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Research Article

Strong Solutions for the Fluid-Particle Interaction Model with Non-Newtonian Potential

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This paper deals with a mathematical fluid-particle interaction model used to describing the evolution of particles dispersed in a viscous compressible non-Newtonian fluid. It is proved that the initial boundary value problems with vacuum admits a unique local strong solution in the dimensional case. The strong nonlinearity of the system brings us difficulties due to the fact that the viscosity term and non-Newtonian gravitational potential term are fully nonlinear.

1. Introduction

Fluid-particle interaction model arises in many practical applications in science and engineering [1–4] and is of primarily importance in the sedimentation analysis of disperse suspensions of particles in fluids. We focus on the fluid-particle interaction model that describes the evolution of particles dispersed in a viscous fluid.

Carrillo and Goudon first derived a fluid-particle interaction system by formal asymptotics from a mesoscopic description (see [5]). In the microscopic description, the cloud of the particles is related to its distribution function $f(t,x,\xi)$, which is the solution to a dimensionless Vlasov-Fokker-Planck equation. On the other hand, the fluid is described by its density $\rho(x,t) \geq 0$ and its velocity field $u(x,t)$. We assume that the fluid is compressible and isentropic, then (ρ,u) solves the compressible Euler equations for the inviscid case or the Navier-Stokes equations for the viscous case, respectively.

In [5], for the inviscid case, the coupling between the kinetic and the fluid equations is obtained through the friction forces that the fluid and the particles exert mutually. The friction force is assumed to follow the Stokes law and thus is proportional to the relative velocity of the fluid and the particles given by $\xi - u_\epsilon(t,x)$. Furthermore, both phases are affected by external forces, which are supposed to derive from a time independent potential $\Phi(x)$. The system was given as follows:

$$\begin{aligned}
&f_t + \theta\xi\cdot\nabla_x f - \kappa\nabla_x\Phi\cdot\nabla_\xi f \\
&\quad = \frac{1}{\epsilon}\mathrm{div}_\xi\left(\left(\xi - \frac{1}{\theta}u\right)f + \nabla_\xi f\right) \\
&\rho_t + \mathrm{div}_x(\rho u) = 0 \\
&(\rho u)_t + \mathrm{div}_x(\rho u\otimes u) + \chi\nabla_x P(\rho) + \alpha\theta\kappa\rho\nabla_x\Phi \\
&\quad = \frac{1}{\epsilon}\frac{\rho_P}{\rho_F}(J - \eta u)
\end{aligned} \tag{1}$$

where

$$\begin{aligned}
\eta(t,x) &= \int_{\mathbb{R}} f(t,x,\xi)\,\mathrm{d}\xi, \\
J(t,x) &= \theta\int_{\mathbb{R}} \xi f(t,x,\xi)\,\mathrm{d}\xi
\end{aligned} \tag{2}$$

$\theta,\kappa,\epsilon,\alpha,\chi$ are some related dimensionless parameters, $P(\rho)$, ρ_P, ρ_F are the pressure, the mass density of particles and

fluid, respectively. Setting $\rho_P/\rho_F = 1/\theta^2$, $\kappa = \theta$, $\theta = 1/\sqrt{\epsilon}$, $\alpha = \mathrm{sign}(\alpha)\epsilon$ with $\mathrm{sign}(\alpha) = \pm 1$, then (1) becomes

$$\begin{aligned} & f_t^\epsilon + \frac{1}{\sqrt{\epsilon}}\left(\xi\cdot\nabla_x f^\epsilon - \nabla_x\Phi\cdot\nabla_\xi f^\epsilon\right) \\ & \quad = \frac{1}{\epsilon}\mathrm{div}_\xi\left(\left(\xi - \sqrt{\epsilon}u^\epsilon\right) f^\epsilon + \nabla_\xi f^\epsilon\right) \\ & \rho_t^\epsilon + \mathrm{div}_x\left(\rho^\epsilon u^\epsilon\right) = 0 \\ & \left(\rho^\epsilon u^\epsilon\right)_t + \mathrm{div}_x\left(\rho^\epsilon u^\epsilon\otimes u^\epsilon\right) + \chi\nabla_x P\left(\rho^\epsilon\right) + \alpha\theta\kappa\rho^\epsilon\nabla_x\Phi \\ & \quad = \left(J^\epsilon - \eta^\epsilon u^\epsilon\right) \end{aligned} \tag{3}$$

Finally, letting $\epsilon \longrightarrow 0$, then (3) converges to system

$$\begin{aligned} & \eta_t + \mathrm{div}_x\left(\eta\left(u - \nabla_x\Phi\right) - \nabla_x\eta\right) = 0 \\ & \rho_t + \mathrm{div}_x\left(\rho u\right) = 0 \\ & \left(\rho u\right)_t + \mathrm{div}_x\left(\rho u\otimes u\right) + \nabla_x\left(\eta + \chi P\left(\rho\right)\right) \\ & \quad + \left(\mathrm{sign}\left(\alpha\right)\rho + \eta\right)\nabla_x\Phi = 0 \end{aligned} \tag{4}$$

For the viscous case, with the dynamic viscosity terms taken into consideration, then $(4)_3$ will be have an additional term of $\mathrm{div}_x\mathbb{S}$. In [6], the viscous stress tensor $\mathbb{S} = \mathbb{S}(\nabla_x u)$ is assumed to satisfy Newton's Law for viscosity which requires that

$$\mathbb{S} = \mu\left(\nabla u + \nabla u^T\right) + \lambda\,\mathrm{div}_x\, u\mathbb{I}, \tag{5}$$

where μ and λ are constant viscosity coefficients satisfying

$$\begin{aligned} & \mu > 0, \\ & \lambda + \frac{2}{3}\mu \geq 0. \end{aligned} \tag{6}$$

Then

$$\mathrm{div}_x\mathbb{S}\left(\nabla_x u\right) = \mu\Delta_x u + \lambda\nabla_x\,\mathrm{div}\, u \tag{7}$$

and thus the system turns into the following equation

$$\begin{aligned} & \rho_t + \mathrm{div}_x\left(\rho u\right) = 0 \\ & \left(\rho u\right)_t + \mathrm{div}_x\left(\rho u\otimes u\right) + \nabla_x\left(P\left(\rho\right) + \eta\right) - \mu\Delta_x u \\ & \quad - \lambda\nabla_x\mathrm{div}_x u = -\left(\eta + \beta\rho\right)\nabla_x\Phi \\ & \eta_t + \mathrm{div}_x\left(\eta\left(u - \nabla_x\Phi\right)\right) - \Delta_x\eta = 0 \end{aligned} \tag{8}$$

Moreover, if the influence of gravitational potential $\Psi = \Psi(x,t)$ was taken into consideration, there will be a Poisson equation coupled to the above system, as for the Navier-Stokes-Poisson equation in [7].

Carrillo et al. obtained the global existence and asymptotic behavior of the weak solutions and stability properties to (8). Subsequently, Fang et al. [8] studied the existence of global classical solutions in dimension one. In [9, 10], Balew and Trivisa obtained the existence of global weak solutions and weakly dissipative solutions by entropy method in dimension three. The two-phase flow hydrodynamic models have been proposed in [3]. For some mathematical results on the Navier-Stokes coupled equations such as the nematic liquid crystal flows models where viscous effects are included, for more details, we refer to [11–13] and references therein.

On the other hand, as we know, the viscous stress tensor $\mathbb{S}$ is depends on the rate of strain $\mathbb{E}_{ij}(\nabla_x u)$, where

$$\mathbb{E}_{ij}\left(\nabla_x u\right) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \tag{9}$$

If the stress and rate of strain satisfy the following linear relation

$$\mathbb{S} = \mu\mathbb{E}_{ij}\left(\nabla u\right). \tag{10}$$

then the fluid is called Newtonian. The coefficient of proportionality μ is called the viscosity coefficient, and it is a characteristic material quantity for the fluid concerned, which in general depends on density, temperature, and pressure. The governing equations of motions of them will be the Navier-Stokes equations. If the relation is not linear, the fluid is called non-Newtonian. Examples of non-Newtonian fluids are molten plastics, polymer solutions, dyes, varnishes, suspensions, adhesives, paints, greases, paper pulp, and biological fluids like blood. The simplest model of the stress-strain relation for such fluids given by the power laws, which states that

$$\mathbb{S} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^q, \tag{11}$$

for $0 < q < 1$ (see [14]). Ladyzhenskaya (see [15]) proposed a special form for $\mathbb{S}$ on the incompressible model:

$$\mathbb{S}_{ij} = \left(\mu_0 + \mu_1\left|\mathbb{E}\left(\nabla_x u\right)\right|^{p-2}\right)\mathbb{E}_{ij}\left(\nabla_x u\right) \tag{12}$$

These models are called

$$\begin{aligned} & \textit{Newtonian}, \quad \textit{for } \mu_0 > 0,\ \mu_1 = 0; \\ & \textit{Rabinowitsch}, \quad \textit{for } \mu_0, \mu_1 > 0,\ p = 4; \\ & \textit{Eills}, \quad \textit{for } \mu_0, \mu_1 > 0,\ p > 2; \\ & \textit{Ostwald} - \textit{deWaele}, \quad \textit{for } \mu_0 = 0,\ \mu_1 > 0,\ p > 1; \\ & \textit{Bingham}, \quad \textit{for } \mu_0, \mu_1 > 0,\ p = 1. \end{aligned} \tag{13}$$

For $\mu_0 = 0$, if $p < 2$ then it is a pseudo-plastic fluid, and if $p > 2$ then it is a dilatant fluid (see [14]). In the view of physics, the model captures the shear thinning fluid for the case of $1 < p < 2$, and captures the shear thickening fluid for the case of $p > 2$.

