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Research Article

Equations with Peakon Solutions in the Negative Order Camassa-Holm Hierarchy

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The negative order Camassa-Holm (CH) hierarchy consists of nonlinear evolution equations associated with the CH spectral problem. In this paper, we show that all the negative order CH equations admit peakon solutions; the Lax pair of the N-order CH equation given by the hierarchy is compatible with its peakon solutions. Special peakon-antipeakon solutions for equations of orders -3 and -4 are obtained. Indeed, for $N \leq -2$, the peakons of N-order CH equation can be constructed explicitly by the inverse scattering approach using Stieltjes continued fractions. The properties of peakons for N-order CH equation when N is odd are much different from the CH peakons; we present the case $N = -3$ as an example.

1. Introduction

The Camassa-Holm (CH) equation [1, 2],

$$\begin{gathered} m_t + m_x u + 2mu_x = 0, \\ 2m = 4u - u_{xx}, \end{gathered} \tag{1}$$

where $u(x,t)$ may be interpreted as a horizontal fluid velocity, retains higher order terms of derivatives in a small amplitude expansion of incompressible Euler's equations for unidirectional motion of waves at the free surface under the influence of gravity. The CH equation can also arise in the modeling of the propagation of shallow water waves over a flat bed [3], capturing stronger nonlinear effects than the classical nonlinear dispersive Benjamin-Bona-Mahoney and Korteweg-de Vries equations, in particular, tests ideas about wave breaking [3–5]. Mathematically, the CH equation possesses bi-Hamiltonian structure, Lax pair, and peaked soliton solutions (peakons), which were described by a finite dimensional Hamiltonian system [1, 2], the integrability of which was established in the framework of the r-matrix approach [6] and the explicit multipeakons were expressed in terms of the orthogonal polynomials associated with classical moment problem [4]. Since the rediscovery by Camassa and Holm in 1993, a large number of studies related to CH equation have been developed; see [7–12] and references therein for other topics. We remark that the peakon interactions were a key ingredient in the development of the theory of continuation after blow-up of global weak solutions of the CH equation, as developed in the papers [13–15].

In the past two decades, peakons have become a hot subject in the field of integrable system and some relevant branches of mathematics. Some other partial differential equations admitting peakons have been reported [16–21]; these equations introduced many challenging problems, including existence, uniqueness, stability, and breakdown of solutions, for part of which one can see [22–30]. We recall that the CH (periodic) peakons [31, 32] and Degasperis-Procesi peakons [33] are stable in the sense that their shape is stable under small perturbations, which shows the peakons are detectable.

In this paper, we consider equations in the negative CH hierarchy associated with the CH spectral problem with the aim of obtaining more peakon equations and properties of the negative N-order CH equation. Equation (1) can be obtained

as the compatibility condition for an overdetermined system [1, 4]

$$\begin{aligned} \varphi_{xx} &= (zm+1)\varphi, \\ \varphi_t &= \left(\frac{1}{z} - u\right)\varphi_x + \frac{1}{2}u_x\varphi. \end{aligned} \tag{2}$$

Starting from the first equation in (2), one can derive integrable hierarchies of nonlinear evolution equations, which contain the well-known Dym-type equation and CH equation [7, 8, 12]. Qiao [12] derived the negative order CH hierarchy and the positive order CH hierarchies via the spectral gradient method, and Alber et al. [7, 8] gave the hierarchy by the method of generating equations, both of which based on the assumption that the potential m is a smooth function.

We extend the differential operator $J = -(m\partial_x + \partial_x m)$ to $PC^\infty(\mathbb{R}) \cap C(\mathbb{R})$, the space of continuous and piecewise smooth functions on $\mathbb{R}$, where m is a discrete measure, and show that all negative order CH equations admit peakons of the form

$$u(x,t) = \frac{1}{2}\sum_{j=1}^{n} m_j(t)\, e^{-2|x-x_j(t)|}. \tag{3}$$

Besides, we show that the Lax integrability is preserved in the peakon case; some constants of motion for the peakon dynamical systems of N-order CH equations are obtained.

The remainder of this paper is organized as follows. In Section 2, we describe the negative order CH hierarchy using the method of finite power expansion with respect to spectral parameter for the purpose of this paper. In Section 3, we derive the negative order CH hierarchy for discrete potential. In Section 4, we prove that the equations of orders $-3, -4, \ldots$ admit multipeakons and the Lax pair of the N-order CH equation given by the negative order CH hierarchy is compatible with its peakons. In Section 5, we give some examples for peakon solutions of the -3-order equation and the -4-order equation. In Section 6, we give some remarks on the work of this paper and that in [34, 35].

2. Negative Order Camassa-Holm Hierarchy

To make a self-contained discussion on equations in the negative order CH hierarchy, we now make a description for the negative order CH hierarchy in the following way.

Consider the CH spectral problem

$$\varphi_{xx} = (zm+1)\varphi, \tag{4}$$

with potential m and spectral parameter z. The equation above is equivalent to

$$\begin{aligned} \Phi_x &= U\Phi, \\ U &= U(m,z) = \begin{bmatrix} 0 & 1 \\ zm+1 & 0 \end{bmatrix}, \end{aligned} \tag{5}$$

where $\Phi = (\varphi, \varphi_x)^T$.

The negative order CH hierarchy is the following linear system of differential equations

$$\begin{aligned} \Phi_x &= U\Phi = (zA + B)\Phi, \\ \Phi_t &= V\Phi, \end{aligned} \tag{6}$$

where

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0 \\ m(x,t) & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ V &= \sum_{j=N+1}^{1} V_j z^j \quad (N < -1), \\ V_j &= V_j(x,t) \in \mathrm{sl}(2,\mathbb{C}). \end{aligned} \tag{7}$$

The compatibility condition of (6) is $U_t - V_x + [U,V] = 0$, which should hold for all z. Equating like powers of z, we obtain

$$\begin{aligned} [A, V_1] &= 0, \\ [A, V_{j-1}] - V_{j,x} + [B, V_j] &= 0, \quad (N+2 \le j \le 0), \\ [B, V_{N+1}] - V_{N+1,x} &= 0, \end{aligned} \tag{8}$$

and the evolution of matrix A

$$A_t - V_{1,x} + [A, V_0] + [B, V_1] = 0. \tag{9}$$

Solving the recursion (8), we will obtain the evolution of m from (9).

