

## Review Article

# An Information Geometric Perspective on the Complexity of Macroscopic Predictions Arising from Incomplete Information

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Motivated by the presence of deep connections among dynamical equations, experimental data, physical systems, and statistical modeling, we report on a series of findings uncovered by the authors and collaborators during the last decade within the framework of the so-called Information Geometric Approach to Chaos (IGAC). The IGAC is a theoretical modeling scheme that combines methods of information geometry with inductive inference techniques to furnish probabilistic descriptions of complex systems in presence of limited information. In addition to relying on curvature and Jacobi field computations, a suitable indicator of complexity within the IGAC framework is given by the so-called information geometric entropy (IGE). The IGE is an information geometric measure of complexity of geodesic paths on curved statistical manifolds underlying the entropic dynamics of systems specified in terms of probability distributions. In this manuscript, we discuss several illustrative examples wherein our modeling scheme is employed to infer macroscopic predictions when only partial knowledge of the microscopic nature of a given system is available. Finally, we include comments on the strengths and weaknesses of the current version of our proposed theoretical scheme in our concluding remarks.

## 1. Introductory Background

Characterizing and to some degree understanding the emergence and evolutionary development of biological systems represent one of the most compelling motivations to investigate the highly elusive concept of complexity [1–5].

Entropic inference methods [6] have recently been combined with information geometric techniques [7–9] in an effort to uncover quantitative indicators of complexity suitable for application to statistical models used to render probabilistic descriptions of systems about which only limited information is known. Within this approach, the notion of complexity associated with statistical models can be regarded as a measure of the difficulty of inferring macroscopic predictions due to the inherent lack of complete knowledge about the microscopic degrees of freedom of the system being investigated.

This line of research was initially referred to as the *Information Geometric Approach to Chaos* (IGAC) [10]. A

schematic outline of the IGAC theoretical framework is presented as follows: once the microscopic degrees of freedom of a complex system of arbitrary nature are identified and its relevant information constraints are chosen, entropic methods are used to establish an initial, static statistical model of the complex system. The statistical model describing the complex system is defined by means of probability distributions parameterized in terms of statistical macrovariables which, in turn, depend upon the specific functional form of the information constraints assumed to be relevant for the implementation of statistical inferences. The next step in the program concerns the evolution of the complex system. In particular, assuming the complex system evolves, the evolution of the associated statistical model from its initial to final configurations is determined by means of the so-called entropic dynamics (ED, [11]). Entropic Dynamics constitute a variant of information-constrained dynamics that is constructed on statistical manifolds whose elements correspond to probability distributions. Furthermore, these

distributions are in one-to-one relation with a suitable set of statistical macrovariables that define a parameter space which serves to provide a convenient parameterization of points on the original statistical manifold. The ED framework prescribes the evolution of probability distributions by means of an entropic inference principle: in particular, starting from the initial configuration, the motion toward the final configuration occurs via the maximization of the logarithmic relative entropy (maximum relative entropy method—MrE method, [6]) between any two consecutive intermediate configurations. At this juncture, it is worth noting that ED provides only the *expected*, but not the *actual*, trajectories of the system. Inferences within ED rely on the nature of the chosen information constraints that are utilized at the level of the MrE algorithm. The validation of modeling schemes of this type can only be verified *a posteriori*. If discrepancies occur between experimental observations and the inferred predictions, a new set of information constraints must be selected [12–14]. This is an extremely important feature of the MrE algorithm and was recently reconsidered in [15]. The evolution of probability distributions is specified in terms of the results of the maximization procedure described above, namely, a geodesic evolution of the statistical macrovariables [6]. A measure of distance between two different probability distributions is quantified by the Fisher-Rao information metric [7]. This distance can be interpreted as the degree of distinguishability between two distributions. After having determined the information metric, one can readily apply the standard methods of Riemannian differential geometry to investigate the geometric structure of the statistical manifold underlying the entropic motion which characterizes the evolution of the probability distributions. Generally speaking, conventional Riemannian geometric quantities such as Christoffel connection coefficients of the second kind, Ricci tensor, Riemannian curvature tensor, sectional curvatures, scalar curvature, Weyl anisotropy tensor, Killing fields, and Jacobi fields can be computed in the standard manner [16]. More specifically, the chaoticity (i.e., temporal complexity) of such statistical models can be investigated by means of suitably chosen measures, such as the signs of the scalar and sectional curvatures of the statistical manifold, the asymptotic temporal behavior of Jacobi fields, the existence of Killing vectors, and the existence of a nonvanishing Weyl anisotropy tensor. In addition to these measures, complexity within the IGAC approach can also be quantified by means of the so-called information geometric entropy (IGE), originally presented in [10].

For a review of the MrE inference algorithm, we refer to [6, 17–19]. Furthermore, for a detailed presentation on the ED approach used in this manuscript, we refer to [11]. Finally, for the sake of brevity, we have omitted mathematical details in our main presentation. For the sake of self-consistency, however, we have added useful mathematical details on the notions of curvature, information geometric entropy, and Jacobi fields in Appendices A, B, and C, respectively.

## 2. Illustrative Examples

In this section, we outline all available applications concerning the characterization of the complexity of geodesic paths

on curved statistical manifolds within the IGAC framework. For a conventional approach to the Riemannian geometrization of classical Newtonian dynamics, we refer to [20–23]. We remark that in what follows we only report on the asymptotic behavior of the chosen complexity indicators. These complexity indicators are parameterized with respect to the statistical affine parameter used to quantify temporal change in the geodesic analysis on the underlying statistical manifold. Furthermore, the order in which these illustrative examples are presented in this manuscript is chronological.

Very preliminary concepts and applications of the IGAC appeared originally in [24–26]. For a more recent review on the IGAC and the IGE with more detailed computations, we refer to [27, 28] and [29], respectively.