Followed by the Ladyzhenskaya model, in this paper, we investigate the compressible non-Newtonian fluid-particle

interaction model in one-dimensional case, then system (8) changes to be

$$\begin{aligned}
&\rho_t + (\rho u)_x = 0,\\
&(\rho u)_t + (\rho u^2)_x + \rho\Psi_x - \lambda\left[\left(u_x^2+\mu_1\right)^{(p-2)/2} u_x\right]_x\\
&\quad + (P+\eta)_x = -\eta\Phi_x, \quad (x,t)\in\Omega_T\\
&\left[\left(\Psi_x^2+\mu_2\right)^{(q-2)/2}\Psi_x\right]_x = 4\pi g\left(\rho - \frac{1}{|\Omega|}\int_\Omega \rho\,\mathrm{d}x\right),\\
&\eta_t + [\eta(u-\Phi_x)]_x = \eta_{xx}
\end{aligned} \tag{14}$$

with the initial and boundary conditions

$$\begin{aligned}
(\rho,u,\eta)|_{t=0} &= (\rho_0,u_0,\eta_0), \quad x\in\Omega,\\
u|_{\partial\Omega} = \Psi|_{\partial\Omega} &= 0, \quad t\in[0,T],
\end{aligned} \tag{15}$$

and the no-flux condition for the density of particles

$$(\eta_x + \eta\Phi_x)|_{\partial\Omega} = 0, \quad t\in[0,T]. \tag{16}$$

where Ω is a bounded interval, $\Omega_T = \Omega\times[0,T]$. ρ, u, η denote the fluid density, velocity, and the density of particle in the mixture, respectively. $P(\rho) = a\rho^\gamma$ is the pressure where $a > 0, \gamma > 1$, $\Psi(x,t)$ stands for the non-Newtonian gravitational potential, and the given function $\Phi(x)$ denotes the external potential. $\lambda > 0$ is the viscosity coefficient and $\mu_1 > 0, \mu_2 > 0$, $p > 2, q > 2$ are constants.

As to the non-Newtonian fluids, there has been much research both theoretically and experimentally, see ([15–23]). Indeed, we have investigated the existence results of solutions for $p > 2$ and $1 < p < 2$ in the absence of term $\Psi(x)$ for (14) in [22, 23]. However, the influence of non-Newtonian gravitational potential for a practical model was not taken into consideration there. To our knowledge, there seems very few mathematical results for the case of the fluid-particle interaction model systems with non-Newtonian gravitational potential, even in dimension one. The existence results to problem (14)-(16) when $p, q > 2$ which describes the motion of the compressible viscous isentropic gas flow is driven by a non-Newtonian gravitational force is still open up to now. We are interested in the existence and uniqueness of strong solutions on a one-dimensional bounded domain. In fact, the strong nonlinearity of (14) brings us new difficulties in getting the upper bound of ρ and the method used in [8] does not suitable for us. Motivated by Cho *etal*'s [24, 25] work on Navier-Stokes equations, we establish local existence and uniqueness of strong solutions by the iteration techniques.

Throughout the paper we assume that $a = \lambda = 1$. In the following sections, we will use simplified notations for standard Sobolev spaces and Bochner spaces, such as $L^p = L^p(\Omega)$, $H_0^1 = H_0^1(\Omega)$, $C(0,T;H^1) = C(0,T;H^1(\Omega))$.

1.1. Main Results

Theorem 1. *Let $\mu_1 > 0$ be a positive constant and $\Phi\in C^2(\Omega)$, and assume that the initial data (ρ_0,u_0,η_0) satisfy the following conditions:*

$$\begin{aligned}
&0\le\rho_0\in H^1(\Omega),\\
&u_0\in H_0^1(\Omega)\cap H^2(\Omega),\\
&\eta_0\in H^2(\Omega)
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
&-\left[\left(u_{0x}^2+\mu_1\right)^{(p-2)/2}u_{0x}\right]_x + (P(\rho_0)+\eta_0)_x + \eta_0\Phi_x\\
&\quad = \rho_0^{1/2}(g+\beta\Phi_x),
\end{aligned} \tag{18}$$

for some $g\in L^2(\Omega)$, where $\beta\neq 0$ is a constant. Then there exist a time $T_*\in(0,+\infty)$ and a unique strong solution (ρ,u,η) to (14)-(16) such that

$$\begin{aligned}
&\rho\in L^\infty\left(0,T_*;H^1(\Omega)\right),\\
&u\in L^\infty\left(0,T_*;W_0^{1,p}(\Omega)\cap H^2(\Omega)\right),\\
&\eta\in L^\infty\left(0,T_*;H^2(\Omega)\right)\\
&\rho_t\in L^\infty\left(0,T_*;L^2(\Omega)\right),\\
&u_t\in L^2\left(0,T_*;H_0^1(\Omega)\right),\\
&\eta_t\in L^\infty\left(0,T_*;L^2(\Omega)\right)\\
&\sqrt{\rho}u_t\in L^\infty\left(0,T_*;L^2(\Omega)\right),\\
&\left(\left(u_x^2+\mu_1\right)^{(p-2)/2}u_x\right)_x\in L^2\left(0,T_*;L^2(\Omega)\right),
\end{aligned} \tag{19}$$

2. A Priori Estimates for Smooth Solutions

In this section, we will prove the local existence of strong solutions. Provided that (ρ,u,η) is a smooth solution of (14)-(16) and $\rho_0\ge\delta$, where $0<\delta\ll 1$ is a positive number. We denote by $M_0 = 1+\mu_1+\mu_1^{-1}+|\rho_0|_{H^1}+|g|_{L^2}$ and introduce an auxiliary function

$$\begin{aligned}
Z(t) = \sup_{0\le s\le t}&\left(1+|\rho(s)|_{H^1}+|u(s)|_{W_0^{1,p}}+|\eta_t(s)|_{L^2}\right.\\
&\left.+|\eta(s)|_{H^1}+|\sqrt{\rho}u_t(s)|_{L^2}\right)
\end{aligned} \tag{20}$$

Then we estimate each term of $Z(t)$ in terms of some integrals of $Z(t)$ and apply arguments of Gronwall-type and thus prove that $Z(t)$ is locally bounded.

2.1. Estimate for $|\rho|_{H^1}$. First we need to do the following estimates. Using $(14)_1$, we rewritten $(14)_2$ as

$$\begin{aligned}
&\rho u_t + \rho u u_x + \rho\Psi_x - \left[\left(u_x^2+\mu_1\right)^{(p-2)/2}u_x\right]_x\\
&\quad + (P+\eta)_x = -\eta\Phi_x
\end{aligned} \tag{21}$$

By virtue of

$$\left|\left[\left(u_x^2+\mu_1\right)^{(p-2)/2}u_x\right]_x\right| \geq \mu_1^{(p-2)/2}\left|u_{xx}\right| \tag{22}$$

Then

$$\left|u_{xx}\right| \leq C\left|\rho u_t+\rho u u_x+\rho\Psi_x+\left(P+\eta\right)_x+\eta\Phi_x\right| \tag{23}$$

Taking the above inequality by L^2 norm, we get

$$\begin{aligned}\left|u_{xx}\right|_{L^2} &\leq C\left|\rho u_t+\rho u u_x+\rho\Psi_x+\left(P+\eta\right)_x+\eta\Phi_x\right|_{L^2}\\ &\leq C\left(\left|\rho\right|_{L^\infty}^{1/2}\left|\sqrt{\rho}u_t\right|_{L^2}+\left|\rho\right|_{L^\infty}\left|u\right|_{L^\infty}\left|u_x\right|_{L^2}\right.\\ &\quad\left.+\left|\rho\right|_{L^2}\left|\Psi_x\right|_{L^2}+\left|P_x\right|_{L^2}+\left|\eta_x\right|_{L^2}+\left|\eta\right|_{L^2}\left|\Phi_x\right|_{L^2}\right)\end{aligned} \tag{24}$$

We deal with the term of $|\Psi_x|_{L^2}$.