Let

$$\begin{aligned} K &= -\frac{1}{2}\partial_x^3 + 2\partial_x, \\ J &= -(m\partial_x + \partial_x m). \end{aligned} \tag{10}$$

Formally, we have

$$\begin{aligned} K^{-1} &= -2\partial_x^{-1} e^{2x} \partial_x^{-1} e^{-4x} \partial_x^{-1} e^{2x}, \\ J^{-1} &= -\frac{1}{2} m^{-1/2} \partial_x^{-1} m^{-1/2}, \end{aligned} \tag{11}$$

where ∂_x^{-1} is an integral operator. Set $\mathscr{L} = J^{-1}K$, then

$$\mathscr{L}^{-1} = K^{-1}J = 2\partial_x^{-1} e^{2x} \partial_x^{-1} e^{-4x} \partial_x^{-1} e^{2x} (m\partial_x + \partial_x m). \tag{12}$$

Remark 1. In general, the differential operator K is not invertible; (12) just gives an integrodifferential operator formally. In Section 3, we shall see that $\mathscr{L}^{-1}$ is actually an operator on the function space $PC^\infty(\mathbb{R}) \cap C(\mathbb{R})$ when m is a finite discrete measure.

In terms of the components, (8) shows that

$$\begin{aligned} &V_1^{11} = V_1^{12} = V_1^{22} = 0, \\ &KV_{N+1}^{12} = 0, \\ &V_{N+1}^{21} = V_{N+1}^{12} - \frac{1}{2}V_{N+1,xx}^{12}, \\ &V_j^{11} = -\frac{1}{2}V_{j,x}^{12}, \quad (N+1 \le j \le 0), \\ &V_j^{12} = \mathscr{L}^{-1}V_{j-1}^{12}, \quad (N+2 \le j \le 0), \\ &V_j^{21} = -\frac{1}{2}V_{j,xx}^{12} + V_j^{12} + mV_{j-1}^{12}, \quad (N+2 \le j \le 0), \end{aligned} \tag{13}$$

where the superscripts denote the position of the entry in the matrix; that is, V_j^{12} is the $(1,2)$ entry of V_j.

Let $G_{-1} = V_{N+1}^{12}$, then $G_{-1} \in \operatorname{Ker} K$. Define the following recursion sequence (called Lenard's sequence in [12]):

$$G_j = \mathscr{L}^{j+1}G_{-1}, \quad j \le -1. \tag{14}$$

Substituting (14) into (13) and (9), we have

$$\begin{aligned} V_{N+1} &= \begin{bmatrix} -\frac{1}{2}G_{-1,x} & G_{-1} \\ G_{-1} - \frac{1}{2}G_{-1,xx} & \frac{1}{2}G_{-1,x} \end{bmatrix}, \\ V_1 &= \begin{bmatrix} 0 & 0 \\ mG_N & 0 \end{bmatrix}, \\ V_j &= \begin{bmatrix} -\frac{1}{2}G_{N-j,x} & G_{N-j} \\ G_{N-j} - \frac{1}{2}G_{N-j,xx} + mG_{N-j+1} & \frac{1}{2}G_{N-j,x} \end{bmatrix}, \\ &\qquad (N+2 \le j \le 0), \end{aligned} \tag{15}$$

and the nonlinear evolution equation

$$m_t + JG_N = 0. \tag{16}$$

With the discussion above, we can define the N-order CH equation ($N < -1$) as follows.

Definition 2. The N-order CH equation ($N < -1$) is the zero curvature equation $U_t - V_x + [U, V] = 0$, where $U = \left[\begin{smallmatrix} 0 & 1 \\ zm+1 & 0 \end{smallmatrix}\right]$ and V is a $\mathrm{sl}(2,\mathbb{C})$-valued Laurent polynomial of z with the lowest degree term $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] z^{N+1}$ and the highest degree term (i.e., z^1-term) with nonvanishing $(2,1)$ entry.

Remark 3. Note that

$$\begin{aligned} \operatorname{Ker} K = \{&\alpha_1(t) + \alpha_2(t)e^{2x} + \alpha_3(t)e^{-2x} \mid \alpha_i \\ &\in C^\infty(\mathbb{R}),\ i = 1,2,3\}, \end{aligned} \tag{17}$$

$1 \in \operatorname{Ker} K$, and the N-order CH equation is given by (16) with $G_{-1} = 1$. The -2-order equation is (1), where $G_{-2} = -u$ by setting $2m = 4u - u_{xx}$. For $N < -2$, the N-order CH equation is an integrodifferential equation in general.

Remark 4. Choosing other $G_{-1} \in \operatorname{Ker} K$, for example, $G_{-1} = e^{2x}$, the first equation in our negative order CH hierarchy is an integrodifferential equation; one can see examples in [12].

3. Hierarchy Associated with Discrete Potential

In the description of the negative order CH hierarchy, we have tacitly assumed that the potential m in (5) is a smooth function. In the remainder, we suppose that m is a finite discrete measure given by

$$m = \sum_{k=1}^{n} m_k \delta_{x_k}, \quad -\infty < x_1 < \cdots < x_n < +\infty, \tag{18}$$

where $\delta_{x_k} := \delta(x - x_k)$ is a Dirac delta distribution supported at the point x_k. In this case, to define G_N ($N < -2$) from (14), distributional calculus will be needed, we take derivatives as distributional derivatives, and D_x will be used to denote distributional derivative with respect to x. Besides, we extend the definition of $J = -(m\partial_x + \partial_x m)$ to $f \in PC^\infty(\mathbb{R}) \cap C(\mathbb{R})$ as follows:

$$\begin{aligned} Jf &:= -D_x(mf) - \langle f_x \rangle m \\ &= -D_x(mf) - \sum_{k=1}^{n} \langle f(x_k) \rangle (x_k) m_k \delta_{x_k}, \end{aligned} \tag{19}$$

where $\langle f_x(x_k) \rangle$ is the average of f at x_k. As we shall see in Section 4, (19) makes the distributional Lax pair of the N-order CH equation compatible with its peakons.

The distributional derivatives we need in this paper can be calculated from the following lemma on piecewise smooth functions.

Lemma 5. *Suppose $f(x)$ is a piecewise smooth function and has discontinuities at x_i ($i = 1, \ldots, n$), then we have*

$$\begin{aligned} D_x^k f = f_{kx} &+ \sum_{i=1}^{n} [f_x(x_i)] D^{k-1}\delta_{x_i} + \cdots \\ &+ \sum_{i=1}^{n} [f_{(k-1)x}(x_i)] \delta_{x_i}, \end{aligned} \tag{20}$$

where $f_{kx} = d^k f/dx^k$ and f_t are ordinary partial derivatives; $[f(x_i)] = f(x_i+) - f(x_i-)$ denotes the jump of f at x_i.

