Research Article

The Johnson Equation, Fredholm and Wronskian Representations of Solutions, and the Case of Order Three

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We construct solutions to the Johnson equation (J) first by means of Fredholm determinants and then by means of Wronskians of order $2N$ giving solutions of order N depending on $2N - 1$ parameters. We obtain N order rational solutions that can be written as a quotient of two polynomials of degree $2N(N + 1)$ in x, t and $4N(N + 1)$ in y depending on $2N - 2$ parameters. This method gives an infinite hierarchy of solutions to the Johnson equation. In particular, rational solutions are obtained. The solutions of order 3 with 4 parameters are constructed and studied in detail by means of their modulus in the (x, y) plane in function of time t and parameters a_1, a_2, b_1 , and b_2 .

1. Introduction

The Johnson equation was introduced in 1980 by Johnson [1] to describe waves surfaces in shallow incompressible fluids [2, 3]. This equation was derived for internal waves in a stratified medium [4]. The Johnson equation is dissipative; it is well known that there is no solution with a linear front localized along straight lines in the (x, y) plane. This Johnson equation is, for example, able to explain the existence of the horseshoe-like solitons and multisoliton solutions quite naturally.

We consider the Johnson equation (J) in the following normalization:

$$\left(u_t + 6uu_x + u_{xxx} + \frac{u}{2t}\right)_x - 3\frac{u_{yy}}{t^2} = 0, \quad (1)$$

where as usual subscripts x, y , and t mean partial derivatives.

The first solutions were constructed in 1980 by Johnson [1]. Other types of solutions were found in [5]. A new approach to solve this equation was given in 1986 [6] by giving a link between solutions of the Kadomtsev-Petviashvili (KP) [7] and solutions of the Johnson equation. In 2007, other types of solutions were obtained by using the Darboux transformation [8]. More recently, in 2013, other extensions have been considered as the elliptic case [9].

Here, we consider the famous Kadomtsev-Petviashvili (KPI), which can be written in the following form:

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0. \quad (2)$$

The KPI equation first appeared in 1970 [7] in a paper written by Kadomtsev and Petviashvili. This equation is considered as a model for surface and internal water waves [10] and in nonlinear optics [11].

In the following, we will use the KPI equation to construct solutions to the Johnson equation but in another way different from this used in [6]. Indeed, these last authors consider another representation of KPI equation given by

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \quad (3)$$

and so the transformations between solutions of (3) and (1) are different from those we use to transform solutions to (2) in solutions to (1).

In fact, to obtain solutions to (1) from solutions to (2), we use the following transformation:

$$(x; y; t) \mapsto \left(x_1 = -ix - i\frac{y^2 t}{12}; y_1 = yt; t_1 = 4it\right) \quad (4)$$

In this paper, we give solutions by means of Fredholm determinants of order $2N$ depending on $2N - 1$ parameters

and then by means of Wronskians of order $2N$ with $2N - 1$ parameters. So we construct an infinite hierarchy of solutions to the Johnson equation, depending on $2N - 1$ real parameters.

New rational solutions depending a priori on $2N - 2$ parameters at order N are constructed, when one parameter tends to 0.

We obtain families depending on $2N - 2$ parameters for the N th order as a ratio of two polynomials of degree $2N(N + 1)$ in x , t and of degree $4N(N + 1)$ in y .

In this paper, we construct only rational solutions of order 3, depending on 4 real parameters; we construct the representations of their modulus in the plane of the coordinates (x, y) according to the four real parameters a_i and b_i for $1 \leq i \leq 2$ and time t .

2. Solutions to Johnson Equation Expressed by Means of Fredholm Determinants

Some notations are given. We define first real numbers λ_j such that $-1 < \lambda_j < 1$, $j = 1, \dots, 2N$; they depend on a parameter ϵ and can be written as

$$\begin{aligned}\lambda_j &= 1 - 2\epsilon^2 j^2, \\ \lambda_{N+j} &= -\lambda_j, \\ 1 &\leq j \leq N.\end{aligned}\quad (5)$$

Then, we define $\kappa_j, \delta_j, \gamma_j$, and $x_{r,j}$; they are functions of λ_j , $1 \leq j \leq 2N$, and are defined by the following formulas:

$$\begin{aligned}\kappa_j &= 2\sqrt{1 - \lambda_j^2}, \\ \delta_j &= \kappa_j \lambda_j, \\ \gamma_j &= \sqrt{\frac{1 - \lambda_j}{1 + \lambda_j}}, \\ x_{r,j} &= (r - 1) \ln \frac{\gamma_j - i}{\gamma_j + i}, \quad r = 1, 3, \\ \tau_j &= -12i\lambda_j^2 \sqrt{1 - \lambda_j^2} - 4i(1 - \lambda_j^2) \sqrt{1 - \lambda_j^2}, \\ \kappa_{N+j} &= \kappa_j, \\ \delta_{N+j} &= -\delta_j, \\ \gamma_{N+j} &= \gamma_j^{-1}, \\ x_{r,N+j} &= -x_{r,j}, \\ \tau_{N+j} &= \tau_j\end{aligned}\quad (6)$$

$$j = 1, \dots, N.$$

e_ν , $1 \leq \nu \leq 2N$ are defined by

$$\begin{aligned}e_j &= 2i \left(\sum_{k=1}^{1/2M-1} a_k (je)^{2k+1} - i \sum_{k=1}^{1/2M-1} b_k (je)^{2k+1} \right), \\ e_{N+j} &= 2i \left(\sum_{k=1}^{1/2M-1} a_k (je)^{2k+1} + i \sum_{k=1}^{1/2M-1} b_k (je)^{2k+1} \right), \\ 1 &\leq j \leq N,\end{aligned}\quad (7)$$

$$a_k, b_k \in \mathbf{R}, \quad 1 \leq k \leq N.$$

ϵ_ν , $1 \leq \nu \leq 2N$ are defined by

$$\begin{aligned}\epsilon_j &= 1, \\ \epsilon_{N+j} &= 0 \\ 1 &\leq j \leq N.\end{aligned}\quad (8)$$

As usual I is the unit matrix and $D_r = (d_{jk})_{1 \leq j, k \leq 2N}$ is the matrix defined by the following:

$$\begin{aligned}d_{\nu\mu} &= (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right) \\ &\cdot \exp \left(\kappa_\nu x + \left(\frac{\kappa_\nu \gamma}{12} - 2\delta_\nu \right) yt + 4i\tau_\nu t + x_{r,\nu} + e_\nu \right).\end{aligned}\quad (9)$$

Then we get the following theorem.

Theorem 1. *The function v defined by*

$$v(x, y, t) = -2 \frac{|n(x, y, t)|^2}{d(x, y, t)^2} \quad (10)$$

with

$$n(x, y, t) = \det(I + D_3(x, y, t)), \quad (11)$$

$$d(x, y, t) = \det(I + D_1(x, y, t)), \quad (12)$$

and $D_r = (d_{jk})_{1 \leq j, k \leq 2N}$ is the matrix

$$\begin{aligned}d_{\nu\mu} &= (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right) \\ &\cdot \exp \left(\kappa_\nu x + \left(\frac{\kappa_\nu \gamma}{12} - 2\delta_\nu \right) yt + 4i\tau_\nu t + x_{r,\nu} + e_\nu \right)\end{aligned}\quad (13)$$

is a solution to (1), depending on $2N - 1$ parameters a_k, b_k , $1 \leq k \leq N - 1$ and ϵ .

Proof. The solution v to the KPI equation can be written as follows by using [12]:

$$v(x, y, t) = -2 \frac{|n(x, y, t)|^2}{d(x, y, t)^2} \quad (14)$$

where

$$n(x, y, t) = \det(I + D_3(x, y, t)), \quad (15)$$

$$d(x, y, t) = \det(I + D_1(x, y, t)), \quad (16)$$

and $D_r = (d_{jk})_{1 \leq j, k \leq 2N}$ is the matrix

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right) \cdot \exp(i\kappa_\nu x - 2\delta_\nu y + \tau_\nu t + x_{r,\nu} + e_\nu). \quad (17)$$

where κ_ν , δ_ν , $x_{r,\nu}$, γ_ν , τ_ν , e_ν , and ϵ_ν are defined in (6), (5), (7), and (8).

