta\left(w_n(\beta) \left(\frac{x}{2}\right)^\beta\right) \\ &= -\sin(\pi x). \end{aligned} \quad (62)$$

Via the inner product we obtain the coefficients  $c_n$  for  $n = 1, 2, 3, \dots$  as follows:

$$c_n = \left\langle -\sin(\pi x), \sin_\beta\left(w_n(\beta) \left(\frac{x}{2}\right)^\beta\right) \right\rangle, \quad (63)$$

$n = 1, 2, 3, \dots$

$$\begin{aligned}
c_n &= - \int \sin(\pi x) \sin\left(\frac{n\pi x}{2}\right) dx, \quad n \neq 2 \\
&= \frac{1}{(n-2)\pi} \sin\left(\frac{(n-2)\pi x}{2}\right) \\
&\quad - \frac{1}{(n+2)\pi} \sin\left(\frac{(n+2)\pi x}{2}\right).
\end{aligned} \tag{64}$$

Replacing  $\beta$  back and substituting upper and lower limits give us the result of the new inner product.

$$\begin{aligned}
c_n &= \frac{1}{w_{n-2}} \sin_\beta \left( w_{n-2}(\beta) \left( \frac{x}{2} \right)^\beta \right) \\
&\quad - \frac{1}{w_{n+2}} \sin_\beta \left( w_{n+2}(\beta) \left( \frac{x}{2} \right)^\beta \right) \Big|_{x=0}^{x=2} \\
&= \frac{1}{w_{n-2}} \sin_\beta(w_{n-2}(\beta)) - \frac{1}{w_{n+2}} \sin_\beta(w_{n+2}(\beta))
\end{aligned} \tag{65}$$

Thus  $c_n = 0, n \neq 2$ .

For  $n = 2$  we get

$$\begin{aligned}
c_2 &= \left\langle -\sin(\pi x), \sin_\beta \left( w_2(\beta) \left( \frac{x}{2} \right)^\beta \right) \right\rangle, \\
&\quad n = 1, 2, 3, \dots \\
&= - \int \sin^2(\pi x) dx = -\frac{1}{2} \left( x + \frac{\sin 2\pi x}{4\pi} \right) \\
&= -\frac{1}{2} x^\beta + \frac{\sin_\beta(w_2(\beta) (x/2)^\beta)}{w_2(\beta)} \Big|_{x=0}^{x=2} = -2^{\beta-1}.
\end{aligned} \tag{66}$$

Thus

$$\begin{aligned}
u(x, t; \alpha, \beta) &= \sum_{n=1}^{\infty} u_n(x, t; \alpha, \beta) \\
&= -2^{\beta-1} \sin_\beta \left( w_2(\beta) \left( \frac{x}{2} \right)^\beta \right) \\
&\quad \cdot E_\alpha \left( - \left( \frac{w_2(\beta)}{2^{\beta+1}} \right)^2 t^\alpha \right).
\end{aligned} \tag{67}$$

Notice that as  $\beta$  and  $\alpha$  go to 1, solution (67) goes to the solution (44).

## 6. Conclusion

In this paper, we establish a new inner product which plays the remarkable role in the determination of the solution for the one-dimensional time and space-fractional diffusion problem which is an open problem. Moreover this inner product can be applied in many related problems and brings new insights into the understanding of time and space-fractional problems.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

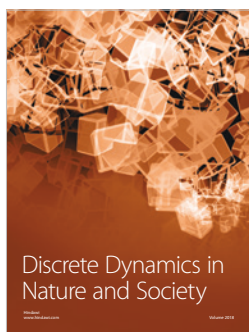
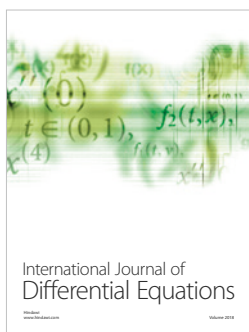
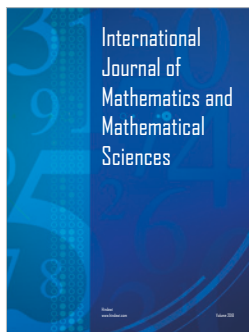
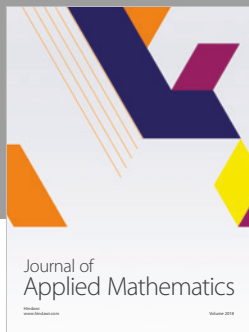
## Authors' Contributions

All authors read and approved the final manuscript.

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