In the remainder, we take the convention that $x_0 = -\infty$, $x_{n+1} = +\infty$.

In this section, we show that G_N ($N < -2$) can be defined by formula (14) recursively for m given by (18).

The following proposition gives the well-defined G_{-2}.

Proposition 6. *For m given by (18), there exists a unique continuous function in $PC^\infty(\mathbb{R})$ solving the following problem:*

$$\left(-\frac{1}{2}D_x^3 + 2D_x\right) f = -D_x m, \tag{21}$$

$$f \sim -\left(\frac{1}{2}\sum_{i=1}^{n} m_i e^{-2x_i}\right) e^{2x}, \quad x \longrightarrow -\infty. \tag{22}$$

Proof. According to Lemma 5, (21) is equivalent to

$$\begin{aligned} &-\frac{1}{2} f_{xxx} + 2 f_x - \frac{1}{2}\sum_{k=1}^{n} [f(x_k)] D^2 \delta_{x_k} \\ &\quad - \frac{1}{2}\sum_{k=1}^{n} [f_x(x_k)] D\delta_{x_k} - \frac{1}{2}\sum_{k=1}^{n} [f_{xx}(x_k)] \delta_{x_k} \\ &\quad + 2\sum_{k=1}^{n} [f(x_k)] \delta_{x_k} = -\sum_{k=1}^{n} m_k D\delta_{x_k}. \end{aligned} \tag{23}$$

Therefore, $f \in PC^{\infty}(\mathbb{R})$ satisfies (21) if and only if $-(1/2) f_{xxx} + 2 f_x = 0$ when $x \neq x_k$ and

$$\begin{aligned} [f(x_k)] &= 0, \\ [f_x(x_k)] &= 2m_k, \\ [f_{xx}(x_k)] &= 0, \\ &(1 \le k \le n). \end{aligned} \tag{24}$$

Thus, we can set

$$f(x) = A_k e^{2x} + B_k e^{-2x} + C_k, \quad x_k < x < x_{k+1}. \tag{25}$$

A straightforward calculation translates (24) into the following relation:

$$\begin{bmatrix} A_k \\ B_k \\ C_k \end{bmatrix} = \begin{bmatrix} A_{k-1} \\ B_{k-1} \\ C_{k-1} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} m_k e^{-2x_k} \\ -\frac{1}{2} m_k e^{2x_k} \\ 0 \end{bmatrix}. \tag{26}$$

The asymptotic condition (22) shows that $B_0 = C_0 = 0$, $A_0 = -(1/2)\sum_{i=1}^{n} m_i e^{-2x_i}$, and substituting into (26) yields

$$\begin{aligned} A_k &= -\frac{1}{2}\sum_{i=k+1}^{n} m_i e^{-2x_i}, \\ B_k &= -\frac{1}{2}\sum_{i=1}^{k} m_i e^{2x_i}, \\ C_k &= 0, \\ &k = 0, 1, \ldots, n. \end{aligned} \tag{27}$$

Hence,

$$f(x) = -\frac{1}{2}\sum_{i=k+1}^{n} m_i e^{2(x-x_i)} - \frac{1}{2}\sum_{i=1}^{k} m_i e^{2(x_i - x)}, \quad x_k < x < x_{k+1}. \tag{28}$$

Define a function on $\mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x_k+), & \text{when } x = x_k, \\ f(x), & \text{when } x \neq x_k \end{cases} \tag{29}$$

by the first condition in (24), $\tilde{f} \in C(\mathbb{R})$, which completes the proof. □

Denote the unique solution to (21) and (22) by G_{-2}, and with Lemma 5 and (19), we have the following theorem.

Theorem 7. *For m given by (18), G_N $(N < -2)$ is well defined by (14), and $G_N \in PC^{\infty}(\mathbb{R}) \cap C(\mathbb{R})$.*

Proof. According to formula (14), $KG_j = JG_{j+1}$; in the distribution sense, we have the following equation formally:

$$\left(-\frac{1}{2}D_x^3 + 2D_x\right) G_j = -D_x\left(mG_{j+1}\right) - mD_x G_{j+1}. \tag{30}$$

Thus, with the definition (19) of J, G_N can be defined recursively. We now prove this theorem by induction.

First, Proposition 6 shows that G_{-2} is a continuous function in $PC^{\infty}(\mathbb{R})$. Consider the following problem:

$$\left(-\frac{1}{2}D_x^3 + 2D_x\right) f = -D_x(mG_{-2}) - mD_x G_{-2}, \tag{31}$$

$$f \sim -e^{2x}\sum_{i=1}^{n}\left(\frac{1}{2}G_{-2}(x_i) m_i + \frac{1}{4}\langle G_{-2,x}(x_i)\rangle m_i\right) e^{-2x_i}, \quad x \longrightarrow -\infty, \tag{32}$$

where m is given by (18). With (19) and Lemma 5 at hand, (31) is equivalent to

$$\begin{aligned} &-\frac{1}{2} f_{xxx} + 2 f_x + 2\sum_{k=1}^{n} [f(x_k)] \delta_{x_k} \\ &\quad - \frac{1}{2}\sum_{k=1}^{n} [f_x(x_k)] D\delta_{x_k} - \frac{1}{2}\sum_{k=1}^{n} [f_{xx}(x_k)] \delta_{x_k} \\ &\quad - \frac{1}{2}\sum_{k=1}^{n} [f(x_k)] D^2 \delta_{x_k} \\ &= -\sum_{k=1}^{n} \langle G_{-2,x}(x_k)\rangle m_k \delta_{x_k} - \sum_{k=1}^{n} G_{-2}(x_k) m_k D\delta_{x_k}. \end{aligned} \tag{33}$$

Therefore, $f \in PC^{\infty}(\mathbb{R})$ satisfies (31) if and only if $-(1/2) f_{xxx} + 2 f_x = 0$ when $x \neq x_k$ and

$$\begin{aligned} [f(x_k)] &= 0, \\ [f_x(x_k)] &= 2G_{-2}(x_k) m_k, \\ [f_{xx}(x_k)] &= 2\langle G_{-2,x}(x_k)\rangle m_k, \\ &(1 \le k \le n). \end{aligned} \tag{34}$$