The connection between the solutions to the Johnson equation and these to the KPI equation was already explained in [6] but with another expression of the KPI equation (3).

Here, the knowledge of a solution u to the KPI equation (2) gives a solution to the Johnson equation (1). Let us consider $u(x, y, t)$ a solution of the KPI equation (2), then the function

$$\tilde{u}(x_1, y_1, t_1) \quad (18)$$

for

$$\begin{aligned} x_1 &= -ix - i\frac{y^2 t}{12}, \\ y_1 &= yt, \\ t_1 &= 4it \end{aligned} \quad (19)$$

is a solution to the KPI equation (2). Using this crucial transformation, the solution to the Johnson equation takes the form

$$v(x, y, t) = -2 \frac{|\det(I + D_3(x, y, t))|^2}{\det(I + D_1(x, y, t))^2} \quad (20)$$

with the matrix D_r defined in (17).

So we get the solutions to (14) by means of Fredholm determinants. $\square$

3. Solutions to the Johnson Equation by Means of Wronskians

We use the following notations:

$$\phi_{r,\nu} = \sin \Theta_{r,\nu}, \quad 1 \leq \nu \leq N, \quad (21)$$

$$\phi_{r,\nu} = \cos \Theta_{r,\nu}, \quad N+1 \leq \nu \leq 2N, \quad r = 1, 3,$$

with

$$\begin{aligned} \Theta_{r,\nu} &= \frac{-i\kappa_\nu x}{2} + i \left(\frac{-\kappa_\nu y}{24} + \delta_\nu \right) yt - i \frac{x_{r,\nu}}{2} + 2\tau_\nu t \\ &\quad + \gamma_\nu w - i \frac{e_\nu}{2}, \quad 1 \leq \nu \leq 2N. \end{aligned} \quad (22)$$

$W_r(w)$ is the Wronskian of the functions $\phi_{r,1}, \dots, \phi_{r,2N}$ defined by

$$W_r(w) = \det \left[\left(\partial_w^{\mu-1} \phi_{r,\nu} \right)_{\nu, \mu \in [1, \dots, 2N]} \right]. \quad (23)$$

We consider the matrix $D_r = (d_{\nu\mu})_{\nu, \mu \in [1, \dots, 2N]}$ defined in (17).

Then we have the following result.

Theorem 2.

$$\det(I + D_r) = k_r(0) \times W_r(\phi_{r,1}, \dots, \phi_{r,2N})(0), \quad (24)$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_{r,\nu})}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}. \quad (25)$$

Proof. First, we remove the factor $(2i)^{-1} e^{i\Theta_{r,\nu}}$ in each row ν in the Wronskian $W_r(w)$ for $1 \leq \nu \leq 2N$.

Then

$$W_r = \prod_{\nu=1}^{2N} e^{i\Theta_{r,\nu}} (2i)^{-N} (2)^{-N} \times \widetilde{W}_r, \quad (26)$$

with

$$\widetilde{W}_r = \begin{vmatrix} (1 - e^{-2i\Theta_{r,1}}) & i\gamma_1 (1 + e^{-2i\Theta_{r,1}}) & \dots & (i\gamma_1)^{2N-1} (1 + (-1)^{2N} e^{-2i\Theta_{r,1}}) \\ (1 - e^{-2i\Theta_{r,2}}) & i\gamma_2 (1 + e^{-2i\Theta_{r,2}}) & \dots & (i\gamma_2)^{2N-1} (1 + (-1)^{2N} e^{-2i\Theta_{r,2}}) \\ \vdots & \vdots & \vdots & \vdots \\ (1 - e^{-2i\Theta_{r,2N}}) & i\gamma_{2N} (1 + e^{-2i\Theta_{r,2N}}) & \dots & (i\gamma_{2N})^{2N-1} (1 + (-1)^{2N} e^{-2i\Theta_{r,2N}}) \end{vmatrix} \quad (27)$$

The determinant $\widetilde{W}_r$ can be written as

$$\widetilde{W}_r = \det(\alpha_{jk}e_j + \beta_{jk}), \quad (28)$$

where $\alpha_{jk} = (-1)^k(i\gamma_j)^{k-1}$, $e_j = e^{-2i\Theta_{r,j}}$, and $\beta_{jk} = (i\gamma_j)^{k-1}$, $1 \leq j \leq N$, $1 \leq k \leq 2N$;

$\alpha_{jk} = (-1)^{k-1}(i\gamma_j)^{k-1}$, $e_j = e^{-2i\Theta_{r,j}}$, and $\beta_{jk} = (i\gamma_j)^{k-1}$, $N+1 \leq j \leq 2N$, $1 \leq k \leq 2N$.

We have to calculate $\widetilde{W}_r$. So, we use the following lemma.

Lemma 3. Let $A = (a_{ij})_{i,j \in [1, \dots, N]}$, let $B = (b_{ij})_{i,j \in [1, \dots, N]}$, and let $(H_{ij})_{i,j \in [1, \dots, N]}$ be the matrix formed by replacing in A the j th row of A by the i th row of B . Then

$$\det(a_{ij}x_i + b_{ij}) = \det(a_{ij}) \times \det\left(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(a_{ij})}\right) \quad (29)$$

Proof. Let $\widetilde{A} = (\widetilde{a}_{ji})_{i,j \in [1, \dots, N]}$ be the transposed matrix in cofactors of A . Then $A \times \widetilde{A} = \det A \times I$.

So $\det(\widetilde{A}) = (\det(A))^{N-1}$.

Then the general term of the product $(c_{ij})_{i,j \in [1, \dots, N]} = (a_{ij}x_i + b_{ij})_{i,j \in [1, \dots, N]} \times (\widetilde{a}_{ji})_{i,j \in [1, \dots, N]}$ can be expressed by

$$c_{ij} = \sum_{s=1}^N (a_{is}x_i + b_{is}) \times \widetilde{a}_{js} = x_i \sum_{s=1}^N a_{is}\widetilde{a}_{js} + \sum_{s=1}^N b_{is}\widetilde{a}_{js} \quad (30)$$

$$= \delta_{ij} \det(A) x_i + \det(H_{ij}).$$

We obtain

$$\begin{aligned} \det(c_{ij}) &= \det(a_{ij}x_i + b_{ij}) \times (\det(A))^{N-1} \\ &= (\det(A))^N \times \det\left(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)}\right). \end{aligned} \quad (31)$$

So $\det(a_{ij}x_i + b_{ij}) = \det(A) \times \det(\delta_{ij}x_i + \det(H_{ij})/\det(A))$. $\square$

We use the notations $U = (\alpha_{ij})_{i,j \in [1, \dots, 2N]}$ and $V = (\beta_{ij})_{i,j \in [1, \dots, 2N]}$.

Using the preceding lemma, we get

$$\begin{aligned} \widetilde{W}_r &= \det(\alpha_{ij}e_i + \beta_{ij}) \\ &= \det(\alpha_{ij}) \times \det\left(\delta_{ij}e_i + \frac{\det(H_{ij})}{\det(\alpha_{ij})}\right) \\ &= \det(U) \times \det\left(\delta_{ij}e_i + \frac{\det(H_{ij})}{\det(U)}\right), \end{aligned} \quad (32)$$

where $(H_{ij})_{i,j \in [1, \dots, N]}$ is the matrix obtained by replacing in U the j th row of U by the i th row of V defined previously.

U is the classical Vandermonde determinant that is equal to

$$\det(U) = i^{N(2N-1)} \prod_{2N \geq l > m \geq 1} (\gamma_l - \gamma_m). \quad (33)$$

We have to compute $\det(H_{ij})$ to evaluate the determinant $\widetilde{W}_r$. To do that, we study two cases.