*2.1. Example 1: Uncorrelated Gaussian Statistical Models.* Previously, the IGAC modeling framework was used to investigate the information geometric properties of a system of arbitrary nature containing  $l$  degrees of freedom, each one characterized by two pieces of relevant information, namely, its mean and its variance [30, 31]. From an information geometric perspective, it was determined that the family of statistical models corresponding to such a system is Gaussian in form. This set of Gaussian distributions generate a non-maximally symmetric  $2l$ -dimensional statistical manifold  $\mathcal{M}_s$  exhibiting a constant negative scalar curvature  $\mathcal{R}_{\mathcal{M}_s}$  proportional to the number of degrees of freedom of the system (Appendix A),

$$\mathcal{R}_{\mathcal{M}_s} = -l. \quad (1)$$

From a dynamical standpoint, it was found that the system explores statistical volume elements on  $\mathcal{M}_s$  at an exponential rate. Specifically, in the asymptotic limit, the IGE  $\mathcal{S}_{\mathcal{M}_s}$  increases linearly with respect to the statistical affine parameter  $\tau$  and is proportional to the number of degrees of freedom  $l$  according to Appendix B:

$$\mathcal{S}_{\mathcal{M}_s}(\tau) \stackrel{\tau \rightarrow \infty}{\sim} l\lambda\tau, \quad (2)$$

where  $\lambda$  represents the maximal positive Lyapunov exponent [32] characterizing the statistical model. Furthermore, the geodesic paths on  $\mathcal{M}_s$  were shown to be hyperbolic trajectories. Finally, by solving the geodesic deviation equations, it was determined that, in the asymptotic limit, the Jacobi vector field intensity  $J_{\mathcal{M}_s}$  exhibits exponential divergence and is proportional to the number of degrees of freedom  $l$  (Appendix C):

$$J_{\mathcal{M}_s}(\tau) \stackrel{\tau \rightarrow \infty}{\sim} l \exp(\lambda\tau). \quad (3)$$

Since the exponential divergence of the Jacobi vector field intensity  $J_{\mathcal{M}_s}$  is a known classical feature of chaos, from (1), (2), and (3), the authors propose that  $\mathcal{R}_{\mathcal{M}_s}$ ,  $\mathcal{S}_{\mathcal{M}_s}$ , and  $J_{\mathcal{M}_s}$  each behave as proper indicators of chaoticity, with each being proportional to the number of Gaussian-distributed microstates of the system. Despite being verified in this special scenario, this proportionality constitutes the first known example appearing in the literature of a possible substantive connection among information geometric indicators

of chaoticity deduced from the probabilistic descriptions of dynamical systems.

**2.2. Example 2: Correlated Gaussian Statistical Models.** In [33], the IGAC framework was applied to analyze the information-constrained dynamics of a system with two correlated, Gaussian-distributed microscopic degrees of freedom. As a working hypothesis, the degrees of freedom were assumed to be characterized by the same variance. The presence of microscopic correlations leads to the emergence of asymptotic information geometric compression of the statistical macrostates explored by the system at a faster rate than that observed in the absence of microscopic correlations. In particular, it was found that in the asymptotic limit (Appendix B)

$$\left[ \exp(\mathcal{S}_{\mathcal{M}_s}(\tau)) \right]_{\text{correlated}} \stackrel{\tau \rightarrow \infty}{\sim} \mathcal{F}(\rho) \cdot \left[ \exp(\mathcal{S}_{\mathcal{M}_s}(\tau)) \right]_{\text{uncorrelated}}, \quad (4)$$

where  $\mathcal{F}(\rho)$  in (4) with  $0 \leq \mathcal{F}(\rho) \leq 1$  is defined as

$$\mathcal{F}(\rho) \stackrel{\text{def}}{=} \frac{1}{2^{5/2}} \left[ \sqrt{\frac{4(4-\rho^2)}{(2-2\rho^2)^2}} \left( \frac{2+\rho}{4(1-\rho^2)} \right)^{-3/2} \right]. \quad (5)$$

The function  $\mathcal{F}(\rho)$  is a monotonic decreasing compression factor defined for any value of the correlation coefficient  $\rho \in (0, 1)$ . This result constitutes an explicit connection between correlations at the *microscopic* level and complexity at the *macroscopic* level [34–36]. This result provides a concise and transparent description of the behavioral change of the macroscopic complexity of a statistical model caused by the presence of microscopic correlations.

**2.3. Example 3: Inverted Harmonic Oscillators.** In a broad sense, it is known that the primary issues addressed by the General Theory of Relativity are twofold: first, one is interested in the manner that space-time geometry evolves in response to mass-energy; second, one seeks to understand how mass-energy configurations move in such a space-time geometry. Within the IGAC approach, one focuses on how systems move in a given statistical geometry, while the evolution of the statistical geometry itself is neglected. The realization that there are two distinct and separate aspects to this scenario served as a turning point in the development of the IGAC framework that led to an intriguing result. The first formal result in this research direction was presented in [37], where the possibility of exploiting well-established principles of inference to derive Newtonian dynamics from relevant prior information codified in an appropriate statistical manifold was explored. The key working hypothesis in that derivation was the assumed existence of an irreducible uncertainty in the location of particles which requires the state of a particle to be characterized by a probability distribution. The corresponding configuration space is therefore a curved statistical manifold whose Riemannian geometry is defined by the Fisher-Rao information metric. The expected trajectory follows from the MrE method viewed as a principle of inference.

In this approach, there is no need for additional physical postulates such as an action principle or equation of motion, nor for the concept of mass, momentum, or of phase space, not even the notion of external, absolute time. Newton's mechanics for any number of particles interacting among themselves and with external fields are completely reproduced by the resulting entropic dynamics. Furthermore, both the interactions between particles and their masses are explained as a consequence of the underlying statistical manifold.

From a more applied perspective, building upon the results found in [37], an information geometric analogue of the Zurek-Paz quantum chaos criterion in the classical reversible limit [38–40] was presented in [41, 42]. In these works, the IGAC framework was used to investigate a set of  $l$  three-dimensional, uncoupled, anisotropic, inverted harmonic oscillators (IHO) characterized by an Ohmic distributed frequency spectrum. Omitting technical details, in [41, 42] it was shown that the asymptotic temporal behavior of the IGE for such a system is given by (Appendix B)

$$\mathcal{S}_{\text{IHO}}^{(l)}(\tau; \omega_1, \dots, \omega_l) \stackrel{\tau \rightarrow \infty}{\sim} \Omega \tau, \quad (6)$$

where

$$\Omega \stackrel{\text{def}}{=} \sum_{i=1}^l \omega_i \quad (7)$$

and  $\omega_i$  with  $1 \leq i \leq l$  denotes the frequency of the  $i$ -th inverted harmonic oscillator. Equation (6) displays an asymptotic, linear IGE growth for the set of inverted harmonic oscillators and can be viewed as an extension of the result of Zurek and Paz presented in [39, 40] to an ensemble of anisotropic, inverted harmonic oscillators within the IGAC framework. Specifically, in [39, 40], Zurek and Paz investigated the effects of decoherence in quantum chaos by considering a single unstable harmonic oscillator with frequency  $\Omega$  characterized by a potential  $V(x)$ ,

$$V(x) \stackrel{\text{def}}{=} -\frac{\Omega^2 x^2}{2}, \quad (8)$$

coupled to an external environment. They determined that, in the reversible classical limit, the von Neumann entropy of such a system increases linearly at a rate determined by the Lyapunov exponent  $\Omega$  according to

$$\mathcal{S}_{\text{quantum}}^{(\text{chaotic})}(\tau) \stackrel{\tau \rightarrow \infty}{\sim} \Omega \tau. \quad (9)$$

In [41, 42], (6) was essentially proposed as the classical information geometric analogue of (9).