Multiplying $(14)_3$ by Ψ and integrating over Ω, we obtain

$$\begin{aligned}&\int_\Omega\left[\left(\Psi_x^2+\mu_2\right)^{(q-2)/2}\Psi_x\right]_x\Psi\,\mathrm{d}x\\ &\quad=4\pi g\left(\int_\Omega\rho\Psi\,\mathrm{d}x-m_0\int_\Omega\Psi\,\mathrm{d}x\right)\\ &\int_\Omega\left|\Psi_x\right|^q\mathrm{d}x\leq\int_\Omega\left(\Psi_x^2+\mu_2\right)^{(q-2)/2}\Psi_x^2\mathrm{d}x\\ &\quad=-\int_\Omega\left(\left(\Psi_x^2+\mu_2\right)^{(q-2)/2}\Psi_x\right)_x\Psi\,\mathrm{d}x\\ &\quad=-4\pi g\left(\int_\Omega\rho\Psi\,\mathrm{d}x-m_0\int_\Omega\Psi\,\mathrm{d}x\right)\\ &\quad\leq\left|\Psi\right|_{L^\infty}8\pi g m_0\leq\left|\Psi_x\right|_{L^q}8\pi g m_0\\ &\quad\leq\frac{1}{q}\left|\Phi_x\right|_{L^q}^q+\frac{1}{p}\left(8\pi g m_0\right)^p\end{aligned} \tag{25}$$

then we have

$$\int_\Omega\left|\Psi_x\right|^q\mathrm{d}x\leq C\left(m_0\right),\quad q>2. \tag{26}$$

Hence, we deduce that

$$\left|u_{xx}\right|_{L^2}\leq CZ^{\gamma+2}\left(t\right) \tag{27}$$

From $(14)_3$, taking it by L^2 norm, we get

$$\begin{aligned}\left|\eta_{xx}\right|_{L^2}&\leq\left|\eta_t+\left(\eta\left(u-\Phi_x\right)\right)_x\right|_{L^2}\\ &\leq\left|\eta_t\right|_{L^2}+\left|\eta_x\right|_{L^2}\left|u\right|_{L^\infty}+C\left|\eta_x\right|_{L^2}+\left|\eta\right|_{H^1}\left|u_x\right|_{L^p}\\ &\quad+C\left|\eta\right|_{H^1}\leq CZ^2\left(t\right)\end{aligned} \tag{28}$$

Multiplying $(14)_1$ by ρ, integrating over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left|\rho\right|^2\mathrm{d}s+\int_\Omega\left(\rho u\right)_x\rho\,\mathrm{d}x=0 \tag{29}$$

Integrating by parts, using Sobolev inequality, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\rho\left(t\right)\right|_{L^2}^2\leq\int_\Omega\left|u_x\right|\left|\rho\right|^2\mathrm{d}x\leq\left|u_{xx}\right|_{L^2}\left|\rho\right|_{L^2}^2 \tag{30}$$

Differentiating $(14)_1$ with respect to x, and multiplying it by ρ_x, integrating over Ω, using Sobolev inequality, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left|\rho_x\right|^2\mathrm{d}x&=-\int_\Omega\left[\frac{3}{2}u_x\left(\rho_x\right)^2+\rho\rho_x u_{xx}\right]\left(t\right)\mathrm{d}x\\ &\leq C\left[\left|u_x\right|_{L^\infty}\left|\rho_x\right|_{L^2}^2+\left|\rho\right|_{L^\infty}\left|\rho_x\right|_{L^2}\left|u_{xx}\right|_{L^2}\right]\\ &\leq C\left|\rho\right|_{H^1}^2\left|u_{xx}\right|_{L^2}\end{aligned} \tag{31}$$

From (30) and (31), by Gronwall's inequality, it follows that

$$\begin{aligned}\sup_{0\leq t\leq T}\left|\rho\left(t\right)\right|_{H^1}^2&\leq\left|\rho_0\right|_{H^1}^2\exp\left\{C\int_0^t\left|u_{xx}\right|_{L^2}\mathrm{d}s\right\}\\ &\leq C\exp\left(C\int_0^t Z^{\gamma+2}\left(s\right)\mathrm{d}s\right)\end{aligned} \tag{32}$$

Besides, we can also get the following estimates. By using $(14)_1$,

$$\begin{aligned}\left|\rho_t\left(t\right)\right|_{L^2}&\leq\left|\rho_x\left(t\right)\right|_{L^2}\left|u\left(t\right)\right|_{L^\infty}+\left|\rho\left(t\right)\right|_{L^\infty}\left|u_x\left(t\right)\right|_{L^2}\\ &\leq CZ^2\left(t\right)\end{aligned} \tag{33}$$

From $(14)_3$, we have

$$\mu_1^{(q-2)/2}\Psi_{xx}\leq\left[\left(\Psi_x^2+\mu_1\right)^{(q-2)/2}\Psi_x\right]_x=4\pi g\left(\rho-m_0\right) \tag{34}$$

then

$$\left|\Psi_{xx}\right|_{L^2}\leq C\left(m_0\right) \tag{35}$$

Differentiating $(14)_3$ with respect to time t, multiplying it by Ψ_t, integrating over Ω to x, and using Young's inequality, we have

$$\begin{aligned}\mu_1^{(q-2)/2}\int_\Omega\Psi_{xt}^2\mathrm{d}x&\leq\int_\Omega\left[\left(\Psi_x^2+\mu_1\right)^{(q-2)/2}\Psi_x\right]_t\Psi_{xt}\mathrm{d}x\\ &=-4\pi g\int_\Omega\rho_t\Psi_t\mathrm{d}x\\ &\leq\int_\Omega\left|\rho_t\right|_{L^2}\left|\Psi_{xt}\right|_{L^2}\\ &\leq C\left|\rho_t\right|_{L^2}^2+\eta\left|\Psi_{xt}\right|_{L^2}^2\end{aligned} \tag{36}$$

thus, we get

$$\left|\Psi_{xt}\right|_{L^2}^2\leq CZ^4\left(t\right) \tag{37}$$

where C is a positive constant, depending only on M_0.

2.2. Estimate for $|u|_{W_0^{1,p}}$. Multiplying (21) by u_t, integrating (by parts) over Ω_T, we have

$$\begin{aligned}&\int\int_{\Omega_T}\rho\left|u_t\right|^2\mathrm{d}x\mathrm{d}s\\ &\quad+\int\int_{\Omega_T}\left(u_x^2+\mu_1\right)^{(p-2)/2}u_x u_{xt}\mathrm{d}x\mathrm{d}s\\ &=-\int\int_{\Omega_T}\left(\rho u u_x+\rho\Psi_x+P_x+\eta_x+\eta\Phi_x\right)u_t\mathrm{d}x\mathrm{d}s\end{aligned} \tag{38}$$

We deal with each term as follows:

$$\begin{aligned}
&\int_\Omega \left(u_x^2+\mu_1\right)^{(p-2)/2} u_x u_{xt}\mathrm{d}x \\
&\quad = \frac{1}{2}\int_\Omega \left(u_x^2+\mu_1\right)^{(p-2)/2}\left(u_x^2\right)_t \mathrm{d}x \\
&\quad = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\int_0^{u_x^2}(s+\mu_1)^{(p-2)/2}\mathrm{d}s\right)\mathrm{d}x \\
&\int_0^{u_x^2}(s+\mu_1)^{(p-2)/2}\mathrm{d}s = \int_{\mu_1}^{u_x^2+\mu_1} t^{(p-2)/2}\mathrm{d}t \\
&\quad = \frac{2}{p}\left[\left(u_x^2+\mu_1\right)^{p/2}-\mu_1^{p/2}\right] \geq \frac{2}{p}\left|u_x\right|^p - \frac{2}{p}\mu_1^{p/2} \\
&-\int\int_{\Omega_T} P_x u_t \mathrm{d}x\mathrm{d}s = \int\int_{\Omega_T} P u_{xt}\mathrm{d}x\mathrm{d}s \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}\int\int_{\Omega_T} P u_x \mathrm{d}x\mathrm{d}s - \int\int_{\Omega_T} P_t u_x \mathrm{d}x\mathrm{d}s
\end{aligned} \tag{39}$$

From $(14)_1$ we get

$$P_t = -\gamma P u_x - P_x u \tag{40}$$

$$\begin{aligned}
&-\int\int_{\Omega_T}\left(\eta_x+\eta\Phi_x\right)u_t\mathrm{d}x\mathrm{d}s \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}\int\int_{\Omega_T}\left(\eta_x+\eta\Phi_x\right)u\,\mathrm{d}x\mathrm{d}s \\
&\qquad -\int\int_{\Omega_T}\left(\eta_x+\eta\Phi_x\right)_t u\,\mathrm{d}x\mathrm{d}s \\
&-\int\int_{\Omega_T}\left(\eta_x+\eta\Phi_x\right)_t u\,\mathrm{d}x\mathrm{d}s \\
&\quad = \int\int_{\Omega_T}\eta_t\left(u_x-\Phi_x u\right)\mathrm{d}x\mathrm{d}s \\
&\quad = -\int\int_{\Omega_T}\left[\eta_x-\eta\left(u-\Phi_x\right)\right]\left(u_x-\Phi_x u\right)_x\mathrm{d}x\mathrm{d}s
\end{aligned} \tag{41}$$