Hence,

$$f(x) = A_k e^{2x} + B_k e^{-2x} + C_k, \quad x_k < x < x_{k+1}, \tag{35}$$

where A_k, B_k, C_k satisfy

$$\begin{bmatrix} A_k \\ B_k \\ C_k \end{bmatrix} = \begin{bmatrix} A_{k-1} \\ B_{k-1} \\ C_{k-1} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} G_{-2}(x_k) m_k e^{-2x_k} + \frac{1}{4} \langle G_{-2,x}(x_k) \rangle m_k e^{-2x_k} \\ -\frac{1}{2} G_{-2}(x_k) m_k e^{2x_k} + \frac{1}{4} \langle G_{-2,x}(x_k) \rangle m_k e^{2x_k} \\ -\frac{1}{2} \langle G_{-2,x}(x_k) \rangle m_k \end{bmatrix}. \tag{36}$$

The asymptotic condition (32) shows that $B_0 = C_0 = 0$ and

$$A_0 = -\sum_{i=1}^{n} \left(\frac{1}{2} G_{-2}(x_i) m_i + \frac{1}{4} \langle G_{-2,x}(x_i) \rangle m_i \right) e^{-2x_i}. \tag{37}$$

Thus, (36) defines a function $f \in PC^{\infty}(\mathbb{R})$ satisfying the asymptotic condition; using the first condition in (34), we can define a unique continuous function $\widetilde{f}$ as we have done in the proof of Proposition 6. Thus, G_{-3} is well defined by (14) and $G_{-3} \in PC^{\infty}(\mathbb{R}) \cap C(\mathbb{R})$; we have so far proved the conclusion for the case $N = -3$.

Next, we assume that $G_{N+1} (N \leq -4)$ is well defined by (14) and $G_{N+1} \in PC^{\infty}(\mathbb{R}) \cap C(\mathbb{R})$. Consider the following problem:

$$\left(-\frac{1}{2} D_x^3 + 2D_x \right) f = -D_x (m G_{N+1}) - m D_x G_{N+1}, \tag{38}$$

$$f \sim -e^{2x} \sum_{i=1}^{n} \left(\frac{1}{2} G_{N+1}(x_i) m_i + \frac{1}{4} \langle G_{N+1,x}(x_i) \rangle m_i \right) \cdot e^{-2x_i}, \quad x \longrightarrow -\infty, \tag{39}$$

where m is given by (18). Similarly, we can obtain a unique continuous function in $PC^{\infty}(\mathbb{R})$ solving (38) and (39).

By induction, for any $N < -2$, G_N is well defined by (14) and $G_N \in PC^{\infty}(\mathbb{R}) \cap C(\mathbb{R})$, which completes the proof. □

Remark 8. In Proposition 6 and the proof of Theorem 7, we have chosen different exponential decay conditions at negative infinity for different G_N; however, we can replace them by boundary condition: $f \to 0, |x| \to \infty$; see Remark 13.

4. Peakons and Lax Integrability

The N-order CH equation can be written formally as

$$m_t - (\partial_x m + m \partial_x) G_N = 0, \qquad 2m = 4u - u_{xx}, \tag{40}$$

where $u = -G_{-2}$ and G_N given by (14). The -2-order equation, that is, CH equation (1), admits peakons (3), where x_k, m_k obey the following Hamiltonian system:

$$\dot{x}_k = u(x_k), \qquad \dot{m}_k = -\langle u_x(x_k) \rangle m_k. \tag{41}$$

In Section 3, we define G_N by (14) for discrete potential; based on this, we have the following theorem.

Theorem 9. *The N-order CH equation given by Definition 2 admits peakons taking the form (3).*

Proof. Substituting (3) into the second equation in (40), we obtain

$$m(x,t) = 2u - \frac{1}{2} D^2 u = \sum_{k=1}^{n} m_k(t) \delta(x - x_k(t)). \tag{42}$$

By Proposition 6 and Theorem 7, m above makes G_N a function in $PC^{\infty}(\mathbb{R}) \cap C(\mathbb{R})$, which is expressed by elementary functions composed of power functions and exponential functions. In the sense of distribution, the N-order CH equation (40) is equivalent to the following dynamic system:

$$\dot{x}_k = -G_N(x_k), \qquad \dot{m}_k = m_k \langle G_{N,x}(x_k) \rangle, \tag{43}$$

for $k = 1, \ldots, n$. Hence, for the initial condition

$$x_1(t_0) < \cdots < x_n(t_0), \tag{44}$$

(43) has a unique solution locally, which completes the proof. □

We shall see that the Lax pair of the N-order CH equation (40) given by (6) is compatible with its peakons. Recall that for the case $N = -2$ the same result had been used in [4]; we will prove the compatibility just for the case $N < -2$.

In the remainder of this section, D_t denotes the distributional derivatives in t, the subscripts of functions denoting the usual partial derivatives, and for simplicity, we will write $\sum$ instead of $\sum_{k=1}^{n}$. We first present a lemma on the calculus of piecewise smooth function.

Lemma 10. *Given smooth functions $x_1(t) < \cdots < x_n(t)$, suppose f and g are piecewise smooth functions with jumps at $x_k(t)$ $(k = 1, \ldots, n)$, then*

$$\begin{aligned} [fg(x_k)] &= \langle f(x_k) \rangle [g(x_k)] + [f(x_k)] \langle g(x_k) \rangle, \\ \langle fg(x_k) \rangle &= \langle f(x_k) \rangle \langle g(x_k) \rangle + \frac{1}{4} [f(x_k)] [g(x_k)], \\ \frac{d}{dt} [f(x_k)] &= [f_x(x_k)] \dot{x}_k + [f_t(x_k)], \\ \frac{d}{dt} \langle f(x_k) \rangle &= \langle f_x(x_k) \rangle \dot{x}_k + \langle f_t(x_k) \rangle. \end{aligned} \tag{45}$$