**2.4. Example 4: Quantum Spin Chains.** We recall that it is commonly conjectured that spectral correlations of classically integrable systems are well described by Poisson statistics and that quantum spectra of classically chaotic systems are universally correlated according to Wigner-Dyson statistics. The former and the latter conjectures are known as the BGS (Bohigas-Giannoni-Schmit, [43]) and BTG (Berry-Tabor-Gutzwiller, [44]) conjectures, respectively.

In [45, 46], the IGAC was employed to analyze the entropic dynamics on statistical manifolds induced by classical probability distributions commonly used in the investigation of regular and chaotic quantum energy level statistics. In particular, an information geometric characterization of the chaotic (integrable) energy level statistics of a quantum antiferromagnetic Ising spin chain immersed in a tilted (transverse) external magnetic field was proposed. The IGAC of a Poisson distribution coupled to an exponential bath (representing a spin chain in a *transverse* magnetic field, corresponding to the integrable case) and that of a Wigner-Dyson distribution coupled to a Gaussian bath (representing a spin chain in a *tilted* magnetic field, corresponding to the chaotic case) were studied. Remarkably, it was found that, in the former case, the IGE exhibits asymptotic logarithmic growth (Appendix B):

$$\mathcal{S}_{\mathcal{M}_s}^{(\text{integrable})}(\tau) \stackrel{\tau \rightarrow \infty}{\sim} c \log(\tau) + \bar{c}, \quad (10)$$

while, in the latter case, the IGE exhibits asymptotic linear growth (Appendix B):

$$\mathcal{S}_{\mathcal{M}_s}^{(\text{chaotic})}(\tau) \stackrel{\tau \rightarrow \infty}{\sim} \mathcal{K}\tau. \quad (11)$$

In this manuscript,  $\log$  denotes the natural logarithmic function. The quantities  $c$  and  $\bar{c}$  in (10) are two suitable constants of integration that depend on the dimensionality of the underlying statistical manifold and the boundary conditions on the statistical variables, respectively. The quantity  $\mathcal{K}$  in (11) is a model parameter that describes the asymptotic temporal rate of change of the IGE. In view of these findings, it was conjectured that the IGAC framework may be of some utility when analyzing potential physical applications in the field of quantum energy level statistics. In such cases, the IGE would serve the role of the entanglement entropy defined in standard quantum information theory [47, 48].

**2.5. Example 5: Statistical Embedding and Complexity Reduction.** Reducing the complexity of statistical models is a very active field of research [49–51]. Building upon the exploratory analysis presented in [52], the IGAC approach was utilized in [53] to investigate the  $2l$ -dimensional Gaussian statistical model  $\mathcal{M}_s$  that is induced by an appropriate embedding within a larger  $4l$ -dimensional Gaussian manifold endowed with a Fisher-Rao information metric  $g_{\mu\nu}$  containing non-trivial off-diagonal terms. These terms arise due to the presence of the correlational structure coefficients  $\rho_k$  with  $1 \leq k \leq l$  that characterize the embedding constraints among the statistical variables on the larger manifold. Two main findings were obtained. First, a power law decay of the IGE (Appendix B) at a rate determined by the coefficients  $\rho_k$  was observed:

$$\mathcal{S}_{\mathcal{M}_s}(\tau; l, \lambda_k, \rho_k) \stackrel{\tau \rightarrow \infty}{\sim} \log \left[ \Lambda(\rho_k) + \frac{\bar{\Lambda}(\rho_k, \lambda_k)}{\tau} \right]^l, \quad (12)$$

where  $\lambda_k$  are suitable constants of integration that specify the problem being investigated, while the functional forms of the bounded functions  $\Lambda(\rho_k)$  and  $\bar{\Lambda}(\rho_k, \lambda_k)$  can be found

in [53]. Second, in addition to (12), it was found that the asymptotic exponential divergence of the Jacobi vector field intensity was attenuated. Due to this observed attenuation, the authors concluded that the presence of such embedding constraints leads to the emergence of an asymptotic information geometric compression of the explored macrostates on the curved statistical manifold  $\mathcal{M}_s$ . These results serve as a further nontrivial step toward the characterization of the complexity of microscopically correlated multidimensional Gaussian statistical models of relevance in the mathematical modeling of realistic physical systems.

**2.6. Example 6: Entanglement Induced via Scattering.** Inspired by the preliminary analysis presented in [54], the IGAC was used to provide an information geometric perspective of the quantum entanglement generated by  $s$ -wave scattering [55] between two Gaussian wave packets in [56, 57]. Within the IGAC framework, the pre- and postcollisional quantum dynamical scenarios related to an elastic, head-on collision were conjectured to be macroscopic manifestations emerging from microscopic statistical structures. Exploiting such a working hypothesis, the pre- and postcollisional scenarios were described by uncorrelated and correlated Gaussian statistical models, respectively. As a consequence, the authors were capable of expressing the entanglement strength in terms of scattering potential and incident particle energies. Furthermore, the manner in which the entanglement duration is related to the scattering potential and incident particle energies was explained. Additionally, the connection between entanglement and complexity of motion was discussed. In particular, it was shown that, in the asymptotic limit (Appendix B),

$$\left[ \exp \left( \mathcal{S}_{\mathcal{M}_s}(\tau) \right) \right]_{\text{correlated}} \stackrel{\tau \rightarrow \infty}{\sim} \mathcal{F}(\rho) \cdot \left[ \exp \left( \mathcal{S}_{\mathcal{M}_s}(\tau) \right) \right]_{\text{uncorrelated}}, \quad (13)$$

where  $\mathcal{F}(\rho)$  in (13) with  $0 \leq \mathcal{F}(\rho) \leq 1$  is defined as

$$\mathcal{F}(\rho) \stackrel{\text{def}}{=} \sqrt{\frac{1-\rho}{1+\rho}}. \quad (14)$$

The function  $\mathcal{F}(\rho)$  is a monotonic decreasing compression factor for any value of the correlation coefficient  $\rho \in (0, 1)$ . The work presented in [56, 57] represents significant progress toward the goal of understanding the relationship between entanglement and statistical microcorrelations on the one hand and the effect of microcorrelations on the complexity of informational geodesic paths on the other. Finally, due to the consequences arising from (13), the IGAC framework was proposed as a potentially suitable platform to establish a sound information geometric interpretation of quantum entanglement, including its connection to complexity of motion in general physical scenarios.