(1) For $1 \leq j \leq N$, the matrix H_{ij} is a Vandermonde matrix, where the j th row of U in U is replaced by the i th row of V . Then we have

$$\det(H_{ij}) = (-1)^{N(2N+1)+N-1} (i)^{N(2N-1)} \times M, \quad (34)$$

with $M = M(m_1, \dots, m_{2N})$ being the determinant defined by $m_k = \gamma_k$ for $k \neq j$ and $m_j = -\gamma_i$. Thus we get

$$\begin{aligned} \det(H_{ij}) &= - (i)^{N(2N-1)} \times \prod_{2N \geq l > k \geq 1,} (m_l - m_k) \\ &= - (i)^{N(2N-1)} \times \prod_{2N \geq l > m \geq 1, l \neq j, m \neq j} (\gamma_l - \gamma_m) \\ &\quad \times \prod_{l < j} (-\gamma_i - \gamma_l) \times \prod_{l > j} (\gamma_l + \gamma_i), \\ &= (-1)^j (i)^{N(2N-1)} \\ &\quad \times \prod_{2N \geq l > m \geq 1, l \neq j, m \neq j} (\gamma_l - \gamma_m) \\ &\quad \times \prod_{l \neq j} (\gamma_l + \gamma_i). \end{aligned} \quad (35)$$

To compute $\widetilde{W}_r$, we have to simplify the quotient $q_{ij} := \det(H_{ij})/\det(U)$:

$$\begin{aligned} q_{ij} &= \frac{(-1)^j (i)^{N(2N-1)} \times \prod_{2N \geq l > m \geq 1, l \neq j, m \neq j} (\gamma_l - \gamma_m) \times \prod_{l \neq j} (\gamma_l + \gamma_i)}{i^{N(2N-1)} \prod_{2N \geq l > m \geq 1} (\gamma_l - \gamma_m)} \\ &= \frac{(-1)^j \prod_{l \neq j} (\gamma_l + \gamma_i)}{\prod_{l < j} (\gamma_j - \gamma_l) \prod_{l > j} (\gamma_l - \gamma_j)} = \frac{(-1)^j \prod_{l \neq j} (\gamma_l + \gamma_i)}{(-1)^{j-1} \prod_{l \neq j} (\gamma_l - \gamma_j)} \\ &= - \frac{\prod_{l \neq j} (\gamma_l + \gamma_i)}{\prod_{l \neq j} (\gamma_l - \gamma_j)}. \end{aligned} \quad (36)$$

q_{ij} is equal to r_{ij} defined by $-\prod_{l \neq j} (\gamma_l + \gamma_i)/\prod_{l \neq i} (\gamma_l - \gamma_i)$, because $\det(\delta_{ij}x_i + \det(q_{ij})/\det(A)) = \det(\delta_{ij}x_i + \det(r_{ij})/\det(A))$.

Thus r_{ij} can be written as

$$r_{ij} = (-1)^{\epsilon_i} \left(\prod_{l \neq j} \frac{\gamma_l + \gamma_i}{\gamma_l - \gamma_j} \right) = c_{ij} e^{-2i\Theta_{r,i}(0)}, \quad (37)$$

with the notations given in (17).

(2) We can do the same estimations for $N+1 \leq j \leq 2N$ are made; $\det(H_{ij})$ is first as follows:

$$\det(H_{ij}) = (-1)^{N(2N+1)+N-1} (i)^{N(2N-1)} \times M, \quad (38)$$

with $M = M(m_1, \dots, m_{2N})$ being the determinant defined by $m_k = \gamma_k$ for $k \neq j$ and $m_j = -\gamma_i$. Then we get

$$\begin{aligned} \det(H_{ij}) &= (i)^{N(2N-1)} \times \prod_{2N \geq l > k \geq 1,} (m_l - m_k) \\ &= (i)^{N(2N-1)} \times \prod_{2N \geq l > m \geq 1, l \neq j, m \neq j} (\gamma_l - \gamma_m) \\ &\quad \times \prod_{l < j} (-\gamma_l - \gamma_i) \times \prod_{l > j} (\gamma_l + \gamma_i), \end{aligned}$$

$$\begin{aligned} &= (-1)^{j-1} (i)^{N(2N-1)} \\ &\quad \times \prod_{2N \geq l > m \geq 1, l \neq j, m \neq j} (\gamma_l - \gamma_m) \\ &\quad \times \prod_{l \neq j} (\gamma_l + \gamma_i). \end{aligned} \quad (39)$$

Then $q_{ij} := \det(H_{ij}) / \det(U)$ can be expressed as

$$\begin{aligned} q_{ij} &= \frac{(-1)^{j-1} (i)^{N(2N-1)} \times \prod_{2N \geq l > m \geq 1, l \neq j, m \neq j} (\gamma_l - \gamma_m) \times \prod_{l \neq j} (\gamma_l + \gamma_i)}{i^{N(2N-1)} \prod_{2N \geq l > m \geq 1} (\gamma_l - \gamma_m)} = \frac{(-1)^{j-1} \prod_{l \neq j} (\gamma_l + \gamma_i)}{\prod_{l < j} (\gamma_j - \gamma_l) \prod_{l > j} (\gamma_l - \gamma_j)} \\ &= \frac{(-1)^{j-1} \prod_{l \neq j} (\gamma_l + \gamma_i)}{(-1)^{j-1} \prod_{l \neq j} (\gamma_l - \gamma_j)} = \frac{\prod_{l \neq j} (\gamma_l + \gamma_i)}{\prod_{l \neq j} (\gamma_l - \gamma_j)}. \end{aligned} \quad (40)$$

q_{ij} is replaced by r_{ij} defined by $\prod_{l \neq j} (\gamma_l + \gamma_i) / \prod_{l \neq i} (\gamma_l - \gamma_i)$, for the same reason as previously exposed.

Then r_{ij} can be written as

$$\begin{aligned} r_{ij} &= (-1)^{\epsilon_i} \prod_{l \neq j} \left(\frac{\gamma_l + \gamma_i}{\gamma_l - \gamma_i} \right) = (-1)^{\epsilon(i)} \frac{\prod_{l \neq j} (\gamma_l + \gamma_i)}{\prod_{l \neq i} (\gamma_l - \gamma_i)} \\ &= c_{ij} e^{-2i\Theta_{r,i}(0)}, \end{aligned} \quad (41)$$

with notations given in (17).

e_i is replaced by $e^{-2i\Theta_{r,i}}$. Then $\det \widetilde{W}_r$ can be rewritten as

$$\begin{aligned} \det \widetilde{W}_r &= \det(U) \times \det \left(\delta_{ij} e_i + \frac{\det(H_{ij})}{\det(U)} \right) \\ &= \det(U) \times \det(\delta_{ij} e_i + r_{ij}) = \det(U) \\ &\quad \cdot \prod_{i=1}^{2N} e^{-2i\Theta_i} \det \left(\delta_{ij} + (-1)^{\epsilon(i)} \prod_{l \neq i} \left| \frac{\gamma_l + \gamma_i}{\gamma_l - \gamma_i} \right| e^{2i\Theta_{r,i}} \right). \end{aligned} \quad (42)$$

We compute the two members of the last relation (42) in $y = 0$. Using (33), we get

$$\begin{aligned} \det \widetilde{W}_r(0) &= i^{N(2N-1)} \prod_{2N \geq l > m \geq 1} (\gamma_l - \gamma_m) \prod_{i=1}^{2N} e^{-2i\Theta_{r,i}(0)} \\ &\quad \times \det \left(\delta_{ij} + (-1)^{\epsilon(i)} \prod_{l \neq i} \left| \frac{\gamma_l + \gamma_i}{\gamma_l - \gamma_i} \right| e^{2i\Theta_{r,i}(0)} \right) \\ &= i^{N(2N-1)} \prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i) e^{-2i \sum_{i=1}^{2N} \Theta_{r,i}(0)} \det(\delta_{ij} + c_{ij}) \end{aligned}$$

$$\begin{aligned} &= i^{N(2N-1)} \prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i) e^{-2i \sum_{i=1}^{2N} \Theta_{r,i}(0)} \det(I + C_r) \\ &= i^{N(2N-1)} \prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i) e^{-2i \sum_{i=1}^{2N} \Theta_{r,i}(0)} \det(I + D_r). \end{aligned} \quad (43)$$

Thus, the Wronskian W_r given by (26) can be rewritten as

$$\begin{aligned} W_r(\phi_{r,1}, \dots, \phi_{r,2N})(0) &= \prod_{j=1}^{2N} e^{i\Theta_{r,j}(0)} (2)^{-2N} (i)^{-N} \\ &\quad \times \widetilde{W}_r = \prod_{j=1}^{2N} e^{i\Theta_{r,j}(0)} (2)^{-2N} (i)^{-N} \\ &\quad \cdot i^{N(2N-1)} \prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i) e^{-2i \sum_{i=1}^{2N} \Theta_{r,i}(0)} \det(I + D_r) \\ &= (2)^{-2N} \prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i) e^{-i \sum_{i=1}^{2N} \Theta_{r,i}(0)} \det(I + D_r). \end{aligned} \quad (44)$$

Then

$$\det(I + D_r) = k_r(0) W_r(\phi_1, \dots, \phi_{2N})(0). \quad (45)$$

This finishes the proof of Theorem 2.