Substituting the above into (38), and using Young's inequality, we obtain

$$\begin{aligned}
&\int_0^t\left|\sqrt{\rho}u_t(s)\right|_{L^2}^2\mathrm{d}s+\left|u_x(t)\right|_{L^p}^p \leq C+\int_\Omega\left|Pu_x\right|\mathrm{d}x \\
&\quad +\int\int_{\Omega_T}\left(\left|\rho u u_x u_t\right|+\left|\rho\Psi_x u_t\right|+\left|\gamma P u_x^2\right|\right. \\
&\quad \left.+\left|P_x u u_x\right|\right)\mathrm{d}x\mathrm{d}s+\int\int_{\Omega_T}\left(\left|\eta_x u_{xx}\right|\right. \\
&\quad +\left|\eta_x\Phi_x u_x\right|+\left|\eta_x\Phi_{xx}u\right|+\left|\eta u u_{xx}\right|+\left|\eta u^2\Phi_{xx}\right| \\
&\quad +\left|\eta u\Phi_x u_x\right|+\left|\eta\Phi_x u_{xx}\right|+\left|\eta\Phi_x\Phi_{xx}u\right| \\
&\quad \left.+\left|\eta\Phi_x^2 u_x\right|\right)\mathrm{d}x\mathrm{d}s \leq C+C\int_0^t\left(\left|\rho\right|_{L^\infty}\left|u\right|_{L^\infty}^2\right. \\
&\quad \cdot\left|u_x\right|_{L^p}^2+\left|\rho\right|_{L^\infty}\left|\Psi_x\right|_{L^2}^2+\left|P\right|_{L^\infty}\left|u_x\right|_{L^p}^2+\left|P_x\right|_{L^2}\left|u\right|_{L^\infty} \\
&\quad \cdot\left|u_x\right|_{L^p}+\left|\eta_x\right|_{L^2}\left|u_{xx}\right|_{L^2}+\left|\eta_x\right|_{L^2}\left|u_x\right|_{L^p}+\left|\eta_x\right|_{L^2} \\
&\quad \cdot\left|u\right|_{L^\infty}+\left|\eta\right|_{L^\infty}\left|u\right|_{L^\infty}\left|u_{xx}\right|_{L^2}+\left|\eta\right|_{L^\infty}\left|u\right|_{L^\infty}^2+\left|\eta\right|_{L^\infty} \\
&\quad \cdot\left|u\right|_{L^\infty}\left|u_x\right|_{L^p}+\left|\eta\right|_{L^\infty}\left|u_{xx}\right|_{L^2}+\left|\eta\right|_{L^\infty}\left|u\right|_{L^\infty}+\left|\eta\right|_{L^\infty} \\
&\quad \left.\cdot\left|u_x\right|_{L^p}\right)\mathrm{d}s+C\left|P(t)\right|_{L^2}^2+\frac{1}{2}\int_0^t\left|\sqrt{\rho}u_t\right|_{L^2}^2\mathrm{d}s \\
&\quad +\frac{1}{2}\left|u_x(t)\right|_{L^p}^p \leq C\left(1+\int_0^t Z^{\gamma+4}(s)\,\mathrm{d}s+\left|P(t)\right|_{L^2}^2\right)
\end{aligned} \tag{42}$$

Using $(14)_1$, we have

$$\begin{aligned}
&\int_\Omega\left|P(t)\right|^2\mathrm{d}x \\
&\quad = \int_\Omega\left|P(0)\right|^2\mathrm{d}x+\int_0^t\frac{\partial}{\partial s}\left(\int_\Omega P(s)^2\,\mathrm{d}x\right)\mathrm{d}s \\
&\quad \leq \int_\Omega\left|P(0)\right|^2\mathrm{d}x \\
&\qquad +2\int_0^t\int_\Omega P(s)\,\gamma\rho^{\gamma-1}\left(-\rho_x u-\rho u_x\right)\mathrm{d}x\mathrm{d}s \\
&\quad \leq C+C\int_0^t\left|P\right|_{L^\infty}\left|\rho\right|_{L^\infty}^{\gamma-1}\left|\rho\right|_{H^1}\left|u_x\right|_{L^p}\mathrm{d}s \\
&\quad \leq C\left(1+\int_0^t Z^{2\gamma+1}(s)\,\mathrm{d}s\right)
\end{aligned} \tag{43}$$

Combining (42)-(43), yields

$$\begin{aligned}
&\int_0^t\left|\sqrt{\rho}u_t(s)\right|_{L^2}^2(s)\,\mathrm{d}s+\left|u_x(t)\right|_{L^p}^p \\
&\quad \leq C\left(1+\int_0^t Z^{2\gamma+3}(s)\,\mathrm{d}s\right)
\end{aligned} \tag{44}$$

where C is a positive constant, depending only on M_0.

2.3. Estimate for $|\eta_t|_{L^2}$ and $|\eta|_{H^1}$. Multiplying $(14)_4$ by η, integrating the resulting equation over Ω_T, and using the boundary conditions (16), Young's inequality, we have

$$\begin{aligned}
&\int_0^t\left|\eta_x(s)\right|_{L^2}^2\mathrm{d}s+\frac{1}{2}\left|\eta(t)\right|_{L^2}^2 \\
&\quad \leq \int\int_{\Omega_T}\left(\left|\eta u\eta_x\right|+\left|\eta\Phi_x\eta_x\right|\right)\mathrm{d}x\mathrm{d}s \\
&\quad \leq \frac{1}{4}\int_0^t\left|\eta_x(s)\right|_{L^2}^2\mathrm{d}s+C\int_0^t\left|u_x\right|_{L^p}^2\left|\eta\right|_{H^1}^2\mathrm{d}s \\
&\qquad +C\int_0^t\left|\eta\right|_{H^1}^2+C \\
&\quad \leq \frac{1}{4}\int_0^t\left|\eta_x(s)\right|_{L^2}^2\mathrm{d}s+C\left(1+\int_0^t Z^4(t)\,\mathrm{d}s\right)
\end{aligned} \tag{45}$$

Multiplying $(14)_4$ by η_t, integrating (by parts) over Ω_T, and using the boundary conditions (16), Young's inequality, we have

$$\begin{aligned}
&\int_0^t |\eta_t(s)|_{L^2}^2 \,\mathrm{d}s + \frac{1}{2}|\eta_x(t)|_{L^2}^2 \\
&\quad \le \int\int_{\Omega_T} |\eta(u-\Phi_x)\eta_{xt}|\,\mathrm{d}x\mathrm{d}s \\
&\quad \le \frac{1}{4}\int_0^t |\eta_{xt}(s)|_{L^2}^2 \,\mathrm{d}s + C\int_0^t |\eta|_{H^1}^2 |u_x|_{L^p}^2 \,\mathrm{d}s \\
&\qquad + C\int_0^t |\eta|_{H^1}^2 \,\mathrm{d}s + C \\
&\quad \le \frac{1}{4}\int_0^t |\eta_{xt}(s)|_{L^2}^2 \,\mathrm{d}s + C\left(1+\int_0^t Z^4(t)\,\mathrm{d}s\right)
\end{aligned} \tag{46}$$

Differentiating $(14)_4$ with respect to t, multiplying the resulting equation by η_t, and integrating (by parts) over Ω_T, we get

$$\begin{aligned}
&\int_0^t |\eta_{xt}(s)|_{L^2}^2 \,\mathrm{d}s + \frac{1}{2}|\eta_t(t)|_{L^2}^2 = \int\int_{\Omega_T} (\eta(u-\Phi_x))_t \\
&\quad \cdot \eta_{xt}\mathrm{d}x\mathrm{d}s \le C + \int\int_{\Omega_T} (|\eta_t u\eta_{xt}| + |\eta_t\Phi_x\eta_{xt}| \\
&\quad + |\eta_x u_t\eta_t| + |\eta u_{xt}\eta_t|)\,\mathrm{d}x\mathrm{d}s \le C\Bigg(1 \\
&\quad + \int_0^t \left(|\eta_t|_{L^2}^2 |u_x|_{L^p}^2 + |\eta_t|_{L^2}^2 + |\eta_x|_{L^2}^2 |\eta_t|_{L^2}^2\right. \\
&\quad \left. + |\eta|_{H^1}^2 |\eta_t|_{L^2}^2\right)\mathrm{d}x\Bigg) + \frac{1}{2}\int_0^t |\eta_{xt}|_{L^2}^2 + \frac{1}{2} \\
&\quad \cdot \int_0^t |u_{xt}|_{L^2}^2 \le C\left(1+\int_0^t Z^{2\gamma+6}(s)\,\mathrm{d}s\right)
\end{aligned} \tag{47}$$