For peakon solutions (3), m is given by (42). By Proposition 6 and Theorem 7, G_j $(j \le -2)$ given by (14) are continuous functions in $PC^{\infty}(\mathbb{R})$ and $[u_x(x_k)] = -2m_k$,

$$\begin{aligned}
\left[G_j(x_k)\right] &= 0,\\
\left[G_{j,x}(x_k)\right] &= 2G_{j+1}(x_k)\, m_k,\\
\left[G_{j,xx}(x_k)\right] &= 2\left\langle G_{j+1,x}(x_k)\right\rangle m_k,
\end{aligned}$$

$$(N \le j \le -3). \tag{46}$$

The derivatives in (15) should be replaced by corresponding distributional derivatives. Thus, (6) can be rewritten as the following well-defined distributional Lax pair:

$$\begin{aligned}
D_x\Phi &= \widetilde{U}\Phi,\\
D_t\Phi &= \widetilde{V}\Phi,
\end{aligned} \tag{47}$$

where

$$\begin{aligned}
\widetilde{U} &= X + z\left(\sum m_k\delta_{x_k}\right)S,\\
X &= \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix},\\
S &= \begin{bmatrix} 0 & 0\\ 1 & 0\end{bmatrix},\\
\widetilde{V} &= T + z\left(\sum m_k G_N(x_k)\,\delta_{x_k}\right)S,\\
T &= \begin{bmatrix} \frac{1}{2}u_x z^{N+2} - \frac{1}{2}\sum_{j=N}^{-3} G_{j,x}z^{N-j} & z^{N+1} - uz^{N+2} + \sum_{j=N}^{-3} G_j z^{N-j}\\ z^{N+1} + uz^{N+2} + \sum_{j=N}^{-3}\left(G_j - \frac{1}{2}G_{j,xx}\right)z^{N-j} & -\frac{1}{2}u_x z^{N+2} + \frac{1}{2}\sum_{j=N}^{-3} G_{j,x}z^{N-j}\end{bmatrix}.
\end{aligned} \tag{48}$$

Note that $u_{xx} = 4u$ and $2G_{j,x} - (1/2)G_{j,xxx} = 0$ when $x \ne x_k$; it is easy to see that $T_x = XT - TX$.

Theorem 11. *With u and m given by (3) and (42) and G_j defined by (14) and (19), the Lax pair (47) satisfies the compatibility condition: $D_tD_x\Phi = D_xD_t\Phi$ if and only if*

$$\begin{aligned}
\dot{x}_k &= -G_N(x_k),\\
\dot{m}_k &= \left\langle G_{N,x}(x_k)\right\rangle m_k.
\end{aligned} \tag{49}$$

Proof. Computing the distributional derivatives of Φ, comparing the coefficients of δ_{x_k} and the regular parts in (47) leads to

$$\begin{aligned}
[\Phi(x_k)] &= zm_kS\Phi(x_k),\\
\dot{x}_k &= -G_N(x_k),
\end{aligned} \tag{50}$$

$$\begin{aligned}
\Phi_x &= X\Phi,\\
\Phi_t &= T\Phi.
\end{aligned} \tag{51}$$

Next we compute $D_tD_x\Phi$ and $D_xD_t\Phi$; using (50) we have

$$\begin{aligned}
D_tD_x\Phi &= X\left(T + zS\sum m_kG_N(x_k)\,\delta_{x_k}\right)\Phi\\
&\quad + zS\sum \frac{d}{dt}\left(m_k\Phi(x_k)\right)\delta_{x_k}\\
&\quad + zS\sum m_k\Phi(x_k)\,G_N(x_k)\,D\delta_{x_k},\\
D_xD_t\Phi &= (T\Phi)_x + \sum\left[T\Phi(x_k)\right]\delta_{x_k}\\
&\quad + zS\sum m_kG_N(x_k)\,\Phi(x_k)\,D\delta_{x_k}.
\end{aligned} \tag{52}$$

Thus, $D_tD_x\Phi = D_xD_t\Phi$ is equivalent to

$$zm_kXSG_N\Phi(x_k) + zS\frac{d}{dt}\left(m_k\Phi(x_k)\right) = \left[T\Phi(x_k)\right]. \tag{53}$$

Computing $[T\Phi(x_k)]$ by using Lemma 10 and relations (50) and (46), we have that (53) is equivalent to

$$m_k\begin{bmatrix} -z^{N+2} - \sum_{j=N}^{-3} G_{j+1}z^{N-j} & 0\\ -\sum_{j=N}^{-3}\left\langle G_{j+1,x}\right\rangle z^{N-j} & z^{N+2} + \sum_{j=N}^{-3} G_{j+1}z^{N-j}\end{bmatrix}_{x_k}$$

$$\cdot \langle \Phi(x_k) \rangle + zm_k \begin{bmatrix} z^{N+1} - uz^{N+2} + \sum_{j=N+1}^{-3} G_j z^{N-j} & 0 \\ -\frac{1}{2}\langle u_x \rangle z^{N+2} + \frac{1}{2}\sum_{j=N}^{-3} \langle G_{j,x} \rangle z^{N-j} & 0 \end{bmatrix}_{x_k} \cdot \Phi(x_k) = z\dot{m}_k S\Phi(x_k) + zm_k S\frac{d}{dt}(\Phi(x_k)). \tag{54}$$

The second component of Φ, that is, φ_x, has jump discontinuities at x_k. By (50) and (51), we have

$$\begin{aligned} S\frac{d}{dt}(\langle \Phi(x_k) \rangle) &= S(\langle \Phi_x(x_k) \rangle \dot{x}_k + \langle \Phi_t(x_k) \rangle) = -G_N(x_k) SX \langle \Phi(x_k) \rangle + S\langle T(x_k) \rangle \langle \Phi(x_k) \rangle \\ &= \begin{bmatrix} 0 & 0 \\ \frac{1}{2}\langle u_x \rangle z^{N+2} - \frac{1}{2}\sum_{j=N}^{-3} \langle G_{j,x} \rangle z^{N-j} & (1-uz) z^{N+1} + \sum_{j=N+1}^{-3} G_j z^{N-j} \end{bmatrix}_{x_k} \langle \Phi(x_k) \rangle. \end{aligned} \tag{55}$$

Therefore,

$$\begin{aligned} m_k S\frac{d}{dt}(\Phi(x_k) - \langle \Phi(x_k) \rangle) &= (m_k \langle G_{N,x}(x_k) \rangle - \dot{m}_k) S\Phi(x_k) \\ &\quad - \frac{1}{2} m_k \langle G_{N,x}(x_k) \rangle S(\Phi(x_k) - \langle \Phi(x_k) \rangle). \end{aligned} \tag{56}$$

The first row of (56) holds naturally for that the first element of $S\Phi$ is zero; by the continuity of φ at the point x_k, the second row holds if and only if $\dot{m}_k = \langle G_{N,x}(x_k) \rangle m_k$, which completes the proof. □

Remark 12. Replacing $\widetilde{V}$ in (47) by $\widetilde{V} - z^{N+1}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\widetilde{V} + z^{N+1}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, the compatibility condition of (47) is also (43).