**2.7. Example 7: Softening of Classical Chaos by Quantization.** Comparing classical and quantum chaoticity (i.e., temporal complexity) and explaining the reason why the former is

stronger than the latter are of great theoretical interest [58–60]. It is usually conjectured that the weakness of quantum chaos may be due to the Heisenberg uncertainty relation. Following the preliminary investigations undertaken in [61–63], the IGAC was used to study both the information geometry and the entropic dynamics of a three-dimensional Gaussian statistical model and a two-dimensional Gaussian statistical model in which the latter is obtained from the former via introduction of a suitable macroscopic information constraint:

$$\sigma_x \sigma_y = \Sigma^2, \quad (15)$$

where  $\Sigma^2 \in \mathbb{R}_0^+$ . The quantities  $x$  and  $y$  denote the microscopic degrees of freedom of the system whose probabilistic description is being investigated. The relation (15) resembles the canonical quantum mechanical minimum uncertainty relation [64]. It was found that the complexity of the two-dimensional Gaussian statistical model, quantified in terms of the IGE, is softened with respect to the complexity of the three-dimensional Gaussian statistical model (Appendix B):

$$\mathcal{S}_{\mathcal{M}_s}^{(2D)}(\tau) \stackrel{\tau \rightarrow \infty}{\sim} \left( \frac{\lambda_{2D}}{\lambda_{3D}} \right) \cdot \mathcal{S}_{\mathcal{M}_s}^{(3D)}(\tau), \quad (16)$$

where  $\lambda_{2D}$  and  $\lambda_{3D}$  are two positive model parameters (satisfying  $\lambda_{2D} \leq \lambda_{3D}$ ) that specify the asymptotic temporal rates of change of the IGE in the two-dimensional and three-dimensional scenarios, respectively. In view of the similarity between the selected macroscopic information constraint on the variances and the phase-space coarse-graining imposed by the Heisenberg uncertainty relations, the authors argued that their work may provide a possible avenue to explain the phenomenon of classical chaos suppression under the operation of quantization within a novel information geometric perspective. We remark that similar investigations were carried out in [65] where the analysis presented in [66] was extended to the case in which, in addition to the macroscopic constraint in (15), the microscopic degrees of freedom  $x$  and  $y$  of the system are also correlated.

**2.8. Example 8: Topologically Distinct Correlational Structures.** In [67], the asymptotic behavior of the IGE (Appendix B) for bivariate and trivariate Gaussian statistical models in both the absence and presence of microcorrelations was investigated. For correlated cases, various correlational structures among the microscopic degrees of freedom of the system were considered. It was determined that the complexity of macroscopic inferences depends not only upon the amount of available microscopic information but also on the manner in which such microscopic information is correlated. In particular, for a trivariate statistical model with two correlations among the three degrees of freedom of the system (referred to as the mildly connected case), it was found that, in the asymptotic limit,

$$\begin{aligned} & \left( \exp \left[ \mathcal{S}_{\text{trivariate}}^{(\text{mildly connected})}(\tau) \right] \right)_{\text{correlated}} \\ & \stackrel{\tau \rightarrow \infty}{\sim} r_{\text{trivariate}}^{(\text{mildly connected})}(\rho) \\ & \cdot \left( \exp \left[ \mathcal{S}_{\text{trivariate}}^{(\text{mildly connected})}(\tau) \right] \right)_{\text{uncorrelated}}, \end{aligned} \quad (17)$$

where

$$r_{\text{trivariate}}^{(\text{mildly connected})}(\rho) = \sqrt{\frac{3(1-2\rho^2)}{3-4\rho}}. \quad (18)$$

The function  $r_{\text{trivariate}}^{(\text{mildly connected})}(\rho)$  exhibits nonmonotonic behavior in the correlation coefficient  $\rho$  and assumes a value equal to zero at the extrema of the allowed range  $\rho \in (-\sqrt{2}/2, \sqrt{2}/2)$ . On the other hand, for closed configurations (i.e., bivariate and trivariate models with all microscopic variables correlated with each other) the complexity ratio between correlated and uncorrelated cases exhibits monotonic behavior in the correlation coefficient  $\rho$ . For instance, in the fully connected bivariate Gaussian case, it was determined that

$$\begin{aligned} & \left( \exp \left[ \mathcal{S}_{\text{bivariate}}^{(\text{fully connected})}(\tau) \right] \right)_{\text{correlated}} \\ & \stackrel{\tau \rightarrow \infty}{\sim} r_{\text{bivariate}}^{(\text{fully connected})}(\rho) \\ & \cdot \left( \exp \left[ \mathcal{S}_{\text{bivariate}}^{(\text{fully connected})}(\tau) \right] \right)_{\text{uncorrelated}}, \end{aligned} \quad (19)$$

where

$$r_{\text{bivariate}}^{(\text{fully connected})}(\rho) = \sqrt{1+\rho}. \quad (20)$$

On the other hand, in the fully connected trivariate Gaussian case, it can be shown that

$$\begin{aligned} & \left( \exp \left[ \mathcal{S}_{\text{trivariate}}^{(\text{fully connected})}(\tau) \right] \right)_{\text{correlated}} \\ & \stackrel{\tau \rightarrow \infty}{\sim} r_{\text{trivariate}}^{(\text{fully connected})}(\rho) \\ & \cdot \left( \exp \left[ \mathcal{S}_{\text{trivariate}}^{(\text{fully connected})}(\tau) \right] \right)_{\text{uncorrelated}}, \end{aligned} \quad (21)$$

where

$$r_{\text{trivariate}}^{(\text{fully connected})}(\rho) = \sqrt{1+2\rho}. \quad (22)$$

It is evident that, in the fully connected bivariate and trivariate cases, the ratios  $r_{\text{bivariate}}^{(\text{fully connected})}(\rho)$  and  $r_{\text{trivariate}}^{(\text{fully connected})}(\rho)$  both exhibit monotonic behavior in  $\rho$  over the open intervals  $(-1, 1)$  and  $(-1/2, 1)$ , respectively. By contrast, in the mildly connected trivariate case depicted in (17), a peak in the function  $r_{\text{trivariate}}^{(\text{mildly connected})}(\rho)$  is observed at  $\rho_{\text{peak}} = 0.5 \geq 0$ .