Combining (45)-(47), we get

$$\begin{aligned}
&|\eta|_{H^1}^2 + |\eta_t|_{L^2}^2 + \int_0^t \left(|\eta_x|_{L^2}^2 + |\eta_t|_{L^2}^2 + |\eta_{xt}|_{L^2}^2\right)(s)\,\mathrm{d}s \\
&\quad \le C\left(1+\int_0^t Z^{2\gamma+6}(s)\,\mathrm{d}s\right)
\end{aligned} \tag{48}$$

2.4. Estimate for $|\sqrt{\rho}u_t|_{L^2}$. Differentiating (21) with respect to t, multiplying the result equation by u_t, and integrating it over Ω with respect to x, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|u_t|^2\,\mathrm{d}x + \int_\Omega \left[\left(u_x^2+\mu_1\right)^{(p-2)/2} u_x\right]_t u_{xt}\mathrm{d}x \\
&\quad = \int_\Omega \left[(\rho u)_x\left(u_t^2 + uu_xu_t + \Psi_x u_t\right) - \rho u_x u_t^2\right. \\
&\quad \left. - \rho\Psi_{xt}u_t - (P+\eta)_t u_{xt} - \eta_t\Phi_x u_t\right]\mathrm{d}x
\end{aligned} \tag{49}$$

Note that

$$\begin{aligned}
&\left[\left(u_x^2+\mu_1\right)^{(p-2)/2} u_x\right]_t u_{xt} \\
&\quad = \left(u_x^2+\mu_1\right)^{(p-4)/2}(p-1)\left(u_x^2+\mu_1\right)u_{xt}^2 \\
&\quad \ge \mu_1^{(p-2)/2} u_{xt}^2
\end{aligned} \tag{50}$$

Combining (40), (49) can be rewritten into

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|u_t|^2\,\mathrm{d}x + \int_\Omega |u_{xt}|^2\,\mathrm{d}x \\
&\quad \le 2\int_\Omega \rho|u||u_t||u_{xt}|\,\mathrm{d}x + \int_\Omega \rho|u||u_x|^2|u_t|\,\mathrm{d}x \\
&\quad + \int_\Omega \rho|u|^2|u_{xx}||u_t|\,\mathrm{d}x + \int_\Omega \rho|u|^2|u_x||u_{xt}|\,\mathrm{d}x \\
&\quad + \int_\Omega \rho|u||\Psi_{xx}||u_t|\,\mathrm{d}x + \int_\Omega \rho|u||\Psi_x||u_{xt}|\,\mathrm{d}x \\
&\quad + \int_\Omega \rho|u_x||u_t|^2\,\mathrm{d}x + \int_\Omega \gamma|P||u_x||u_{xt}|\,\mathrm{d}x \\
&\quad + \int_\Omega |P_x||u||u_{xt}|\,\mathrm{d}x + \int_\Omega |\eta_t||u_{xt}|\,\mathrm{d}x \\
&\quad + \int_\Omega |\eta_t||\Phi_x||u_t|\,\mathrm{d}x + \int_\Omega \rho|\Psi_{xt}||u_t|\,\mathrm{d}x = \sum_{j=1}^{12} I_j
\end{aligned} \tag{51}$$

By using Sobolev inequality, Hölder inequality and Young's inequality, (35), (37), we estimate each term of I_j as follows:

$$\begin{aligned}
I_1 &= 2\int_\Omega \rho|u||u_t||u_{xt}|\,\mathrm{d}x \\
&\le 2|\rho|_{L^\infty}^{1/2}|u|_{L^\infty}|\sqrt{\rho}u_t|_{L^2}|u_{xt}|_{L^2} \\
&\le CZ^5(t) + \frac{1}{7}|u_{xt}|_{L^2}^2 \\
I_2 &= \int_\Omega \rho|u||u_x|^2|u_t|\,\mathrm{d}x \\
&\le |\rho|_{L^\infty}^{1/2}|u|_{L^2}^2|u_x|_{L^\infty}^2|\sqrt{\rho}u_t|_{L^2} \le CZ^{2\gamma+7}(t) \\
I_3 &= \int_\Omega \rho|u|^2|u_{xx}||u_t|\,\mathrm{d}x \\
&\le |\rho|_{L^\infty}^{1/2}|u|_{L^\infty}^2|u_{xx}|_{L^2}|\sqrt{\rho}u_t|_{L^2} \le CZ^{\gamma+6}(t) \\
I_4 &= \int_\Omega \rho|u|^2|u_x||u_{xt}|\,\mathrm{d}x \\
&\le |\rho|_{L^\infty}|u|_{L^\infty}^2|u_x|_{L^2}|u_{xt}|_{L^2} \le CZ^8(t) + \frac{1}{7}|u_{xt}|_{L^2}^2 \\
I_5 &= \int_\Omega \rho|u||\Psi_{xx}||u_t|\,\mathrm{d}x \\
&\le |\rho|_{L^\infty}^{1/2}|u|_{L^2}|\Psi_{xx}|_{L^2}|\sqrt{\rho}u_t|_{L^2} \le CZ^3(t)
\end{aligned}$$

$$
\begin{aligned}
I_6 &= \int_\Omega \rho |u| |\Psi_x| |u_{xt}| \, dx \le |\rho|_{L^\infty} |u|_{L^\infty} |\Psi_x|_{L^2} |u_{xt}|_{L^2} \\
&\le CZ^4(t) + \frac{1}{7} |u_{xt}|_{L^2}^2 \\
I_7 &= \int_\Omega \rho |u_x| |u_t|^2 \, dx \le |u_x|_{L^\infty} |\sqrt{\rho} u_t|_{L^2}^2 \le CZ^{\gamma+4}(t) \\
I_8 &= \int_\Omega \gamma |P| |u_x| |u_{xt}| \, dx \le C |P|_{L^\infty} |u_x|_{L^p} |u_{xt}|_{L^2} \\
&\le CZ^{2\gamma+2}(t) + \frac{1}{7} |u_{xt}|_{L^2}^2 \\
I_9 &= \int_\Omega |P_x| |u| |u_{xt}| \, dx \le |P_x|_{L^2} |u|_{L^\infty} |u_{xt}|_{L^2} \\
&\le CZ^{2\gamma+2}(t) + \frac{1}{7} |u_{xt}|_{L^2}^2 \\
I_{10} &= \int_\Omega |\eta_t| |u_{xt}| \, dx \le |\eta_t|_{L^2} |u_{xt}|_{L^2} \\
&\le CZ^2(t) + \frac{1}{7} |u_{xt}|_{L^2}^2 \\
I_{11} &= \int_\Omega |\eta_t| |\Phi_x| |u_t| \, dx \le CZ^2(t) + \frac{1}{7} |u_{xt}|_{L^2}^2 \\
I_{12} &= \int_\Omega \rho |\Psi_{xt}| |u_t| \, dx \le C |\rho|_{L^\infty}^{1/2} |\Psi_{xt}|_{L^2} |\sqrt{\rho} u_t|_{L^2} \\
&\le CZ^6(t)
\end{aligned}
\tag{52}
$$

Substituting I_j $(j = 1, 2, \ldots, 12)$ into (51), and integrating over $(\tau, t) \subset (0, T)$ on the time variable, we have

$$
\begin{aligned}
&|\sqrt{\rho} u_t(t)|_{L^2}^2 + \int_\tau^t |u_{xt}|_{L^2}^2(s) \, ds \\
&\quad \le |\sqrt{\rho} u_t(\tau)|_{L^2}^2 + C \int_\tau^t Z^{2\gamma+7}(s) \, ds.
\end{aligned}
\tag{53}
$$

To obtain the estimate of $|\sqrt{\rho} u_t(t)|_{L^2}^2$, we need to estimate $\lim_{\tau \longrightarrow 0} |\sqrt{\rho} u_t(\tau)|_{L^2}^2$. Multiplying (21) by u_t and integrating over Ω, we get

$$
\begin{aligned}
\int_\Omega \rho |u_t|^2 \, dx \le 2 \int_\Omega \Big(&\rho |u|^2 |u_x|^2 + \rho |\Psi_x|^2 \\
&+ \rho^{-1} \Big| - \Big[\left(u_x^2 + \mu_1 \right)^{(p-2)/2} u_x \Big]_x + (P + \eta)_x \\
&+ \eta \Phi_x \Big|^2 \Big) \, dx.
\end{aligned}
\tag{54}
$$