Theorem 11 implies that the Lax pair of the N-order CH equation is compatible with its peakons when extending the definition of J by (19). Using this result, we can also obtain some constants of motion for the peakon ODEs (43).

The Lax pair for the N-order CH equation (40) given by the negative order CH hierarchy is equivalent to

$$\begin{aligned} \varphi_{xx} &= (zm+1)\varphi, \\ \varphi_t &= \left(\frac{1}{2}u_x z^{N+2} - \frac{1}{2}\sum_{j=N}^{-3} G_{j,x} z^{N-j}\right)\varphi \\ &\quad + \left(z^{N+1} - uz^{N+2} + \sum_{j=N}^{-3} G_j z^{N-j}\right)\varphi_x. \end{aligned} \tag{57}$$

When m is given by (42), the spatial part leads to

$$\frac{1}{z}\varphi(x,t) = -\frac{1}{2}\sum_{j=1}^{n} m_j e^{-|x-x_j|}\varphi(x_j(t), t). \tag{58}$$

Define $L = (L_{ij})$, $L_{ij} = -(1/2)m_j e^{-|x_i - x_j|}$, then

$$L\Psi = \frac{1}{z}\Psi, \tag{59}$$

where

$$\Psi = (\varphi(x_1(t), t), \ldots, \varphi(x_n(t), t))^T. \tag{60}$$

Taking derivative of $(1/z)\varphi(x_i(t), t)$ with respect to t, using $\dot{x}_k = -G_N(x_k)$, the second equation in (57), and (58), we obtain certain matrix $A = (A_{ij})$ such that

$$\frac{d}{dt}\left(\frac{1}{z}\Psi\right) = A\Psi. \tag{61}$$

Since $E = (E_{ij})$ with $E_{ij} = e^{-|x_i - x_j|}$ is single-pair matrix and is oscillatory [36], so L is nonsingular. Hence, the compatibility condition of (59) and (61) $L_t L = AL - LA = [A, L]$ leads to $(d/dt)(\operatorname{tr} L^s) = \sum_{k=0}^{s-1} \operatorname{tr}(L^k [A, L] L^{s-k-2}) = 0$, where s is arbitrary positive integer. Therefore, $\operatorname{tr}(L^s)$ is a constant of motion for the peakon ODEs of the N-order CH equation.

Remark 13. When $s = 1$, $\operatorname{tr}(L) = -(1/2)(m_1 + \cdots + m_n)$, for any $N \le -2$, the peakon ODEs (43) has a constant of motion: $m_1 + \cdots + m_n$; therefore,

$$\sum_{k=1}^{n} \langle G_{N,x}(x_k) \rangle m_k = \sum_{k=1}^{n} \dot{m}_k = 0. \tag{62}$$

By (62), for m given by (18) and $G_{-1} = 1$, there is a unique G_N defined by (14) and G_N approaches to zero as $|x| \to \infty$. Indeed, only minor modifications are needed. Replacing (22), (32), and (39) by $f \to 0, |x| \to \infty$, which is equivalent to $B_0 = C_0 = C_n = A_n = 0$, then the condition (62) implies the existence and uniqueness of corresponding boundary problems, and G_j $(j \le -2)$ defined in Section 3 are exactly the solutions to the modified problems.

Remark 14. When $s = 2$, $\operatorname{tr}(L^2) = \sum_{i,j=1}^{n} L_{ij} L_{ji}$, particularly, when $n = 2$, we have $\operatorname{tr}(L^2) = (1/4)(m_1^2 + m_2^2) + (1/2)m_1 m_2 e^{-2|x_1 - x_2|}$. Thus, for any N-order equation, the corresponding two-peakon ODEs have a constant of motion: $(1/4)(m_1^2 + m_2^2) + (1/2)m_1 m_2 e^{-2|x_1 - x_2|}$, which equals the Hamiltonian of CH peakon ODEs [1, 4].

5. Examples of Peakon Solutions

In this section, we present some special peakon solutions for the N-order CH equation in the cases $N = -3, -4$ by integrating the ODEs (43).

The -3-order CH equation can be written as the following integrodifferential equation:

$$\begin{aligned} &m_t + 2(\partial m + m\partial)\partial^{-1}e^{2x}\partial^{-1}e^{-4x}\partial^{-1}e^{2x}(m\partial + \partial m)u \\ &\quad = 0, \\ &2m = 4u - u_{xx}. \end{aligned} \tag{63}$$

For $N = -3$, the ODEs (43) can be written as

$$\begin{aligned} \dot{x}_k &= -\frac{1}{2}\sum_{i=1}^{n} m_i u(x_i) e^{-2|x_i - x_k|} + \frac{1}{4}\sum_{i=1}^{n} \operatorname{sgn}(x_k - x_i) m_i \langle u_x(x_i)\rangle e^{-2|x_i - x_k|} - \frac{1}{2}\sum_{i=1}^{k-1} \langle u_x(x_i)\rangle m_i - \frac{1}{4}\langle u_x(x_k)\rangle m_k \\ &= \frac{1}{4}\sum_{i,j=1}^{n} m_i m_j \left(\operatorname{sgn}(x_i - x_k)\operatorname{sgn}(x_i - x_j) - 1\right) e^{-2|x_i - x_k| - 2|x_i - x_j|} + \frac{1}{2}\sum_{i=1}^{k-1}\sum_{j=1}^{n} m_i m_j \operatorname{sgn}(x_i - x_j) e^{-2|x_i - x_j|} \\ &\quad - \frac{1}{4}\sum_{i=1}^{n} m_i m_k \operatorname{sgn}(x_i - x_k) e^{-2|x_i - x_k|}, \\ \dot{m}_k &= m_k \sum_{i=1}^{n} \operatorname{sgn}(x_i - x_k) m_i u(x_i) e^{-2|x_i - x_k|} \\ &\quad + \frac{1}{2} m_k \sum_{i=1}^{n} m_i \langle u_x(x_i)\rangle e^{-2|x_i - x_k|} = \frac{1}{2} m_k \sum_{i,j=1}^{n} m_i m_j \left(\operatorname{sgn}(x_i - x_k) - \operatorname{sgn}(x_i - x_j)\right) e^{-2|x_i - x_k| - 2|x_i - x_j|}. \end{aligned} \tag{64}$$

Example 1. When $n = 1$, (64) is simplified as

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{4}m_1^2, \\ \dot{m}_1 &= 0. \end{aligned} \tag{65}$$

Imposing the initial condition $x_1(0) = 0$, $m_1(0) = c$, we have

$$u(x,t) = \frac{c}{2} e^{-2|x + (c^2/4)t|}, \tag{66}$$

where c is an arbitrary nonzero constant. Thus, (63) admits the peakon solution (66).