We recall that, in an antiferromagnetic triangular Ising model with coupling between neighboring spins equal to  $J = -1$ , any three neighboring spins are frustrated [68, 69]. The frustration arises from the inability of the spin system to find an energetically favorable ordered state. For the sake of reasoning, assume that one spin is in the +1 state. Then, it is energetically favorable for its immediate neighbors to be in the opposite state. However, because of the geometry and/or interactions between the spins, it is impossible to find an energetically optimal configuration in the case of an antiferromagnetic triangular Ising model. At best, one can only have two out of three favorable couplings. Furthermore, when the system is frustrated, the absence of an ordered state

can be described in terms of the absence of a peak in both the standard deviation of the energy and the heat capacity of the system as a function of its temperature. Instead, a peak in such thermodynamical quantities is present when considering a ferromagnetic triangular Ising model with  $J = +1$ . Within the IGAC, one would desire a configuration of minimum complexity in order to make reliable macroscopic predictions. This requirement is the analogue of the ideal scenario of minimum energy spin configurations in statistical physics [68–70]. Our results in (17) and (21) exhibit a dramatically distinct behavior between the mildly connected and the fully connected trivariate Gaussian configurations. This behavior is due to the fact that when carrying out statistical inferences with positively correlated Gaussian random variables, the system seems to be frustrated in the fully connected case. This happens because the maximum entropy favorable scenario appears to be incompatible with the ideal scenario of minimum complexity. Certain lattice configurations in the presence of correlations are not especially favorable from a statistical inference perspective of minimum complexity, just like certain spin configurations are not particularly favorable from an energy standpoint.

Based on these findings, it was argued in [67] that the impossibility of attaining the most favorable configuration for certain correlational structures among microscopic degrees of freedom (from an entropic inference viewpoint) leads to an information geometric analogue of the frustration effect that occurs in statistical physics in the presence of loops [70].

### 3. Conclusions

In this manuscript, we presented a brief survey of the main results uncovered by the authors and collaborators within the framework of the IGAC over the past decade. As pointed out in the Introductory Background, for the sake of brevity, we have omitted mathematical details in our main presentation. However, for the sake of self-consistency, we have added a number of Appendices covering the basic mathematical details necessary to critically follow the content of the manuscript. For an extended review with more mathematical details and physical interpretations on the IGAC, we refer to the recent work appearing in [71].

We provided here several illustrative examples of entropic dynamical models employed to infer macroscopic predictions when only limited information of the microscopic nature of a system is available. In the first example, we considered the IGAC applied to a high-dimensional Gaussian statistical model. In particular, we reported the scaling of the scalar curvature with the microscopic degrees of freedom of the system in (1), the asymptotic temporal linear growth of the IGE in (2), and, finally, the asymptotic exponential growth of the Jacobi vector field intensity on such a curved manifold in (3). In the second example, we studied the IGAC of a low-dimensional correlated Gaussian statistical model and showed that, compared to the uncorrelated scenario where correlations among microscopic degrees of freedom are absent, the IGE decreases. This decrease is quantified in terms of a monotonic decreasing compression factor that depends on the correlation coefficient. This finding appears

in (4). In the third example, we investigated the IGAC of a set of uncoupled and anisotropic inverted harmonic oscillators. In particular, we demonstrated the asymptotic temporal growth of the IGE with proportionality constant given by the sum of all the frequencies of the oscillators. This finding is reported in (6). In the fourth example, we analyzed the IGAC of integrable and chaotic quantum spin chains. Specifically, we found that the IGE exhibits asymptotic temporal logarithmic and linear growth, respectively. The former and latter results appear in (10) and (11), respectively. In the fifth example, we studied the information geometric complexity reduction in the presence of a statistical embedding of a lower-dimensional statistical manifold in a higher-dimensional one. The observed power law decay of the IGE in terms of the correlation coefficients that specify the correlational structure that characterizes the statistical embedding is reported in (12). In the sixth illustrative example, we investigated the IGAC applied to a scattering process between two Gaussian wave packets where quantum entanglement is generated. Conjecturing that the pre- and postcollisional quantum dynamical scenarios related to an elastic head-on collision are macroscopic manifestations emerging from microscopic statistical structures, the IGAC allows linking the behavior of the complexity of motion to the presence of entanglement. Equation (13) illustrates this statement. In the seventh example, we compared the IGAC applied to a three-dimensional Gaussian statistical model with that of a two-dimensional Gaussian model obtained from the former model upon exploitation of a suitable macroscopic information constraint (see (15)) that resembles the Heisenberg uncertainty relation. In particular, we found that the IGE of the lower-dimensional statistical model is softened with respect to the higher-dimensional one. This finding appears in (16). Finally, in the eighth example, we analyzed the information geometric complexity behavior of topologically distinct statistical correlational structures for underlying curved statistical manifolds of different dimensionality. The outcomes of this particular investigation are presented in (17), (19), and (21).

We are aware of several unresolved issues within the IGAC. In what follows, we outline in a more systematic fashion a number of strengths and weaknesses of the IGAC theoretical scheme.

*Strengths.* IGAC is characterized by a number of very convenient features:

- (i) No arbitrariness or lack of explanation of how macrostates of a system leading to the formation of geodesic paths on the curved statistical manifold is present. Within the IGAC, the transition (that is, the updating) from an initial macrostate to a final macrostate occurs by navigating through a continuous sequence of intermediate macrostates chosen by maximizing the relative entropy between any two consecutive intermediate macrostates subject to the available information constraints.
- (ii) All the dynamical information is collected into a single geometric quantity where all the available symmetries are retained: the curved statistical manifold

on which the geodesic flow is induced. For instance, the sensitive dependence of trajectories on initial conditions can be analyzed from the geodesic deviation equation. Furthermore, the nonintegrability (chaoticity) of the system can be studied by investigating the existence (absence) of Killing tensors on the curved manifold.

- (iii) IGAC offers a unifying theoretical setting wherein both curvature and entropic indicators of complexity are available.
- (iv) IGAC represents a convenient platform for enhancing our comprehension of the role played by statistical curvature in modeling realistic processes by linking it to conventionally accepted quantities, including entropy.
- (v) From a more foundational perspective, provided that the true degrees of freedom of the system are identified, IGAC presents a serious opportunity to uncover deep insights into the foundations of modeling and inductive reasoning together with the relationship to each other.

*Weaknesses.* Despite its strengths, the current version of the IGAC needs to be improved since it exhibits several weak points, including the following:

- (i) IGE lacks a detailed comparison with other entropic complexity indicators of geometric flavor.
- (ii) Despite the interpretational power of the Riemannian geometrization of dynamics, the integration of geodesic equations together with computations involving curvatures and Jacobi fields can become quite challenging, especially for higher-dimensional statistical manifolds lacking any particular degree of symmetry.
- (iii) There is no fully developed quantum mechanical IGAC framework suitable for characterizing the complexity of quantum evolution.
- (iv) IGAC lacks experimental evidence in support of theoretical macroscopic predictions advanced within its setting.
- (v) General results with a wide range of applicability are absent. Most macroscopic predictions are limited to specific classes of physical systems in the presence of very peculiar functional forms of relevant available information.

Despite these weaknesses, we are truly gratified that the IGAC is gradually gaining attention within the scientific community. Indeed, there seems to be an increasing number of scientists who either actively make use of, or whose work is related to, the IGAC [72–94].