According to the smoothness of (ρ, u, η), we obtain

$$
\begin{aligned}
&\lim_{\tau \longrightarrow 0} \int_\Omega \Big(\rho |u|^2 |u_x|^2 + \rho |\Psi_x|^2 \\
&\quad + \rho^{-1} \Big| - \Big[\left(u_x^2 + \mu_1 \right)^{(p-2)/2} u_x \Big]_x + (P + \eta)_x + \eta \Phi_x \Big|^2 \Big) \, dx \\
&= \int_\Omega \Big(\rho |u_0|^2 |u_{0x}|^2 + \rho_0 |\Phi_x|^2 \\
&\quad + \rho_0^{-1} \Big| - \Big[\left(u_{0x}^2 + \mu_1 \right)^{(p-2)/2} u_{0x} \Big]_x + (P_0 + \eta_0)_x \\
&\quad + \eta_0 \Phi_x \Big|^2 \Big) \, dx \le |\rho_0|_{L^\infty} |u_0|_{L^\infty}^2 |u_{0x}|_{L^2}^2 + |\rho_0|_{L^\infty} |\Phi_x|_{L^2}^2 \\
&\quad + |g|_{L^2}^2 + \beta |\Phi_x|_{L^2}^2 \le C.
\end{aligned}
\tag{55}
$$

Therefore, taking a limit on τ in (53), as $\tau \longrightarrow 0$, we conclude that

$$
\begin{aligned}
&|\sqrt{\rho} u_t(t)|_{L^2}^2 + \int_0^t |u_{xt}|_{L^2}^2(s) \, ds \\
&\quad \le C \left(1 + \int_0^t Z^{2\gamma+7}(s) \, ds \right),
\end{aligned}
\tag{56}
$$

where C is a positive constant, depending only on M_0.

Combining the estimates of (27), (28), (32), (33), (44), (48), (56) and the definition of $Z(t)$, we conclude that

$$
Z(t) \le \widetilde{C} \exp \left(\widetilde{\widetilde{C}} \int_0^t Z^{2\gamma+7}(s) \, ds \right)
\tag{57}
$$

where $\widetilde{C}, \widetilde{\widetilde{C}}$ are positive constants, depending only on M_0. This means that there exist a time $T_1 > 0$ and a constant $C > 0$, such that

$$
\begin{aligned}
&\operatorname{ess\,sup}_{0 \le t \le T_1} \Big(|\rho|_{H^1} + |u|_{W_0^{1,p} \cap H^2} + |\eta|_{H^2} + |\eta_t|_{L^2} \\
&\quad + |\sqrt{\rho} u_t|_{L^2} + |\rho_t|_{L^2} \Big) + \int_0^{T_1} \Big(|\sqrt{\rho} u_t|_{L^2}^2 + |u_{xt}|_{L^2}^2 \\
&\quad + |\eta_x|_{L^2}^2 + |\eta_t|_{L^2}^2 + |\eta_{xt}|_{L^2}^2 \Big) \, ds \le C
\end{aligned}
\tag{58}
$$

3. Proof of the Main Theorem

In this section, our proof will be based on the usual iteration argument and some ideas developed in [24, 25]. We construct the approximate solutions, by using the iterative scheme, inductively, as follows: first define $u^0 = 0$ and assuming that u^{k-1} was defined for $k \ge 1$, let ρ^k, u^k, η^k be the unique smooth solution to the following problems:

$$
\rho_t^k + \rho_x^k u^{k-1} + \rho^k u_x^{k-1} = 0 \tag{59}
$$

$$
\rho^k u_t^k + \rho^k u^{k-1} u_x^k + \rho^k \Psi_x^k + L_p u^k + P_x^k + \eta_x^k = -\eta^k \Phi_x \tag{60}
$$

$$
L_q \Psi^k = -4\pi g \left(\rho^k - m_0 \right) \tag{61}
$$

$$
\eta_t^k + \left(\eta^k \left(u^{k-1} - \Phi_x \right) \right)_x = \eta_{xx}^k \tag{62}
$$

with the initial and boundary conditions

$$
\begin{aligned}
\left(\rho^k, u^k, \eta^k \right) \big|_{t=0} &= (\rho_0, u_0, \eta_0) \\
u^k \big|_{\partial\Omega} &= \Psi^k \big|_{\partial\Omega} = 0 \\
\left(\eta_x^k + \eta^k \Phi_x \right) \big|_{\partial\Omega} &= 0
\end{aligned}
\tag{63}
$$

where

$$L_p u^k = -\left[\left(\left(u_x^k\right)^2 + \mu_1\right)^{(p-2)/2} u_x^k\right]_x$$
$$L_q \Psi^k = -\left[\left(\left(\Psi_x^k\right)^2 + \mu_2\right)^{(q-2)/2} \Psi_x^k\right]_x \tag{64}$$

We directly construct approximate solutions of the problem (59)–(63). More precisely, we first find ρ^k from (59) and (63) with smooth function u^{k-1}, i.e.,

$$\rho_t^k + \rho_x^k u^{k-1} + \rho^k u_x^{k-1} = 0, \tag{65}$$

and

$$\rho^k\big|_{t=0} = \rho_0, \quad \rho_0 \geq \delta. \tag{66}$$

It follows from the classical linear hyperbolic theory that there is a unique solution ρ^k on this above initial problem. Using the method of characteristics, we have

$$\frac{d\rho^k}{dt} = -\rho^k u_x^{k-1}, \tag{67}$$

$$\frac{dx}{dt} = u^{k-1}(x,t), \tag{68}$$

$$x|_{t=0} = x_0,$$
$$\rho^k\big|_{t=0} = \rho_0. \tag{69}$$

By (68) and (69), we have

$$x(t) = x_0 + \int_0^t u^{k-1}(x(s), s)\,\mathrm{d}s = U(x_0, t) \tag{70}$$

Using (67), then

$$d\left(\ln \rho^k\right) = -u_x^{k-1} dt \tag{71}$$

which means

$$\rho^k(x,t) \geq \delta \exp\left[-\int_0^{T_1} \left|u_x^{k-1}(\cdot, s)\right|_{L^\infty} \mathrm{d}s\right] > 0, \quad \text{for all } t \in (0, T_1). \tag{72}$$

Next, combining the classical stableness results of the elliptic equation, the existence of Ψ^k can be obtained by (61) and (63), then by (62) and (63) we get η^k. The last, with ρ^k, Ψ^k, η^k being given, by virtue of (72), from (60) and (63), according to the classical theorem of quasi-linear parabolic equation (see [17], Chapter VI, Theorem 5.2), there exists a unique smooth solution u^k. With the process, the nonlinear coupled system has been deduced into a sequence of decoupled problems and each problem admits a smooth solution. And the following estimates hold:

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0\leq t\leq T_1} \Big(\left|\rho^k\right|_{H^1} + \left|u^k\right|_{W_0^{1,p}\cap H^2} + \left|\eta^k\right|_{H^2} + \left|\eta_t^k\right|_{L^2} \\
&\quad + \left|\sqrt{\rho^k} u_t^k\right|_{L^2} + \left|\rho_t^k\right|_{L^2}\Big) + \int_0^{T_1} \Big(\left|\sqrt{\rho^k} u_t^k\right|_{L^2}^2 + \left|u_{xt}^k\right|_{L^2}^2 \\
&\quad + \left|\eta_x^k\right|_{L^2}^2 + \left|\eta_t^k\right|_{L^2}^2 + \left|\eta_{xt}^k\right|_{L^2}^2\Big)\,\mathrm{d}s \leq C
\end{aligned} \tag{73}$$

where C is a generic constant depending only on M_0, but independent of k.

Next, we have to prove that the approximate solution (ρ^k, u^k, η^k) converges to a solution to the original problem (14) in a strong sense. To this end, let us define

$$\begin{aligned}
\overline{\rho}^{k+1} &= \rho^{k+1} - \rho^k, \\
\overline{u}^{k+1} &= u^{k+1} - u^k, \\
\overline{\Psi}^{k+1} &= \Psi^{k+1} - \Psi^k, \\
\overline{\eta}^{k+1} &= \eta^{k+1} - \eta^k
\end{aligned} \tag{74}$$

then we can verify that the functions $\overline{\rho}^{k+1}, \overline{u}^{k+1}, \overline{\eta}^{k+1}$ satisfy the system of equations

$$\overline{\rho}_t^{k+1} + \left(\overline{\rho}^{k+1} u^k\right)_x + \left(\rho^k \overline{u}^k\right)_x = 0 \tag{75}$$

$$\begin{aligned}
&\rho^{k+1}\overline{u}_t^{k+1} + \rho^{k+1} u^k \overline{u}_x^{k+1} + \left(L_p u^{k+1} - L_p u^k\right) \\
&\quad = -\overline{\rho}^{k+1}\left(u_t^k + u^k u_x^k + \Psi_x^{k+1}\right) - \left(P_x^{k+1} - P_x^k\right) \\
&\qquad + \rho^k\left(\overline{u}^k u_x^k - \overline{\Psi}_x^{k+1}\right) - \overline{\eta}_x^{k+1} - \overline{\eta}^{k+1}\Phi_x
\end{aligned} \tag{76}$$

$$L_q \Psi^{k+1} - L_q \Psi^k = -4\pi g \overline{\rho}^{k+1} \tag{77}$$

$$\overline{\eta}_t^{k+1} + \left(\eta^k \overline{u}^k\right)_x + \left(\overline{\eta}^{k+1}\left(u^k - \Phi_x\right)\right)_x = \overline{\eta}_{xx}^{k+1} \tag{78}$$