Then the solution v to the Johnson equation can be rewritten as

$$v(x, y, t) = -2 \frac{|\det(I + D_3(x, y, t))|^2}{(\det(I + D_1(x, y, t)))^2}. \quad (46)$$

With (24), the following link between Fredholm determinants and Wronskians is obtained:

$$\det(I + D_3) = k_3(0) \times W_3(\phi_{r,1}, \dots, \phi_{r,2N})(0) \quad (47)$$

and

$$\det(I + D_1) = k_1(0) \times W_1(\phi_{r,1}, \dots, \phi_{r,2N})(0). \quad (48)$$

As $\Theta_{3,j}(0)$ contains N terms $x_{3,j}$ $1 \leq j \leq N$ and N terms $-x_{3,j}$ $1 \leq j \leq N$, we have the relation $k_3(0) = k_1(0)$, and we get the following theorem.

Theorem 4. *The function v defined by*

$$v(x, y, t) = -2 \frac{|W_3(\phi_{3,1}, \dots, \phi_{3,2N})(0)|^2}{(W_1(\phi_{1,1}, \dots, \phi_{1,2N})(0))^2} \quad (49)$$

is a solution of the Johnson equation which depends on $2N - 1$ real parameters a_k , b_k , and ϵ , with ϕ_v^r defined in (21).

$$\begin{aligned} \phi_{r,v} &= \sin\left(\frac{-i\kappa_v x}{2} + i\left(\frac{-\kappa_v y}{24} + \delta_v\right)yt - i\frac{x_{r,v}}{2} + 2\tau_v t\right. \\ &\quad \left.+ \gamma_v w - i\frac{e_v}{2}\right), \quad 1 \leq v \leq N, \\ \phi_{r,v} &= \cos\left(\frac{-i\kappa_v x}{2} + i\left(\frac{-\kappa_v y}{24} + \delta_v\right)yt - i\frac{x_{r,v}}{2} + 2\tau_v t\right. \\ &\quad \left.+ \gamma_v w - i\frac{e_v}{2}\right), \quad N + 1 \leq v \leq 2N, \quad r = 1, 3, \end{aligned} \quad (50)$$

where κ_v , δ_v , $x_{r,v}$, γ_v , and e_v are defined in (6), (5), and (7).

4. Study of the Limit Case When ϵ Tends to 0

4.1. Rational Solutions of Order N Depending on $2N - 2$ Parameters. An infinite hierarchy of rational solutions to the Johnson equation depending on $2N - 2$ parameters is obtained. For this, we take the limit when the parameter ϵ tends to 0.

We get the following statement.

Theorem 5. *The function v*

$$v(x, y, t) = \lim_{\epsilon \rightarrow 0} -2 \frac{|W_3(x, y, t)|^2}{(W_1(x, y, t))^2} \quad (51)$$

is a rational solution to the Johnson equation. It is a quotient of two polynomials $n(x, y, t)$ and $d(x, y, t)$ depending on $2N - 2$ real parameters $\tilde{a}_j$ and $\tilde{b}_j$, $1 \leq j \leq N - 1$, of degrees $2N(N + 1)$ in x , t and $4N(N + 1)$ in y .

4.2. Families of Rational Solutions of Order 3 Depending on 4 Parameters. Here we construct families of rational solutions to the Johnson equation of order 3 explicitly; they depend on 4 parameters.

We only give the expression without parameters and we give it in the appendix because of the length of the solutions.

We construct the patterns of the modulus of the solutions in the plane (x, y) of coordinates in functions of parameters a_i , b_i , $1 \leq i \leq 2$, and time t .

The role of the parameters a_i and b_i for the same integer i is the same one; one will be interested primarily only in parameters a_i .

The study of these configurations makes it possible to give the following conclusions. The variation of the configuration of the module of the solutions is very fast according to time t . When time t grows from 0 to 0.01, one passes from a rectilinear structure with a height of 98 to a horseshoe structure with a maximum height equal to 4. The role played by the parameters a_i and b_i is the same for same index i . When variables x , y , and time tend towards infinity, the modulus of the solutions tends towards 2 in accordance with the structure of the polynomials which will be studied in a forthcoming article.

5. Conclusion

We have constructed solutions to the Johnson equation, starting from the solutions of the KPI equation, what makes it possible to obtain rational solutions. These solutions are expressed by means of quotients of two polynomials of degree $2N(N + 1)$ in x , t and $4N(N + 1)$ in y depending on $2N - 2$ parameters.

Here we have given a new method to construct solutions to the Johnson equation related to previous results [12–14].

We have given two types of representations of the solutions to the Johnson equation. An expression by means of Fredholm determinants of order $2N$ depending on $2N - 1$ real parameters is given. Another expression by means of Wronskians of order $2N$ depending on $2N - 1$ real parameters is also constructed. Also rational solutions to the Johnson equation depending on $2N - 2$ real parameters are obtained when one of parameters (ϵ) tends to zero.

The patterns of the modulus of the solutions in the plane (x, y) and their evolution according to time and parameters have been studied in Figures 1, 2, 3, 4, and 5.

In another study, we will give a more general representation of rational solutions to the Johnson equation. It can be written without limit at order N depending on $2N - 2$ real parameters. We will prove that these solutions can be written as a quotient of polynomials of degree $2N(N + 1)$ in x , t and $4N(N + 1)$ in y .

Appendix

The solutions to the Johnson equation can be written as

$$\begin{aligned} v_3(x, y, t) &= -2 \frac{|n_3(x, y, t)|^2}{(d_3(x, y, t))^2}, \\ n_3(x, y, t) &= A_3(x, y, t) + iB_3(x, y, t), \\ d_3(x, y, t) &= C_3(x, y, t) \end{aligned} \quad (A.1)$$

with

$$\begin{aligned} A_3(x, y, t) &= \sum_{k=0}^{12} a_k(y, t) x^k, \\ B_3(x, y, t) &= \sum_{k=0}^{12} b_k(y, t) x^k, \\ C_3(x, y, t) &= \sum_{k=0}^{12} c_k(y, t) x^k, \end{aligned} \quad (A.2)$$

FIGURE 1: Solution of order 3 to (1), on the left for $t = 0$, $a_1 = 0$, $b_1 = 0$, $a_2 = 0$, and $b_2 = 0$; in the center for $t = 0$, $a_1 = 10^3$, $b_1 = 0$, $a_2 = 0$, and $b_2 = 0$; and on the right for $t = 0$, $a_1 = 0$, $b_1 = 10^3$, $a_2 = 0$, and $b_2 = 0$.

FIGURE 2: Solution of order 3 to (1), on the left for $t = 0$, $a_1 = 10^3$, $b_1 = 0$, $a_2 = 0$, and $b_2 = 0$; in the center for $t = 0$, $a_1 = 0$, $b_1 = 0$, $a_2 = 0$, and $b_2 = 10^6$; and on the right for $t = 0$, $a_1 = 0$, $b_1 = 10^3$, $a_2 = 0$, and $b_2 = 0$.

$$a_{12} = 8916100448256,$$

$$a_{11} = 8916100448256t^2 + 2567836929097728t,$$

$$a_{10}$$

$$= 4086546038784t^2y^4$$

$$+ 2139864107581440t^2y^2$$

$$+ 338954474640900096t^2$$

$$+ 40122452017152,$$

$$a_9$$

$$= 1135151677440t^3y^6 + 802449040343040t^3y^4$$

$$+ 231105323618795520t^3y^2$$

$$+ 27116357971272007680t^3$$

$$+ 33435376680960ty^2$$

$$+ 13909116699279360t,$$

$$a_8$$

$$= 212840939520t^4y^8 + 178322008965120t^4y^6$$

$$+ 69545583496396800t^4y^4$$

$$+ 14790740711602913280t^4y^2$$

$$+ 12538266255360t^2y^4$$

$$+ 1464283330448688414720t^4$$

$$+ 6419592322744320t^2y^2$$

FIGURE 3: Solution of order 3 to (1), on the left for $t = 0, 01$, $a_1 = 0$, $b_1 = 0$, $a_2 = 10^6$, and $b_2 = 0$; in the center for $t = 0, 1$, $a_1 = 10^3$, $b_1 = 0$, $a_2 = 0$, and $b_2 = 0$; and on the right for $t = 1$, $a_1 = 10^3$, $b_1 = 0$, $a_2 = 0$, and $b_2 = 0$.