Example 2. When $n = 2$, for $x_1(0) < x_2(0)$, in the neighborhood of $t = 0$, (64) is simplified as

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{4}m_1^2 - \frac{1}{2}m_1 m_2 e^{2(x_1 - x_2)} - \frac{1}{4}m_2^2 e^{2(x_1 - x_2)}, \\ \dot{m}_1 &= \frac{1}{2}m_1 m_2 (m_1 + m_2) e^{2(x_1 - x_2)}, \\ \dot{x}_2 &= -\frac{1}{4}m_2^2 - \frac{1}{2}m_1 m_2 e^{2(x_1 - x_2)} - \frac{1}{4}m_1^2 e^{2(x_1 - x_2)}, \\ \dot{m}_2 &= -\frac{1}{2}m_1 m_2 (m_1 + m_2) e^{2(x_1 - x_2)}. \end{aligned} \tag{67}$$

Given the initial condition,

$$\begin{aligned} m_1(0) + m_2(0) &= 0, \\ x_1(0) &< x_2(0). \end{aligned} \tag{68}$$

Note that $m_1 + m_2$ is a constant of motion for system (67); thus, (67) admits the following solution:

$$\begin{aligned} m_1 &= \frac{C_1}{2}, \\ m_2 &= -\frac{C_1}{2}, \\ x_1 &= \frac{1}{2}(C_2 + C_3) + \frac{C_1^2}{16}\left(e^{2C_2} - 1\right)t, \\ x_2 &= \frac{1}{2}(C_3 - C_2) + \frac{C_1^2}{16}\left(e^{2C_2} - 1\right)t, \end{aligned} \tag{69}$$

where C_1, C_2, C_3 are constants and $C_1 \neq 0$, $C_2 < 0$. Therefore, the -3-order CH equation (63) admits the following peakon-antipeakon solution:

$$u(x,t) = \frac{C_1}{4}\left(e^{-2|x - x_1|} - e^{-2|x - x_2|}\right), \tag{70}$$

where x_1, x_2 are given by (69).

Remark 3. Note that $x_1 - x_2 \equiv C_2 < 0$ for all $t \in \mathbb{R}$; this peakon-antipeakon pair can not collide; that is, (70) gives a peakon-antipeakon solution globally. Besides, (70) is a superposition of two traveling waves (with constant amplitudes and the same constant speeds), which is an unusual feature for two-peakon solutions compared with the CH equation (1).

For the case $N = -4$, the peakon ODEs (43) can be written as

$$\begin{aligned}
\dot{x}_k = {} & \frac{1}{8}\sum_{i,j,l=1}^{n} m_i m_j m_l \left(1 + \operatorname{sgn}\left(x_j - x_l\right)\operatorname{sgn}\left(x_i - x_j\right)\right) \\
& \cdot e^{-2|x_i - x_k| - 2|x_i - x_j| - 2|x_j - x_l|} - \frac{1}{4}\sum_{i,l=1}^{n}\sum_{j=1}^{i} m_i m_j m_l \\
& \cdot \operatorname{sgn}\left(x_j - x_l\right) e^{-2|x_i - x_k| - 2|x_j - x_l|} + \frac{1}{8}\sum_{i,j=1}^{n} m_i^2 m_j \\
& \cdot \operatorname{sgn}\left(x_i - x_j\right) e^{-2|x_i - x_k| - 2|x_i - x_j|} - \frac{1}{8}\sum_{i,j,l=1}^{n} m_i m_j m_l \\
& \cdot \operatorname{sgn}\left(x_i - x_k\right)\left(\operatorname{sgn}\left(x_j - x_l\right) + \operatorname{sgn}\left(x_i - x_j\right)\right) \\
& \cdot e^{-2|x_i - x_k| - 2|x_i - x_j| - 2|x_j - x_l|} - \frac{1}{4} \\
& \cdot \sum_{i=1}^{k}\sum_{j,l=1}^{n} m_i m_j m_l \left(\operatorname{sgn}\left(x_j - x_l\right) + \operatorname{sgn}\left(x_i - x_j\right)\right) \\
& \cdot e^{-2|x_i - x_j| - 2|x_j - x_l|} - \frac{1}{8} m_k \sum_{i,j=1}^{n} m_i m_j \left(\operatorname{sgn}\left(x_i - x_k\right)\right. \\
& \left. - \operatorname{sgn}\left(x_i - x_j\right)\right) e^{-2|x_i - x_k| - 2|x_i - x_j|}, \\
\dot{m}_k = {} & \frac{1}{4} m_k \sum_{i,j,l=1}^{n} m_i m_j m_l \left(\operatorname{sgn}\left(x_i - x_j\right)\right. \\
& \left. - \operatorname{sgn}\left(x_i - x_k\right)\right) e^{-2|x_i - x_k| - 2|x_i - x_j| - 2|x_j - x_l|} - \frac{1}{4} \\
& \cdot m_k \sum_{i,j,l=1}^{n} m_i m_j m_l \operatorname{sgn}\left(x_j - x_l\right)\left(\operatorname{sgn}\left(x_i - x_k\right)\right. \\
& \left. \cdot \operatorname{sgn}\left(x_i - x_j\right) - 1\right) e^{-2|x_i - x_k| - 2|x_i - x_j| - 2|x_j - x_l|} + \frac{1}{2} \\
& \cdot m_k \sum_{i,l=1}^{n}\sum_{j=1}^{i} m_i m_j m_l \operatorname{sgn}\left(x_i - x_k\right) \\
& \cdot \operatorname{sgn}\left(x_j - x_l\right) e^{-2|x_i - x_k| - 2|x_j - x_l|} - \frac{1}{4} m_k \sum_{i,j=1}^{n} m_i^2 m_j \\
& \cdot \operatorname{sgn}\left(x_i - x_k\right)\operatorname{sgn}\left(x_i - x_j\right) e^{-2|x_i - x_k| - 2|x_i - x_j|}.
\end{aligned} \tag{71}$$

Example 4. When $n = 1$, (71) is simplified as

$$\begin{aligned}
\dot{x}_1 &= \frac{1}{8} m_1^2, \\
\dot{m}_1 &= 0.
\end{aligned} \tag{72}$$

Imposing the initial condition $x_1(0) = 0$, $m_1(0) = c$, we have

$$u(x,t) = \frac{c}{2} e^{-2|x - (c^3/8)t|}, \tag{73}$$

where c is an arbitrary nonzero constant. Thus, the -4-order CH equation admits the peakon solution (73).