Multiplying (75) by $\overline{\rho}^{k+1}$, integrating over Ω, and using Young's inequality, we obtain

$$\begin{aligned}
\frac{d}{dt}\left|\overline{\rho}^{k+1}\right|_{L^2}^2 &\leq C\left|\overline{\rho}^{k+1}\right|_{L^2}^2 \left|u_x^k\right|_{L^\infty} + \left|\rho^k\right|_{H^1}\left|\overline{u}_x^k\right|_{L^2}\left|\overline{\rho}^{k+1}\right|_{L^2} \\
&\leq C\left|u_{xx}^k\right|_{L^2}\left|\overline{\rho}^{k+1}\right|_{L^2}^2 + C_\zeta\left|\rho^k\right|_{H^1}^2\left|\overline{\rho}^{k+1}\right|_{L^2}^2 \\
&\quad + \zeta\left|\overline{u}_x^k\right|_{L^2}^2 \leq C_\zeta\left|\overline{\rho}^{k+1}\right|_{L^2}^2 + \zeta\left|\overline{u}_x^k\right|_{L^2}^2
\end{aligned} \tag{79}$$

where C_ζ is a positive constant, depending on M_0 and ζ for all $t < T_1$ and $k \geq 1$.

Multiplying (76) by $\overline{u}^{k+1}$, integrating over Ω, and using Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^{k+1}\left|\overline{u}^{k+1}\right|^2 \mathrm{d}x + \int_\Omega \left(L_p u^{k+1} - L_p u^k\right)\overline{u}^{k+1}\mathrm{d}x \\
&\quad \leq C\int_\Omega \Big(\left|\overline{\rho}^{k+1}\right|\left(\left|u_t^k\right| + \left|u^k u_x^k\right| + \left|\Psi_x^{k+1}\right|\right)\left|\overline{u}^{k+1}\right| \\
&\qquad + \left|P_x^{k+1} - P_x^k\right|\left|\overline{u}^{k+1}\right| + \left|\rho^k\overline{u}^k\right|\left|u_x^k\right|\left|\overline{u}^{k+1}\right| \\
&\qquad + \left|\rho^k\right|\left|\overline{\Psi}_x^{k+1}\right|\left|\overline{u}^{k+1}\right| + \left|\overline{\eta}_x^{k+1}\right|\left|\overline{u}^{k+1}\right| \\
&\qquad + \left|\overline{\eta}^{k+1}\Phi_x\right|\left|\overline{u}^{k+1}\right|\Big)\,\mathrm{d}x
\end{aligned} \tag{80}$$

Let

$$\sigma(s) = \left(s^2 + \mu_1\right)^{(p-2)/2} s \tag{81}$$

and then

$$\begin{aligned}\sigma'(s) &= \left(\left(s^2+\mu_1\right)^{(p-2)/2} s\right)' \\ &= \left(s^2+\mu_1\right)^{(p-4)/2}\left((p-1)s^2+\mu_1\right) \geq \mu_1^{(p-2)/2}\end{aligned} \tag{82}$$

We estimate the second term of (80) as follows:

$$\begin{aligned}&\int_\Omega \left(L_p u^{k+1} - L_p u^k\right)\overline{u}^{k+1}\mathrm{d}x \\ &\quad = \int_\Omega \int_0^1 \sigma'\left(\theta u_x^{k+1} + (1-\theta)u_x^k\right)\mathrm{d}\theta \left|\overline{u}_x^{k+1}\right|^2 \mathrm{d}x \\ &\quad \geq \mu_1^{(p-2)/2}\int_\Omega \left|\overline{u}_x^{k+1}\right|^2 \mathrm{d}x\end{aligned} \tag{83}$$

Similarly, we have

$$\begin{aligned}&\mu_2^{(q-2)/2}\int_\Omega \left(\overline{\Psi}_x^{k+1}\right)^2 \mathrm{d}x \\ &\quad \leq \int_\Omega \left(L_q\Psi^{k+1} - L_q\Psi^k\right)\overline{\Psi}^{k+1}\mathrm{d}x \\ &\quad = 4\pi g\int_\Omega \overline{\rho}^{k+1}\overline{\Psi}^{k+1}\mathrm{d}x\end{aligned} \tag{84}$$

and then we have

$$\left|\overline{\Psi}_x^{k+1}\right|_{L^2}^2 \leq C\left|\overline{\rho}^{k+1}\right|_{L^2}^2 \tag{85}$$

Substituting (83) into (80) and using Young's inequality, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \rho^{k+1}\left|\overline{u}^{k+1}\right|^2\mathrm{d}x + \int_\Omega \left|\overline{u}_x^{k+1}\right|^2\mathrm{d}x \\ &\quad \leq C\Big(\left|\overline{\rho}^{k+1}\right|_{L^2}\left|u_{xt}^k\right|_{L^2}\left|\overline{u}_x^{k+1}\right|_{L^2} \\ &\quad + \left|\overline{\rho}^{k+1}\right|_{L^2}\left|u_x^k\right|_{L^p}\left|u_{xx}^k\right|_{L^2}\left|\overline{u}_x^{k+1}\right|_{L^2} \\ &\quad + \left|\overline{\rho}^{k+1}\right|_{L^2}\left|\Psi_x^{k+1}\right|_{L^2}\left|\overline{u}_x^{k+1}\right|_{L^2} + \left|P^{k+1}-P^k\right|_{L^2}\left|\overline{u}_x^{k+1}\right|_{L^2} \\ &\quad + \left|\rho^k\right|_{L^2}^{1/2}\left|\sqrt{\rho^k}\overline{u}^k\right|_{L^2}\left|u_{xx}^k\right|_{L^2}\left|\overline{u}_x^{k+1}\right|_{L^2} \\ &\quad + \left|\rho^k\right|_{L^\infty}\left|\overline{\Psi}_x^{k+1}\right|_{L^2}\left|\overline{u}_x^{k+1}\right|_{L^2} + \left|\overline{\eta}^{k+1}\right|_{L^2}\left|\overline{u}_x^{k+1}\right|_{L^2} \\ &\quad + \left|\overline{\eta}^{k+1}\right|_{L^2}\left|\overline{u}_x^{k+1}\right|_{L^2}\Big) \leq B_\zeta(t)\left|\overline{\rho}^{k+1}\right|_{L^2}^2 \\ &\quad + C\left(\left|\sqrt{\rho^k}\overline{u}^k\right|_{L^2}^2 + \left|\overline{\eta}^{k+1}\right|_{L^2}^2\right) + \zeta\left|\overline{u}_x^{k+1}\right|_{L^2}^2\end{aligned} \tag{86}$$

where $B_\zeta(t) = C(1+|u_{xt}^k(t)|_{L^2}^2)$, for all $t \leq T_1$ and $k \geq 1$. Using (73) we derive

$$\int_0^t B_\zeta(s)\,\mathrm{d}s \leq C + Ct \tag{87}$$

Multiplying (78) by $\overline{\eta}^{k+1}$, integrating over Ω, and using (73) and Young's inequality, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \left|\overline{\eta}^{k+1}\right|^2\mathrm{d}x + \int_\Omega \left|\overline{\eta}_x^{k+1}\right|^2\mathrm{d}x \\ &\quad \leq \int_\Omega \left|\overline{\eta}^{k+1}\right|\left|u^k - \Phi_x\right|\left|\overline{\eta}_x^{k+1}\right|\mathrm{d}x \\ &\quad + \int_\Omega \left(\left|\eta^k\right|\left|\overline{u}^k\right|\right)_x\left|\overline{\eta}^{k+1}\right|\mathrm{d}x \\ &\quad \leq \left|\overline{\eta}^{k+1}\right|_{L^2}\left|u^k - \Phi_x\right|_{L^\infty}\left|\overline{\eta}_x^{k+1}\right|_{L^2} \\ &\quad + \left|\eta_x^k\right|_{L^2}\left|\overline{u}^k\right|_{L^\infty}\left|\overline{\eta}^{k+1}\right|_{L^2} + \left|\eta^k\right|_{L^\infty}\left|\overline{u}_x^k\right|_{L^2}\left|\overline{\eta}^{k+1}\right|_{L^2} \\ &\quad \leq C_\zeta\left|\overline{\eta}^{k+1}\right|_{L^2}^2 + \zeta\left|\overline{\eta}_x^{k+1}\right|_{L^2}^2 + \zeta\left|\overline{u}_x^k\right|_{L^2}^2\end{aligned} \tag{88}$$