FIGURE 4: Solution of order 3 to (1), on the left for $t = 0, 1$, $a_1 = 0$, $b_1 = 0$, $a_2 = 10^6$, and $b_2 = 0$; in the center for $t = 1$, $a_1 = 0$, $b_1 = 0$, $a_2 = 10^6$, and $b_2 = 0$; and on the right for $t = 10$, $a_1 = 10^3$, $b_1 = 0$, $a_2 = 0$, and $b_2 = 0$.

$$\begin{aligned}
 &+ 1964395250759761920t^2 \\
 &- 125382662553600, \\
 \mathbf{a}_7 &= 28378791936t^5y^{10} + 26005292974080t^5y^8 \\
 &+ 12125896609628160t^5y^6 \\
 &+ 3492258223572910080t^5y^4 \\
 &+ 2786281390080t^3y^6 \\
 &+ 621211109887322357760t^5y^2 \\
 &+ 802449040343040t^3y^4 \\
 &+ 56228479889229635125248t^5 \\
 &+ 539245755110522880t^3y^2 \\
 &+ 155302777471830589440t^3 \\
 &- 83588441702400ty^2 \\
 &- 24073471210291200t, \\
 \mathbf{a}_6 &= 2759049216t^6y^{12} + 2600529297408t^6y^{10} \\
 &+ 1352275234652160t^6y^8 \\
 &+ 467916951524474880t^6y^6
 \end{aligned}$$

FIGURE 5: Solution of order 3 to (1), on the left for $t = 10$, $a_1 = 0$, $b_1 = 0$, $a_2 = 10^6$, and $b_2 = 0$; in the center for $t = 100$, $a_1 = 10^6$, $b_1 = 0$, $a_2 = 0$, and $b_2 = 0$; and on the right for $t = 10^3$, $a_1 = 10^5$, $b_1 = 10^3$, $a_2 = 0$, and $b_2 = 0$.

$$\begin{aligned}
 &+ 406332702720t^4y^8 \\
 &+ 112163117062988759040t^6y^4 \\
 &- 104021171896320t^4y^6 \\
 &+ 17890879964754883903488t^6y^2 \\
 &- 18188844914442240t^4y^4 \\
 &+ 1574397436898429783506944t^6 \\
 &+ 25883796245305098240t^4y^2 \\
 &- 24379962163200t^2y^4 \\
 &+ 7765138873591529472000t^4 \\
 &- 18456327927889920t^2y^2 \\
 &- 1714031150172733440t^2 \\
 &+ 376147987660800, \\
 &- 24073471210291200t^5y^6 \\
 &+ 357817599295097678069760t^7y^2 \\
 &- 3030047576335319040t^5y^4 \\
 &- 4063327027200t^3y^6 \\
 &+ 32387604416196269832142848t^7 \\
 &+ 776513887359152947200t^5y^2 \\
 &- 5717449412444160t^3y^4 \\
 &+ 259417759488945816600576t^5 \\
 &- 722204136308736000t^3y^2 \\
 &- 52692013785085378560t^3 \\
 &+ 188073993830400ty^2 \\
 &+ 78238781433446400t,
 \end{aligned}$$

$\mathbf{a}_5$

$$\begin{aligned}
 &= 197074944t^7y^{14} + 180592312320t^7y^{12} \\
 &+ 100553799499776t^7y^{10} \\
 &+ 39230841972326400t^7y^8 \\
 &+ 40633270272t^5y^{10} \\
 &+ 11298482488030003200t^7y^6 \\
 &- 45509262704640t^5y^8 \\
 &+ 2402016291564313116672t^7y^4
 \end{aligned}$$

$\mathbf{a}_4$

$$\begin{aligned}
 &= 10264320t^8y^{16} + 8599633920t^8y^{14} \\
 &+ 5056584744960t^8y^{12} \\
 &+ 2139864107581440t^8y^{10} \\
 &+ 2821754880t^6y^{12} \\
 &+ 697595699071549440t^8y^8 \\
 &- 6501323243520t^6y^{10} \\
 &+ 17748888539234959360t^8y^6
 \end{aligned}$$

$$\begin{aligned}
& - 3845068318310400t^6y^8 \\
& + 34787822153690052034560t^8y^4 \\
& - 1044253684499742720t^6y^6 \\
& - 423263232000t^4y^8 \\
& + 4907212790332768156385280t^8y^2 \\
& - 109389853179563212800t^6y^4 \\
& - 947335672627200t^4y^6 \\
& + 485814066242944047482142720t^8 \\
& + 14909066637295736586240t^6y^2 \\
& - 102312252643737600t^4y^4 \\
& + 5903990388369111688151040t^6 \\
& + 13866319417127731200t^4y^2 \\
& + 39182082048000t^2y^4 \\
& - 249593749508299161600t^4 \\
& + 34104084214579200t^2y^2 \\
& + 9605315012906188800t^2 \\
& + 493694233804800,
\end{aligned}$$

 $\mathbf{a}_3$

$$\begin{aligned}
& = 380160t^9y^{18} + 268738560t^9y^{16} \\
& + 171992678400t^9y^{14} \\
& + 75951966781440t^9y^{12} + 134369280t^7y^{14} \\
& + 27104945362698240t^9y^{10} \\
& - 511678218240t^7y^{12} \\
& + 7806224264457093120t^9y^8 \\
& - 288534917283840t^7y^{10} \\
& + 1814330860623290695680t^9y^6 \\
& - 100216969038397440t^7y^8 \\
& - 28217548800t^5y^{10} \\
& + 340778665995331121971200t^9y^4 \\
& - 20508457607060520960t^7y^6 \\
& - 91715095756800t^5y^8 \\
& + 44164915112994913407467520t^9y^2
\end{aligned}$$

$$\begin{aligned}
& - 1804470366815555420160t^7y^4 \\
& - 3789342690508800t^5y^6 \\
& + 5182016706591403173142855680t^9 \\
& + 178908799647548839034880t^7y^2 \\
& + 8152882249885286400t^5y^4 \\
& + 4353564672000t^3y^6 \\
& + 90783436621156210893127680t^7 \\
& + 1358899302878517657600t^5y^2 \\
& + 5934779360870400t^3y^4 \\
& + 29285666608973768294400t^5 \\
& + 1420334801407180800t^3y^2 \\
& + 526149786772124467200t^3 \\
& + 164564744601600ty^2 \\
& + 119615060076134400t,
\end{aligned}$$

 $\mathbf{a}_2$

$$\begin{aligned}
& = 9504t^{10}y^{20} + 4976640t^{10}y^{18} \\
& + 3941498880t^{10}y^{16} + 1651129712640t^{10}y^{14} \\
& + 4199040t^8y^{16} + 653847366205440t^{10}y^{12} \\
& - 23648993280t^8y^{14} \\
& + 205426954327818240t^{10}y^{10} \\
& - 10990332149760t^8y^{12} \\
& + 54232715942544015360t^{10}y^8 \\
& - 4695812902748160t^8y^{10} \\
& - 1175731200t^6y^{12} \\
& + 11359288866511037399040t^{10}y^6 \\
& - 1230421861859328000t^8y^8 \\
& - 5224277606400t^6y^{10} \\
& + 2249139195569185405009920t^{10}y^4 \\
& - 212000616866308423680t^8y^6 \\
& + 376147987660800t^6y^8 \\
& + 235546213935972871506493440t^{10}y^2 \\
& - 13134177751903386992640t^8y^4
\end{aligned}$$