In conclusion, we emphasize that it was not our intention to report in this manuscript all the available scientific investigations on complexity based upon the information geometric approach. Instead, we limited our presentation to the findings uncovered within the framework of the so-called IGAC

theoretical framework. For an overview of various methods of information geometry used to quantify the complexity of physical systems in both classical and quantum settings, we refer to the recent review article in [72] and to the works cited therein.

## Appendix

### A. Curvature

In this appendix, we review some basic mathematical details on the concept of curvature. Recall that an  $n$ -dimensional  $C^\infty$  differentiable manifold is a set of points  $\mathcal{M}$  that is endowed with coordinate systems  $\mathcal{C}_\mathcal{M}$  and fulfills the following two conditions: (1) each element  $c \in \mathcal{C}_\mathcal{M}$  is a one-to-one mapping from  $\mathcal{M}$  to some open subset of  $\mathbb{R}^n$ ; (2) for all  $c \in \mathcal{C}_\mathcal{M}$ , given any one-to-one mapping  $\eta$  from  $\mathcal{M}$  to  $\mathbb{R}^n$ , we find that  $\eta \in \mathcal{C}_\mathcal{M} \iff \eta \circ c^{-1}$  is a  $C^\infty$  diffeomorphism.

In this manuscript, the points of  $\mathcal{M}$  are probability distributions. Moreover, we take into consideration Riemannian manifolds  $(\mathcal{M}, g)$ . The structure of  $\mathcal{M}$  as a manifold does not naturally determine the Riemannian metric  $g$ . Formally, an infinite number of Riemannian metrics on  $\mathcal{M}$  can be considered. A key working assumption in the information geometry framework is the choice of the Fisher-Rao information metric as the metric that underlies the Riemannian geometry of probability distributions [7, 95, 96],

$$g_{\mu\nu}(\theta) \stackrel{\text{def}}{=} \int p(x|\theta) \partial_\mu \log p(x|\theta) \partial_\nu \log p(x|\theta) dx, \quad (\text{A.1})$$

with  $\mu, \nu = 1, \dots, n$  for an  $n$ -dimensional manifold and  $\partial_\mu \stackrel{\text{def}}{=} \partial/\partial\theta^\mu$ . The quantity  $x$  in (A.1) labels the microstates of the system. The most compelling support of the choice of the information metric comes from Cencov's characterization theorem [97]. In this theorem, Cencov proves that the information metric is the only Riemannian metric, up to any arbitrary constant scale factor, that remains invariant under a family of probabilistically meaningful mappings (named congruent embeddings) by Markov morphism [97, 98].

Once the Fisher-Rao information metric  $g_{\mu\nu}(\theta)$  in (A.1) has been introduced, we can use standard differential geometry methods applied to the space of probability distributions to characterize the geometric properties of a curved statistical manifold  $\mathcal{M}_s$ . For instance, the Ricci scalar curvature  $\mathcal{R}_{\mathcal{M}_s}$  is given by [99]

$$\mathcal{R}_{\mathcal{M}_s} \stackrel{\text{def}}{=} g^{\mu\nu} \mathcal{R}_{\mu\nu}, \quad (\text{A.2})$$

where  $g^{\mu\nu} g_{\nu\rho} = \delta_\rho^\mu$  so that  $g^{\mu\nu} = (g_{\mu\nu})^{-1}$ . The Ricci tensor  $\mathcal{R}_{\mu\nu}$  in (A.2) is defined as [99]

$$\mathcal{R}_{\mu\nu} \stackrel{\text{def}}{=} \partial_\gamma \Gamma_{\mu\nu}^\gamma - \partial_\nu \Gamma_{\mu\lambda}^\lambda + \Gamma_{\mu\nu}^\gamma \Gamma_{\gamma\eta}^\eta - \Gamma_{\mu\gamma}^\eta \Gamma_{\nu\eta}^\gamma. \quad (\text{A.3})$$

The Christoffel connection coefficients  $\Gamma_{\mu\nu}^\rho$  that appear in the Ricci tensor in (A.3) are defined in the standard manner as [99]

$$\Gamma_{\mu\nu}^\rho \stackrel{\text{def}}{=} \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \quad (\text{A.4})$$

We remark at this point that a geodesic on a  $n$ -dimensional curved statistical manifold  $\mathcal{M}_s$  represents the maximum probability path a complex dynamical system explores in its evolution between initial and final macrostates  $\theta_{\text{initial}}$  and  $\theta_{\text{final}}$ , respectively. Each point of the geodesic represents a macrostate parameterized by the macroscopic dynamical variables  $\theta = (\theta^1, \dots, \theta^n)$  defining the macrostate of the system. In the framework of ED, each component  $\theta^j$  with  $j = 1, \dots, n$  is a solution of the geodesic equation [11]

$$\frac{d^2 \theta^k}{d\tau^2} + \Gamma_{lm}^k \frac{d\theta^l}{d\tau} \frac{d\theta^m}{d\tau} = 0. \quad (\text{A.5})$$

Furthermore, as stated earlier, each macrostate  $\theta$  is in a one-to-one correspondence with the probability distribution  $p(x | \theta)$ . This is a distribution of the microstates  $x$ . It is also convenient to observe that the scalar curvature  $\mathcal{R}_{\mathcal{M}_s}$  can be expressed as the sum of all sectional curvatures  $\mathcal{K}(e_\rho, e_\sigma)$  of planes spanned by pairs of orthonormal basis elements  $\{e_\rho = \partial_{\theta_\rho(p)}\}$  of the tangent space  $T_p \mathcal{M}_s$  with  $p \in \mathcal{M}_s$ :

$$\mathcal{R}_{\mathcal{M}_s} \stackrel{\text{def}}{=} \mathcal{R}_\alpha^\alpha \stackrel{\text{def}}{=} \sum_{\rho \neq \sigma} \mathcal{K}(e_\rho, e_\sigma), \quad (\text{A.6})$$

where  $\mathcal{K}(a, b)$  is defined as [99]

$$\mathcal{K}(a, b) \stackrel{\text{def}}{=} \frac{\mathcal{R}_{\mu\nu\rho\sigma} a^\mu b^\nu a^\rho b^\sigma}{(g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma}) a^\mu b^\nu a^\rho b^\sigma}, \quad (\text{A.7})$$

with

$$\begin{aligned} a &\stackrel{\text{def}}{=} \sum_\rho \langle a, e^\rho \rangle e_\rho, \\ b &\stackrel{\text{def}}{=} \sum_\rho \langle b, e^\rho \rangle e_\rho, \end{aligned} \quad (\text{A.8})$$

$$\langle e_\rho, e^\sigma \rangle \stackrel{\text{def}}{=} \delta_\rho^\sigma.$$