Example 5. When $n = 2$, let $x_1(0) < x_2(0)$; in the neighborhood of $t = 0$, (71) is simplified as

$$\begin{aligned}
\dot{x}_1 &= \frac{1}{8} m_1^3 + \left(\frac{1}{8} m_2^3 + \frac{3}{8} m_1^2 m_2 + \frac{1}{4} m_1 m_2^2\right) e^{2(x_1 - x_2)} \\
&\quad + \frac{1}{8} m_1 m_2^2 e^{4(x_1 - x_2)}, \\
\dot{m}_1 &= -\frac{1}{4} m_1^2 m_2^2 e^{4(x_1 - x_2)} - \frac{1}{4} m_1 m_2^3 e^{2(x_1 - x_2)} \\
&\quad - \frac{1}{4} m_1^2 m_2 \left(m_1 + m_2\right) e^{2(x_1 - x_2)}, \\
\dot{x}_2 &= \frac{1}{8} m_2^3 + \left(\frac{1}{8} m_1^3 + \frac{3}{8} m_1 m_2^2 + \frac{1}{4} m_1^2 m_2\right) e^{2(x_1 - x_2)} \\
&\quad + \frac{1}{8} m_1^2 m_2 e^{4(x_1 - x_2)}, \\
\dot{m}_2 &= \frac{1}{4} m_1^2 m_2^2 e^{4(x_1 - x_2)} + \frac{1}{4} m_2 m_1^3 e^{2(x_1 - x_2)} \\
&\quad + \frac{1}{4} m_1 m_2^2 \left(m_1 + m_2\right) e^{2(x_1 - x_2)}.
\end{aligned} \tag{74}$$

Let

$$\begin{aligned}
p(t) &= m_1(t) + m_2(t), \\
P(t) &= m_1(t) - m_2(t), \\
q(t) &= x_1(t) + x_2(t), \\
Q(t) &= x_1(t) - x_2(t).
\end{aligned} \tag{75}$$

Obviously, $m_1 + m_2$ is a constant of motion for system (74); for $p(0) = 0$, we have that $p(t) \equiv 0$; then $\dot{q} = 0$; thus (74) admits the following solution:

$$\begin{aligned}
m_1 &= -\frac{c}{2\tanh\left(\left(c^3/32\right)\left(t + c_2\right)\right)}, \\
m_2 &= \frac{c}{2\tanh\left(\left(c^3/32\right)\left(t + c_2\right)\right)}, \\
x_1 &= \frac{1}{2} c_1 + \frac{1}{2}\ln\left(\operatorname{sech}\left(\frac{c^3}{32}\left(t + c_2\right)\right)\right), \\
x_2 &= \frac{1}{2} c_1 - \frac{1}{2}\ln\left(\operatorname{sech}\left(\frac{c^3}{32}\left(t + c_2\right)\right)\right),
\end{aligned} \tag{76}$$

where $c > 0, c_1, c_2$ are arbitrary constants. Exchanging the place of x_1 and x_2, we can obtain another solution to (76). Hence, the -4-order CH equation admits symmetric peakon-antipeakon solution

$$u(x,t) = -\frac{e^{-2|x - x_1|} - e^{-2|x - x_2|}}{4\tanh\left(\left(c^3/32\right)\left(t + c_2\right)\right)}, \tag{77}$$

where x_1, x_2 are given by (76) and can be exchanged.

Remark 6. It is easy to find that peakon-antipeakon given by (77) will collide at the position $x = (1/2)c_1$ as $t = -c_2$. This phenomenon is similar to the CH equation (1).

6. Concluding Remarks

In this paper, we extend the negative order CH hierarchy to the case where the potential is a finite discrete measure, showing that the N-order CH equation ($N < -1$) admits peakons and the peakons are compatible with corresponding Lax pairs obtained from negative order CH hierarchy. We have proved the existence of multipeakons of (40), and peakon-antipeakon solutions for -3- and -4-order CH equation were given.

We would like to remark that, due to Remark 12, the peakon ODEs (43) can be solved explicitly by the inverse scattering approach using Stieltjes continued fractions, and the process is similar to that in [4]. Specifically, the N-order CH equation ($N < -1$) admits peakon solutions (3) with

$$x_j = \frac{1}{2}\ln\left(\frac{2\widetilde{\Delta}^0_{n-j+1}}{\Delta^2_{n-j}}\right), \qquad m_j = \frac{2\widetilde{\Delta}^0_{n-j+1}\Delta^2_{n-j}}{\Delta^1_{n-j+1}\Delta^1_{n-j}}, \tag{78}$$

where

$$\Delta^0_k = \widetilde{\Delta}^0_k + \frac{1}{2}\Delta^2_{k-1} \quad (k \geq 1), \qquad \Delta^l_k = \widetilde{\Delta}^l_k \quad (k \geq 1, l \geq 1), \tag{79}$$

$\widetilde{\Delta}^l_k = \det(A_{i+j+l})^{k-1}_{i,j=0}$, $A_k = \int_{-\infty}^{+\infty}\lambda^k d\mu(\lambda)$, $\mu = \sum_{j=1}^n a_j\delta_{-\lambda_j}$, and

$$a_j = a_j(0)\exp\left(-2\lambda_j^{N+1}t\right), \quad a_j(0) > 0. \tag{80}$$

When N is even, the properties of peakons of the N-order CH equation are similar to the CH equation [4], while for the odd cases, there are some new features (see Example 2 and Remark 3 for peakon-antipeakon pair); further studies on multipeakons given above will be needed.

The CH equation (1) can be viewed as isospectral deformation of the string equation [4, 34]. Recently, we found the work [34, 35] of Szmigielski et al., in which they studied the isospectral deformation of mass density of the classical string problem. The N-order CH equation (16) we derived using method of finite power expansion with respect to spectral parameter may be interpreted as limit of small parameter; one can see [35, Example 3.8] for details, where vanishing condition was imposed on associated spectral problem in their setting, which is implied by the fact that $(d/dt)\mathrm{tr}(L) = 0$ holds for $x_i < x_j$ $(i < j)$; see Remark 13.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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