$$\begin{aligned}
& + 1033554363961835520t^6y^6 \\
& + 272097792000t^4y^8 \\
& + 37310520287458102846628560896t^{10} \\
& + 1226803197583192039096320t^8y^2 \\
& + 246897520732746547200t^6y^4 \\
& + 515462057164800t^4y^6 \\
& + 905380759816395724853084160t^8 \\
& + 27954499944929506099200t^6y^2 \\
& + 141431643360460800t^4y^4 \\
& + 862595998300681902489600t^6 \\
& + 44295187026935808000t^4y^2 \\
& + 20570593075200t^2y^4 \\
& + 13630591987036559769600t^4 \\
& - 10155995666841600t^2y^2 \\
& + 8350485326069760000t^2 \\
& - 246847116902400, \\
\mathbf{a}_1 & \\
& = 144t^{11}y^{22} + 41472t^{11}y^{20} + 59719680t^{11}y^{18} \\
& + 17199267840t^{11}y^{16} + 77760t^9y^{18} \\
& + 9906778275840t^{11}y^{14} - 604661760t^9y^{16} \\
& + 2853152143441920t^{11}y^{12} \\
& - 180592312320t^9y^{14} \\
& + 821707817311272960t^{11}y^{10} \\
& - 121358033879040t^9y^{12} - 27993600t^7y^{14} \\
& + 236651851385646612480t^{11}y^8 \\
& - 31028029559930880t^9y^{10} \\
& - 162855567360t^7y^{12} \\
& + 34077866599533112197120t^{11}y^6 \\
& - 6881802969981911040t^9y^8 \\
& + 37614798766080t^7y^{10} \\
& + 9814425580665536312770560t^{11}y^4 \\
& - 1242422219774644715520t^9y^6 \\
& + 47210751873515520t^7y^8 \\
& + 9069926400t^5y^{10} \\
& + 565310913446334891615584256t^{11}y^2 \\
& - 17038933299766556098560t^9y^4 \\
& + 28592864205503201280t^7y^6 \\
& + 22348298649600t^5y^8 \\
& + 162809543072544448785288265728t^{11} \\
& + 3680409592749576117288960t^9y^2 \\
& + 425233795458583756800t^7y^4 \\
& + 16216157690265600t^5y^6 \\
& + 5299789813559389608896102400t^9 \\
& + 188493199628667526840320t^7y^2 \\
& + 3450530873475072000t^5y^4 \\
& + 1142810726400t^3y^6 \\
& + 10121126380061334322544640t^7 \\
& + 654028065841191321600t^5y^2 \\
& - 4184646362726400t^3y^4 \\
& + 164399083009466381107200t^5 \\
& - 675561785838796800t^3y^2 \\
& + 332358343529280307200t^3 \\
& - 41141186150400t^2y^2 \\
& - 59243308056576000t, \\
\mathbf{a}_0 & \\
& = t^{12}y^{24} + 497664t^{12}y^{20} + 648t^{10}y^{20} \\
& + 103195607040t^{12}y^{16} - 6635520t^{10}y^{18} \\
& - 447897600t^{10}y^{16} \\
& + 11412608573767680t^{12}y^{12} \\
& - 1651129712640t^{10}y^{14} - 291600t^8y^{16} \\
& - 113927950172160t^{10}y^{12} \\
& - 2149908480t^8y^{14} \\
& + 709955554156939837440t^{12}y^8 \\
& - 136951302885212160t^{10}y^{10}
\end{aligned}$$

$$\begin{aligned}
& + 935210188800t^8y^{12} \\
& - 6162808629834547200t^{10}y^8 \\
& + 594406696550400t^8y^{10} + 125971200t^6y^{12} \\
& + 23554621393597287150649344t^{12}y^4 \\
& - 3786429622170345799680t^{10}y^6 \\
& + 304663152316907520t^8y^8 \\
& + 386983526400t^6y^{10} \\
& + 153350399697899004887040t^{10}y^4 \\
& + 274450410981965168640t^8y^6 \\
& + 727916013158400t^6y^8 \\
& + 325619086145088897570576531456t^{12} \\
& - 26223983281671965245440t^8y^4 \\
& - 47790298402652160t^6y^6 \\
& + 23808556800t^4y^8 \\
& + 13850117379435204844581814272t^{10} \\
& + 175158576526078771200t^6y^4 \\
& - 208971104256000t^4y^6 \\
& + 45085017511182307436789760t^8 \\
& + 68145477097881600t^4y^4 \\
& + 732141665224344207360000t^6 \\
& - 1714216089600t^2y^4 \\
& + 5839742646192439296000t^4 \\
& - 629671731344179200t^2 - 30855889612800
\end{aligned} \tag{A.3}$$

$$\mathbf{b}_{12} = 0,$$

$$\mathbf{b}_{11} = 0,$$

$$\mathbf{b}_{10} = 213986410758144ty,$$

$$\mathbf{b}_9$$

$$\begin{aligned}
& = 178322008965120t^2y^3 \\
& + 51356738581954560t^2y,
\end{aligned}$$

$$\mathbf{b}_8$$

$$= 66870753361920t^3y^5$$

$$\begin{aligned}
& + 34237825721303040t^3y^3 \\
& + 5546527766851092480t^3y \\
& + 802449040343040ty,
\end{aligned}$$

$$\mathbf{b}_7$$

$$\begin{aligned}
& = 14860167413760t^4y^7 \\
& + 9986032502046720t^4y^5 \\
& + 2875977360589455360t^4y^3 \\
& + 354977777078469918720t^4y \\
& + 534966026895360t^2y^3 \\
& + 154070215745863680t^2y,
\end{aligned}$$

$$\mathbf{b}_6$$

$$\begin{aligned}
& = 2167107747840t^5y^9 \\
& + 1664338750341120t^5y^7 \\
& + 633399775844106240t^5y^5 \\
& + 138046913308293857280t^5y^3 \\
& + 156031757844480t^3y^5 \\
& + 14909066637295736586240t^5y \\
& + 59916195012280320t^3y^3 \\
& + 12941898122652549120t^3y \\
& - 2006122600857600ty,
\end{aligned}$$

$$\mathbf{b}_5$$

$$\begin{aligned}
& = 216710774784t^6y^{11} + 173368619827200t^6y^9 \\
& + 77035107872931840t^6y^7 \\
& + 22186111067404369920t^6y^5 \\
& + 26005292974080t^4y^7 \\
& + 4141407399248815718400t^6y^3 \\
& + 7489524376535040t^4y^5 \\
& + 429381119154117213683712t^6y \\
& + 2978690837753364480t^4y^3 \\
& + 621211109887322357760t^4y \\
& - 1003061300428800t^2y^3 \\
& - 442951870269358080t^2y,
\end{aligned}$$

b₄

$$\begin{aligned}
&= 15049359360t^7y^{13} + 11557907988480t^7y^{11} \\
&\quad + 5646863617228800t^7y^9 \\
&\quad + 1917318240392970240t^7y^7 \\
&\quad + 2708884684800t^5y^9 \\
&\quad + 468373455867425587200t^7y^5 \\
&\quad + 79515022065577261793280t^7y^3 \\
&\quad + 72755379657768960t^5y^5 \\
&\quad + 8587622383082344273674240t^7y \\
&\quad + 98604938077352755200t^5y^3 \\
&\quad - 208971104256000t^3y^5 \\
&\quad + 1863633296619670732800t^5y \\
&\quad - 160489808068608000t^3y^3 \\
&\quad - 17332899271409664000t^3y \\
&\quad + 4513775851929600ty
\end{aligned}$$

b₃

$$\begin{aligned}
&= 716636160t^8y^{15} + 481579499520t^8y^{13} \\
&\quad + 257576235171840t^8y^{11} \\
&\quad + 97007172877025280t^8y^9 \\
&\quad + 180592312320t^6y^{11} \\
&\quad + 27938065788583280640t^8y^7 \\
&\quad - 86684309913600t^6y^9 \\
&\quad + 6152948136026811924480t^8y^5 \\
&\quad - 32811249649582080t^6y^7 \\
&\quad + 954180264786927141519360t^8y^3 \\
&\quad + 410853908655636480t^6y^5 \\
&\quad - 23219011584000t^4y^7 \\
&\quad + 117773106967986435753246720t^8y \\
&\quad + 2662333328088524390400t^6y^3 \\
&\quad - 227360561430528000t^4y^5 \\
&\quad + 357817599295097678069760t^6y \\
&\quad - 7575118940838297600t^4y^3
\end{aligned}$$

$$\begin{aligned}
&+ 332791666011065548800t^4y \\
&+ 1504591950643200t^2y^3 \\
&+ 818498021149900800t^2y
\end{aligned}$$