Note that the sectional curvatures  $\mathcal{K}(e_\rho, e_\sigma)$  completely determine the Riemann curvature tensor  $\mathcal{R}_{\alpha\beta\rho\sigma}$  where [99]

$$\begin{aligned} \mathcal{R}_{\beta\rho\sigma}^\alpha &\stackrel{\text{def}}{=} g^{\alpha\gamma} \mathcal{R}_{\gamma\beta\rho\sigma} \\ &\stackrel{\text{def}}{=} \partial_\sigma \Gamma_{\beta\rho}^\alpha - \partial_\rho \Gamma_{\beta\sigma}^\alpha + \Gamma_{\lambda\sigma}^\alpha \Gamma_{\beta\rho}^\lambda - \Gamma_{\lambda\rho}^\alpha \Gamma_{\beta\sigma}^\lambda. \end{aligned} \quad (\text{A.9})$$

We point out that the negativity of the Ricci curvature  $\mathcal{R}_{\mathcal{M}_s}$  is a strong (that is, sufficient but not necessary) criterion of dynamical instability and that the compactness of the manifold  $\mathcal{M}_s$  is necessary in order to handle true chaotic (i.e., temporally complex) dynamical systems. Specifically, we observe that from (A.6) the negativity of  $\mathcal{R}_{\mathcal{M}_s}$  implies

that negative principal curvatures (that is, extrema of sectional curvatures) dominate over positive ones. Therefore, the negativity of the Ricci scalar is only a sufficient but not necessary condition for local instability of geodesic flows on curved statistical manifolds. Given these observations, we reach the conclusion that the negativity of the Ricci scalar curvature provides a strong criterion of local instability. We also point out the possible occurrence of scenarios where negative sectional curvatures are present, but the positive ones prevail in the sum in (A.6) so that  $\mathcal{R}_{\mathcal{M}_s}$  is nonnegative despite the instability in the flow in those directions. In summary, to properly characterize the chaoticity (i.e., temporal complexity) of complex dynamical systems, the signs of the sectional curvatures are of primary importance. For further mathematical details on the notion of curvature in differential geometry, we refer to [100].

## B. Information Geometric Entropy

In this appendix, we present the concept of the IGE within the IGAC theoretical setting. Assume that the points  $\{p(x; \theta)\}$  of an  $n$ -dimensional curved statistical manifold  $\mathcal{M}_s$  are parameterized by means of  $n$  real valued variables  $(\theta^1, \dots, \theta^n)$ ,

$$\mathcal{M}_s \stackrel{\text{def}}{=} \{p(x; \theta) : \theta = (\theta^1, \dots, \theta^n) \in \mathcal{D}_\theta^{(\text{tot})}\}. \quad (\text{B.1})$$

We remark that the microvariables  $x$  in (B.1) belong to the microspace  $\mathcal{X}$  while the macrovariables  $\theta$  in (B.1) are elements of the parameter space  $\mathcal{D}_\theta^{(\text{tot})}$  given by

$$\mathcal{D}_\theta^{(\text{tot})} \stackrel{\text{def}}{=} \bigotimes_{j=1}^n \mathcal{J}_{\theta^j} = (\mathcal{J}_{\theta^1} \otimes \mathcal{J}_{\theta^2} \cdots \otimes \mathcal{J}_{\theta^n}) \subseteq \mathbb{R}^n. \quad (\text{B.2})$$

The quantity  $\mathcal{J}_{\theta^j}$  with  $1 \leq j \leq n$  in (B.2) is a subset of  $\mathbb{R}^n$  and denotes the entire range of allowable values for the statistical macrovariables  $\theta^j$ . The information geometric entropy is a proposed measure of temporal complexity of geodesic paths within the IGAC. The IGE is defined as

$$\mathcal{S}_{\mathcal{M}_s}(\tau) \stackrel{\text{def}}{=} \log \widetilde{\text{vol}}[\mathcal{D}_\theta(\tau)], \quad (\text{B.3})$$

where the average dynamical statistical volume  $\widetilde{\text{vol}}[\mathcal{D}_\theta(\tau)]$  is given by

$$\widetilde{\text{vol}}[\mathcal{D}_\theta(\tau)] \stackrel{\text{def}}{=} \frac{1}{\tau} \int_0^\tau \text{vol}[\mathcal{D}_\theta(\tau')] d\tau'. \quad (\text{B.4})$$

Observe that the operation of temporal average is denoted with the tilde symbol in (B.4). Moreover, the volume  $\text{vol}[\mathcal{D}_\theta(\tau')]$  in the RHS of (B.4) is defined as

$$\text{vol}[\mathcal{D}_\theta(\tau')] \stackrel{\text{def}}{=} \int_{\mathcal{D}_\theta(\tau')} \rho(\theta^1, \dots, \theta^n) d^n \theta, \quad (\text{B.5})$$

where  $\rho(\theta^1, \dots, \theta^n)$  is the so-called Fisher density and equals the square root of the determinant  $g(\theta)$  of the Fisher-Rao information metric tensor  $g_{\mu\nu}(\theta)$ :

$$\rho(\theta^1, \dots, \theta^n) \stackrel{\text{def}}{=} \sqrt{g(\theta)}. \quad (\text{B.6})$$

We point out that the expression of  $\text{vol}[\mathcal{D}_\theta(\tau')]$  in (B.5) can become more transparent for statistical manifolds with information metric tensor whose determinant can be factorized in the following manner:

$$g(\theta) = g(\theta^1, \dots, \theta^n) = \prod_{j=1}^n g_j(\theta^j). \quad (\text{B.7})$$

With the aid of the factorized determinant, the IGE in (B.3) can be recast as

$$\mathcal{S}_{\mathcal{M}_s}(\tau) = \log \left\{ \frac{1}{\tau} \int_0^\tau \left[ \prod_{j=1}^n \left( \int_{\tau_0}^{\tau_0+\tau'} \sqrt{g_j[\theta^j(\xi)]} \frac{d\theta^j}{d\xi} d\xi \right) \right] d\tau' \right\}. \quad (\text{B.8})$$

We also emphasize that, within the IGAC, the leading asymptotic behavior of  $\mathcal{S}_{\mathcal{M}_s}(\tau)$  is employed to quantify the complexity of the statistical models being investigated. For this reason, it is customary to consider the quantity

$$\mathcal{S}_{\mathcal{M}_s}^{(\text{asymptotic})}(\tau) \sim \lim_{\tau \rightarrow \infty} [\mathcal{S}_{\mathcal{M}_s}(\tau)], \quad (\text{B.9})$$

that is to say, the leading asymptotic term in the IGE expression. The integration space  $\mathcal{D}_\theta(\tau')$  in (B.5) is defined by

$$\mathcal{D}_\theta(\tau') \stackrel{\text{def}}{=} \{\theta : \theta^j(\tau_0) \leq \theta^j \leq \theta^j(\tau_0 + \tau')\}, \quad (\text{B.10})$$

where  $\theta^j = \theta^j(\xi)$  with  $\tau_0 \leq \xi \leq \tau_0 + \tau'$  and  $\tau_0$  denoting the initial value of the affine parameter  $\xi$  such that

$$\frac{d^2\theta^j(\xi)}{d\xi^2} + \Gamma_{ik}^j \frac{d\theta^i}{d\xi} \frac{d\theta^k}{d\xi} = 0. \quad (\text{B.11})$$