Collecting (79), (86), and (88), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\left|\overline{\rho}^{k+1}(t)\right|_{L^2}^2 + \left|\sqrt{\rho^{k+1}}\overline{u}^{k+1}(t)\right|_{L^2}^2 + \left|\overline{\eta}^{k+1}(t)\right|_{L^2}^2\right) \\ &\quad + \left|\overline{u}_x^{k+1}(t)\right|_{L^2}^2 + \left|\overline{\eta}_x^{k+1}\right|_{L^2}^2 \\ &\quad \leq E_\zeta(t)\left|\overline{\rho}^{k+1}(t)\right|_{L^2}^2 + C\left|\sqrt{\rho^k}\overline{u}^k\right|_{L^2}^2 + C_\zeta\left|\overline{\eta}^{k+1}\right|_{L^2}^2 \\ &\quad + \zeta\left|\overline{u}_x^k\right|_{L^2}^2\end{aligned} \tag{89}$$

where $E_\zeta(t)$ depends only on $B_\zeta(t)$ and C_ζ, for all $t \leq T_1$ and $k \geq 1$. Using (73), we have

$$\int_0^t E_\zeta(s)\,\mathrm{d}s \leq C + C_\zeta t \tag{90}$$

Integrating (89) over $(0,t) \subset (0,T_1)$ with respect to t, using Gronwall's inequality, we have

$$\begin{aligned}&\left|\overline{\rho}^{k+1}(t)\right|_{L^2}^2 + \left|\sqrt{\rho^{k+1}}\overline{u}^{k+1}(t)\right|_{L^2}^2 + \left|\overline{\eta}^{k+1}(t)\right|_{L^2}^2 \\ &\quad + \int_0^t \left|\overline{u}_x^{k+1}(t)\right|_{L^2}^2\mathrm{d}s + \int_0^t \left|\overline{\eta}_x^{k+1}\right|_{L^2}^2\mathrm{d}s \\ &\quad \leq C\exp\left(C_\zeta t\right)\int_0^t\left(\left|\sqrt{\rho^k}\overline{u}^k(s)\right|_{L^2}^2 + \left|\overline{u}_x^k(s)\right|_{L^2}^2\right)\mathrm{d}s\end{aligned} \tag{91}$$

from the above recursive relation, choose $\zeta > 0$ and $0 < T_* < T_1$ such that $C\exp(C_\zeta T_*) < 1/2$, using Gronwall's inequality, we deduce that

$$\begin{aligned}&\sum_{k=1}^K\Bigg[\sup_{0\leq t\leq T_*}\left(\left|\overline{\rho}^{k+1}(t)\right|_{L^2}^2 + \left|\sqrt{\rho^{k+1}}\overline{u}^{k+1}(t)\right|_{L^2}^2 + \left|\overline{\eta}^{k+1}(t)\right|_{L^2}^2\right) \\ &\quad + \int_0^{T_*}\left|\overline{u}_x^{k+1}(t)\right|_{L^2}^2\mathrm{d}t + \int_0^{T_*}\left|\overline{\eta}_x^{k+1}(t)\right|_{L^2}^2\mathrm{d}t\Bigg] < C\end{aligned} \tag{92}$$

Since all of the constants do not depend on δ, as $k \longrightarrow \infty$, we conclude that sequence (ρ^k, u^k, η^k) converges to a limit $(\rho^\delta, u^\delta, \eta^\delta)$ in the following convergence:

$$\rho \longrightarrow \rho^\delta \text{ in } L^\infty\left(0, T_*; L^2(\Omega)\right), \tag{93}$$

$$u \longrightarrow u^\delta \text{ in } L^\infty\left(0, T_*; L^2(\Omega)\right) \cap L^2\left(0, T_*; H_0^1(\Omega)\right), \tag{94}$$

$$\eta \longrightarrow \eta^\delta \text{ in } L^\infty\left(0, T_*; L^2(\Omega)\right) \cap L^2\left(0, T_*; H^1(\Omega)\right), \tag{95}$$

and there also holds

$$\begin{aligned} &\operatorname*{ess\,sup}_{0\le t\le T_1}\Big(\left|\rho^\delta\right|_{H^1} + \left|u^\delta\right|_{W_0^{1,p}\cap H^2} + \left|\eta^\delta\right|_{H^2} + \left|\eta_t^\delta\right|_{L^2} \\ &\quad + \left|\sqrt{\rho^\delta}u_t^\delta\right|_{L^2} + \left|\rho_t^\delta\right|_{L^2}\Big) + \int_0^{T_*}\Big(\left|\sqrt{\rho}^\delta u_t^\delta\right|_{L^2}^2 + \left|u_{xt}^\delta\right|_{L^2}^2 \\ &\quad + \left|\eta_x^\delta\right|_{L^2}^2 + \left|\eta_t^\delta\right|_{L^2}^2 + \left|\eta_{xt}^\delta\right|_{L^2}^2\Big)\,\mathrm{d}s \le C. \end{aligned} \tag{96}$$

For each small $\delta > 0$, let $\rho_0^\delta = J_\delta * \rho_0 + \delta$, J_δ is a mollifier on Ω, and $u_0^\delta \in H_0^1(\Omega) \cap H^2(\Omega)$ is a smooth solution of the boundary value problem:

$$\begin{aligned} &L_p u_0^\delta + \left(P\left(\rho_0^\delta\right) + \eta_0^\delta\right)_x + \eta_0^\delta \Phi_x \\ &\quad = \left(\rho_0^\delta\right)^{1/2}\left(g^\delta + \beta\Phi_x\right), \\ &u_0^\delta(0) = u_0^\delta(1) = 0, \end{aligned} \tag{97}$$

where $g^\delta \in C_0^\infty$ and satisfies $|g^\delta|_{L^2} \le |g|_{L^2}$, $\lim_{\delta\longrightarrow 0^+}|g^\delta - g|_{L^2} = 0$.

We deduce that $(\rho^\delta, u^\delta, \eta^\delta)$ is a solution of the following initial boundary value problem:

$$\begin{aligned} &\rho_t + (\rho u)_x = 0, \\ &(\rho u)_t + \left(\rho u^2\right)_x + \rho\Psi_x - \lambda\left[\left(u_x^2 + \mu_1\right)^{(p-2)/2} u_x\right]_x \\ &\quad + (P + \eta)_x = -\eta\Phi_x, \\ &\left[\left(\Psi_x^2 + \mu_2\right)^{(q-2)/2}\Psi_x\right]_x = 4\pi g\left(\rho - \frac{1}{|\Omega|}\int_\Omega \rho\,\mathrm{d}x\right), \\ &\eta_t + \left(\eta\left(u - \Phi_x\right)\right)_x = \eta_{xx}, \\ &(\rho, u, \eta)|_{t=0} = \left(\rho_0^\delta, u_0^\delta, \eta_0^\delta\right), \\ &u|_{\partial\Omega} = \Psi|_{\partial\Omega} = \left(\eta_x + \eta\Phi_x\right)|_{\partial\Omega} = 0. \end{aligned} \tag{98}$$

where $\rho_0^\delta \ge \delta$, $p, q > 2$.

By the proof of Lemma 2.3 in [20], there exists a subsequence $\{u_0^{\delta_j}\}$ of $\{u_0^\delta\}$, as $\delta_j \longrightarrow 0^+$, $u_0^\delta \longrightarrow u_0$ in $H_0^1(\Omega) \cap H^2(\Omega)$, $-(|u_{0x}^{\delta_j}|^{p-2}u_{0x}^{\delta_j})_x \longrightarrow -(|u_{0x}|^{p-2}u_{0x})_x$ in $L^2(\Omega)$, Hence, u_0 satisfies (18) of Theorem 1. By virtue of the lower semicontinuity of various norms, we deduce that (ρ, u, η) satisfies the following uniform estimate:

$$\begin{aligned} &\operatorname*{ess\,sup}_{0\le t\le T_1}\Big(|\rho|_{H^1} + |u|_{W_0^{1,p}\cap H^2} + |\eta|_{H^2} + |\eta_t|_{L^2} \\ &\quad + \left|\sqrt{\rho}u_t\right|_{L^2} + |\rho_t|_{L^2}\Big) + \int_0^{T_*}\Big(\left|\sqrt{\rho}u_t\right|_{L^2}^2 + |u_{xt}|_{L^2}^2 \\ &\quad + |\eta_x|_{L^2}^2 + |\eta_t|_{L^2}^2 + |\eta_{xt}|_{L^2}^2\Big)\,\mathrm{d}s \le C, \end{aligned} \tag{99}$$

where C is a positive constant, depending only on M_0.

The uniqueness of solution can be obtained by the same method as the above proof of convergence; we omit the details here. This completes the proof.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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