b₂

$$\begin{aligned}
&= 22394880t^9y^{17} + 11466178560t^9y^{15} \\
&\quad + 7430083706880t^9y^{13} \\
&\quad + 2853152143441920t^9y^{11} \\
&\quad + 7524679680t^7y^{13} \\
&\quad + 924421294475182080t^9y^9 \\
&\quad - 8668430991360t^7y^{11} \\
&\quad + 236651851385646612480t^9y^7 \\
&\quad - 3358397835509760t^7y^9 \\
&\quad + 51116799899299668295680t^9y^5 \\
&\quad - 616280862983454720t^7y^7 \\
&\quad - 1451188224000t^5y^9 \\
&\quad + 6542950387110357541847040t^9y^3 \\
&\quad + 36976851779007283200t^7y^5 \\
&\quad - 1560317578444800t^5y^7 \\
&\quad + 1059957962711877921779220480t^9y \\
&\quad + 53956622115927427645440t^7y^3 \\
&\quad - 805658836504412160t^5y^5 \\
&\quad + 4293811191541172136837120t^7y \\
&\quad - 80116512187849113600t^5y^3 \\
&\quad + 188073993830400t^3y^5 \\
&\quad + 32613583269084423782400t^5y \\
&\quad + 120367356051456000t^3y^3 \\
&\quad + 34088035233772339200t^3y \\
&\quad + 3949553870438400ty,
\end{aligned}$$

b₁

$$\begin{aligned}
&= 414720t^{10}y^{19} + 119439360t^{10}y^{17} \\
&\quad + 137594142720t^{10}y^{15} \\
&\quad + 39627113103360t^{10}y^{13} + 179159040t^8y^{15}
\end{aligned}$$

$$\begin{aligned}
& + 17118912860651520t^{10}y^{11} \\
& - 361184624640t^8y^{13} \\
& + 4930246903867637760t^{10}y^9 \\
& - 113927950172160t^8y^{11} \\
& + 946607405542586449920t^{10}y^7 \\
& - 32811249649582080t^8y^9 \\
& - 48372940800t^6y^{11} \\
& + 272622932796264897576960t^{10}y^5 \\
& - 2875977360589455360t^8y^7 \\
& - 51081825484800t^6y^9 \\
& + 19628851161331072625541120t^{10}y^3 \\
& + 118325925692823306240t^8y^5 \\
& - 16583946833756160t^6y^7 \\
& + 5653109134463348916155842560t^{10}y \\
& + 647479465391129131745280t^8y^3 \\
& - 19977771308380323840t^6y^5 \\
& + 10448555212800t^4y^7 \\
& + 29443276741996608938311680t^8y \\
& + 2033726847845400576000t^6y^3 \\
& + 3009183901286400t^4y^5 \\
& + 670907998678308146380800t^6y \\
& - 674057193888153600t^4y^3 \\
& + 1063084488646459392000t^4y \\
& + 658258978406400t^2y^3 \\
& - 243743896004198400t^2y, \\
\mathbf{b}_0 & = 3456t^{11}y^{21} + 1433272320t^{11}y^{17} \\
& + 1866240t^9y^{17} + 237762678620160t^{11}y^{13} \\
& - 5733089280t^9y^{15} - 1031956070400t^9y^{13} \\
& + 19720987615470551040t^{11}y^9 \\
& - 475525357240320t^9y^{11} - 671846400t^7y^{13} \\
& - 196867497897492480t^9y^9 \\
& - 619173642240t^7y^{11} \\
& + 817868798388794692730880t^{11}y^5 \\
& + 39441975230941102080t^9y^7 \\
& + 664992491765760t^7y^9 \\
& - 7099555541569398374400t^9y^5 \\
& - 171189128606515200t^7y^7 \\
& + 217678233600t^5y^9 \\
& + 13567461922712037398774022144t^{11}y \\
& + 3271475193555178770923520t^9y^3 \\
& - 122331751302215761920t^7y^5 \\
& - 111451255603200t^5y^7 \\
& + 88329830225989826814935040t^9y \\
& + 35497777707846991872000t^7y^3 \\
& + 222278384175022080t^5y^5 \\
& + 4523836791088020644167680t^7y \\
& - 77035107872931840000t^5y^3 \\
& + 27427457433600t^3y^5 \\
& + 15696673580188591718400t^5y \\
& - 12036735605145600t^3y^3 \\
& - 16213482860131123200t^3y \\
& - 987388467609600ty \\
& \mathbf{c}_{12} = 8916100448256, \\
& \mathbf{c}_{11} = 8916100448256ty^2 + 2567836929097728t, \\
& \mathbf{c}_{10} \\
& = 4086546038784t^2y^4 \\
& + 2139864107581440t^2y^2 \\
& + 338954474640900096t^2 \\
& - 13374150672384, \\
& \mathbf{c}_9 \\
& = 1135151677440t^3y^6 + 802449040343040t^3y^4
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
& + 231105323618795520t^3y^2 \\
& + 27116357971272007680t^3 \\
& - 11145125560320ty^2 + 1069932053790720t,
\end{aligned}$$

 $\mathbf{c}_8$

$$\begin{aligned}
= & 212840939520t^4y^8 + 178322008965120t^4y^6 \\
& + 69545583496396800t^4y^4 \\
& + 14790740711602913280t^4y^2 \\
& - 4179422085120t^2y^4 \\
& + 1464283330448688414720t^4 \\
& + 577763309046988800t^2 \\
& + 75229597532160,
\end{aligned}$$

 $\mathbf{c}_7$

$$\begin{aligned}
= & 28378791936t^5y^{10} + 26005292974080t^5y^8 \\
& + 12125896609628160t^5y^6 \\
& + 3492258223572910080t^5y^4 \\
& - 928760463360t^3y^6 \\
& + 621211109887322357760t^5y^2 \\
& - 267483013447680t^3y^4 \\
& + 56228479889229635125248t^5 \\
& + 231105323618795520t^3y^2 \\
& + 66558333202213109760t^3 \\
& + 50153065021440ty^2 \\
& + 14444082726174720t,
\end{aligned}$$

 $\mathbf{c}_6$

$$\begin{aligned}
= & 2759049216t^6y^{12} + 2600529297408t^6y^{10} \\
& + 1352275234652160t^6y^8 \\
& + 467916951524474880t^6y^6 \\
& - 135444234240t^4y^8 \\
& + 112163117062988759040t^6y^4 \\
& - 104021171896320t^4y^6 \\
& + 17890879964754883903488t^6y^2 \\
& + 28888165452349440t^4y^4
\end{aligned}$$

$$\begin{aligned}
& + 1574397436898429783506944t^6 \\
& + 25883796245305098240t^4y^2 \\
& + 14627977297920t^2y^4 \\
& + 4037872214267595325440t^4 \\
& + 7222041363087360t^2y^2 \\
& + 1521443380490403840t^2 \\
& - 325994922639360
\end{aligned}$$

 $\mathbf{c}_5$

$$\begin{aligned}
= & 197074944t^7y^{14} + 180592312320t^7y^{12} \\
& + 100553799499776t^7y^{10} \\
& + 39230841972326400t^7y^8 \\
& - 13544423424t^5y^{10} \\
& + 11298482488030003200t^7y^6 \\
& - 19503969730560t^5y^8 \\
& + 2402016291564313116672t^7y^4 \\
& - 534966026895360t^5y^6 \\
& + 357817599295097678069760t^7y^2 \\
& + 3749041916482682880t^5y^4 \\
& + 2437996216320t^3y^6 \\
& + 32387604416196269832142848t^7 \\
& + 1397724997246475304960t^5y^2 \\
& + 1504591950643200t^3y^4 \\
& + 152072479700416513179648t^5 \\
& + 741462913276968960t^3y^2 \\
& + 102610763686745210880t^3 \\
& - 162997461319680ty^2 \\
& - 61387351586242560t,
\end{aligned}$$

 $\mathbf{c}_4$

$$\begin{aligned}
= & 10264320t^8y^{16} + 8599633920t^8y^{14} \\
& + 5056584744960t^8y^{12} \\
& + 2139864107581440t^8y^{10} - 940584960t^6y^{12} \\
& + 697595699071549440t^8y^8
\end{aligned}$$

$$\begin{aligned}
& -2167107747840t^6y^{10} \\
& +17748888539234959360t^8y^6 \\
& -501530650214400t^6y^8 \\
& +34787822153690052034560t^8y^4 \\
& +222545867188469760t^6y^6 \\
& +253957939200t^4y^8 \\
& +4907212790332768156385280t^8y^2 \\
& +167936535162991411200t^6y^4 \\
& +167176883404800t^4y^6 \\
& +485814066242944047482142720t^8 \\
& +44727199911887209758720t^6y^2 \\
& +138422459459174400t^4y^4 \\
& +3757084792598525619732480t^6 \\
& +32354745306631372800t^4y^2 \\
& -33957804441600t^2y^4 \\
& +4409489574646618521600t^4 \\
& -18055103407718400t^2y^2 \\
& -5705412676839014400t^2 \\
& +117546246144000, \\
\mathbf{c}_3 & \\
& = 380160t^9y^{18} + 268738560t^9y^{16} \\
& + 171992678400t^9y^{14} \\
& + 75951966781440t^9y^{12} - 44789760t^7y^{14} \\
& + 27104945362698240t^9y^{10} \\
& - 150493593600t^7y^{12} \\
& + 7806224264457093120t^9y^8 \\
& - 55725627801600t^7y^{10} \\
& + 1814330860623290695680t^9y^6 \\
& + 1069932053790720t^7y^8 \\
& + 16930529280t^5y^{10} \\
& + 340778665995331121971200t^9y^4 \\
& + 8662169907489669120t^7y^6 \\
& + 10448555212800t^5y^8 \\
& + 44164915112994913407467520t^9y^2 \\
& + 3756848140747139973120t^7y^4 \\
& + 12259638116352000t^5y^6 \\
& + 5182016706591403173142855680t^9 \\
& + 894543998237744195174400t^7y^2 \\
& + 8666449635704832000t^5y^4 \\
& - 3773089382400t^3y^6 \\
& + 61340159879159601954816000t^7 \\
& + 249593749508299161600t^5y^2 \\
& - 1755357275750400t^3y^4 \\
& + 118740066432748187811840t^5 \\
& - 24073471210291200t^3y^2 \\
& - 333562017089794867200t^3 \\
& + 39182082048000ty^2 \\
& - 12789031580467200t, \\
\mathbf{c}_2 & \\
& = 9504t^{10}y^{20} + 4976640t^{10}y^{18} \\
& + 3941498880t^{10}y^{16} + 1651129712640t^{10}y^{14} \\
& - 1399680t^8y^{16} + 653847366205440t^{10}y^{12} \\
& - 6449725440t^8y^{14} \\
& + 205426954327818240t^{10}y^{10} \\
& - 2941074800640t^8y^{12} \\
& + 54232715942544015360t^{10}y^8 \\
& - 416084687585280t^8y^{10} + 705438720t^6y^{12} \\
& + 11359288866511037399040t^{10}y^6 \\
& + 181888449144422400t^8y^8 \\
& + 348285173760t^6y^{10} \\
& + 2249139195569185405009920t^{10}y^4 \\
& + 142977160212161495040t^8y^6 \\
& + 487599243264000t^6y^8 \\
& + 235546213935972871506493440t^{10}y^2
\end{aligned}$$

$$\begin{aligned}
& + 42242355472337920327680t^8y^4 \\
& + 930840886797926400t^6y^6 \\
& - 235818086400t^4y^8 \\
& + 37310520287458102846628560896t^{10} \\
& + 11041228778248728351866880t^8y^2 \\
& + 256141733677498368000t^6y^4 \\
& - 41794220851200t^4y^6 \\
& + 640391269138426244408279040t^8 \\
& - 18902566629428523171840t^6y^2 \\
& + 57174494124441600t^4y^4 \\
& + 1936048796185974936698880t^6 \\
& + 35050974082183987200t^4y^2 \\
& + 4897760256000t^2y^4 \\
& - 11162387130787823616000t^4 \\
& + 1880739938304000t^2y^2 \\
& - 2193695064037785600t^2 \\
& - 105791621529600,
\end{aligned}$$

 $\mathbf{c}_1$

$$\begin{aligned}
& = 144t^{11}y^{22} + 41472t^{11}y^{20} + 59719680t^{11}y^{18} \\
& + 17199267840t^{11}y^{16} - 25920t^9y^{18} \\
& + 9906778275840t^{11}y^{14} - 156764160t^9y^{16} \\
& + 2853152143441920t^{11}y^{12} \\
& - 77396705280t^9y^{14} \\
& + 821707817311272960t^{11}y^{10} \\
& - 12383472844800t^9y^{12} + 16796160t^7y^{14} \\
& + 236651851385646612480t^{11}y^8 \\
& - 1069932053790720t^9y^{10} \\
& + 4837294080t^7y^{12} \\
& + 34077866599533112197120t^{11}y^6 \\
& + 1746129111786455040t^9y^8 \\
& + 4179422085120t^7y^{10} \\
& + 9814425580665536312770560t^{11}y^4
\end{aligned}$$

$$\begin{aligned}
& + 1360748145467468021760t^9y^6 \\
& + 24742178743910400t^7y^8 \\
& - 7860602880t^5y^{10} \\
& + 565310913446334891615584256t^{11}y^2 \\
& + 187428266297432117084160t^9y^4 \\
& + 23765330778799472640t^7y^6 \\
& + 2612138803200t^5y^8 \\
& + 162809543072544448785288265728t^{11} \\
& + 77288601447741098463068160t^9y^2 \\
& + 3176311567816725626880t^7y^4 \\
& - 3176360784691200t^5y^6 \\
& + 3886512529943552379857141760t^9 \\
& - 501583599011877995151360t^7y^2 \\
& + 5568996339980697600t^5y^4 \\
& + 272097792000t^3y^6 \\
& + 17481945565560486557122560t^7 \\
& + 404434316332892160000t^5y^2 \\
& + 579894814310400t^3y^4 \\
& - 188626316295071953059840t^5 \\
& - 844076084310835200t^3y^2 \\
& - 113674931054995046400t^3 \\
& - 17631936921600ty^2 \\
& - 16362437463244800t,
\end{aligned}$$

 $\mathbf{c}_0$

$$\begin{aligned}
& = t^{12}y^{24} + 497664t^{12}y^{20} - 216t^{10}y^{20} \\
& + 103195607040t^{12}y^{16} - 1658880t^{10}y^{18} \\
& - 806215680t^{10}y^{16} \\
& + 11412608573767680t^{12}y^{12} + 174960t^8y^{16} \\
& - 173368619827200t^{10}y^{12} \\
& + 709955554156939837440t^{12}y^8 \\
& + 68475651442606080t^{10}y^{10} \\
& - 148343685120t^8y^{12}
\end{aligned}$$

$$\begin{aligned}
& -11093055533702184960t^{10}y^8 \\
& -534966026895360t^8y^{10} - 109175040t^6y^{12} \\
& + 23554621393597287150649344t^{12}y^4 \\
& + 7572859244340691599360t^{10}y^6 \\
& + 666300186498170880t^8y^8 \\
& + 116095057920t^6y^{10} \\
& - 51116799899299668295680t^{10}y^4 \\
& + 121612756962068398080t^8y^6 \\
& - 400527949824000t^6y^8 \\
& + 325619086145088897570576531456t^{12} \\
& + 235546213935972871506493440t^{10}y^2 \\
& + 11758638865724316057600t^8y^4 \\
& + 164056248247910400t^6y^6 \\
& + 5668704000t^4y^8 \\
& + 10458251898757195494888308736t^{10} \\
& - 3680409592749576117288960t^8y^2 \\
& + 19932834162121113600t^6y^4 \\
& + 27862813900800t^4y^6 \\
& + 67167475067679764140523520t^8 \\
& - 6123366654603606097920t^6y^2 \\
& + 25013841179443200t^4y^4 \\
& - 1242244730886105480560640t^6 \\
& - 34280623003454668800t^4y^2 \\
& - 734664038400t^2y^4 \\
& - 497165327434933862400t^4 \\
& + 1880739938304000t^2y^2 \\
& - 1126187075056435200t^2 + 4407984230400
\end{aligned} \tag{A.5}$$

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Conflicts of Interest

The author declares that there are no conflicts of interest.

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