The integration domain  $\mathcal{D}_\theta(\tau')$  in (B.10) is an  $n$ -dimensional subspace of  $\mathcal{D}_\theta^{(\text{tot})}$  whose elements are  $n$ -dimensional macrovariables  $\{\theta\}$  with components  $\theta^j$  bounded by given limits of integration  $\theta^j(\tau_0)$  and  $\theta^j(\tau_0 + \tau')$ . The integration of the  $n$ -coupled nonlinear second-order ordinary differential equations in (B.11) specifies the temporal functional form of such limits.

The IGE at a certain instant is essentially the logarithm of the volume of the effective parameter space explored by the system at that very instant. In order to average out the possibly highly complex fine details of the entropic dynamical description of the system on the curved statistical manifold, the temporal average has been taken into consideration. Furthermore, to eliminate the consequences of transient effects which enter the computation of the expected value of the volume of the effective parameter space, the long-term asymptotic temporal behavior is considered to conveniently describe the selected dynamical complexity indicator. In summary, the IGE is constructed to furnish an asymptotic coarse-grained inferential description of the complex dynamics of a system in the presence of only partial knowledge. For further details on the IGE, we refer to [29, 71].

## C. Jacobi Fields

In this appendix, we review some basic mathematical details on the concept of Jacobi vector fields. The investigation of the instability of natural motions by way of the instability of geodesics on a suitable curved manifold is especially advantageous within the Riemannian geometrization of dynamics. In particular, the so-called Jacobi-Levi-Civita (JLC) equation for geodesic spread is a very powerful mathematical tool employed to study the stability/instability of a geodesic flow. This equation is a familiar quantity both in theoretical physics (in General Relativity, for instance) and in Riemannian geometry. The JLC equation covariantly describes how neighboring geodesics locally scatter. More specifically, the JLC equation connects curvature properties of the ambient manifold to the stability/instability of a geodesic flow. It paves the way to a wide and largely unexplored field of investigation that concerns the links among geometry, topology, and geodesic instability and therefore to chaoticity and complexity. To the best of our knowledge, the use of the JLC equation in the framework of information geometry appeared originally in [30].

Let us consider two neighboring geodesic paths  $\theta^\alpha(\tau)$  and  $\theta^\alpha(\tau) + \delta\theta^\alpha(\tau)$ , with  $\tau$  denoting the affine parameter, that satisfy the following geodesic equations of motion:

$$\frac{d^2\theta^\alpha}{d\tau^2} + \Gamma_{\beta\gamma}^\alpha(\theta) \frac{d\theta^\beta}{d\tau} \frac{d\theta^\gamma}{d\tau} = 0, \quad (\text{C.1})$$

and

$$\begin{aligned} & \frac{d^2[\theta^\alpha + \delta\theta^\alpha]}{d\tau^2} \\ & + \Gamma_{\beta\gamma}^\alpha(\theta + \delta\theta) \frac{d[\theta^\beta + \delta\theta^\beta]}{d\tau} \frac{d[\theta^\gamma + \delta\theta^\gamma]}{d\tau} = 0, \end{aligned} \quad (\text{C.2})$$

respectively. Observing that, to first order in  $\delta\theta^\alpha$ ,

$$\Gamma_{\beta\gamma}^\alpha(\theta + \delta\theta) \approx \Gamma_{\beta\gamma}^\alpha(\theta) + \partial_\xi \Gamma_{\beta\gamma}^\alpha \delta\theta^\xi, \quad (\text{C.3})$$

And, after some algebra, to first order in  $\delta\theta^\alpha$  (C.2) becomes

$$\begin{aligned} & \frac{d^2\theta^\alpha}{d\tau^2} + \frac{d^2(\delta\theta^\alpha)}{d\tau^2} + \Gamma_{\beta\gamma}^\alpha(\theta) \frac{d\theta^\beta}{d\tau} \frac{d\theta^\gamma}{d\tau} \\ & + 2\Gamma_{\beta\gamma}^\alpha(\theta) \frac{d\theta^\beta}{d\tau} \frac{d(\delta\theta^\gamma)}{d\tau} \\ & + \partial_\xi \Gamma_{\beta\gamma}^\alpha(\theta) \delta\theta^\xi \frac{d\theta^\beta}{d\tau} \frac{d\theta^\gamma}{d\tau} = 0. \end{aligned} \quad (\text{C.4})$$

The geodesic deviation equation can be obtained by subtracting (C.1) from (C.4):

$$\begin{aligned} & \frac{d^2(\delta\theta^\alpha)}{d\tau^2} + 2\Gamma_{\beta\gamma}^\alpha(\theta) \frac{d\theta^\beta}{d\tau} \frac{d(\delta\theta^\gamma)}{d\tau} \\ & + \partial_\xi \Gamma_{\beta\gamma}^\alpha(\theta) \delta\theta^\xi \frac{d\theta^\beta}{d\tau} \frac{d\theta^\gamma}{d\tau} = 0. \end{aligned} \quad (\text{C.5})$$

Equation (C.5) can be rewritten in a more convenient form in terms of covariant derivatives (see [101], for instance) along the curve  $\theta^\alpha(\tau)$ ,

$$\begin{aligned} \frac{D^2(\delta\theta^\alpha)}{D\tau^2} &= \frac{d^2(\delta\theta^\alpha)}{d\tau^2} + \partial_\beta \Gamma_{\rho\sigma}^\alpha \frac{d\theta^\beta}{d\tau} \delta\theta^\rho \frac{d\theta^\sigma}{d\tau} \\ &+ 2\Gamma_{\rho\sigma}^\alpha \frac{d(\delta\theta^\rho)}{d\tau} \frac{d\theta^\sigma}{d\tau} \\ &+ -\Gamma_{\rho\sigma}^\alpha \Gamma_{\kappa\lambda}^\sigma \delta\theta^\rho \frac{d\theta^\kappa}{d\tau} \frac{d\theta^\lambda}{d\tau} \\ &+ \Gamma_{\rho\sigma}^\alpha \Gamma_{\kappa\lambda}^\rho \delta\theta^\kappa \frac{d\theta^\lambda}{d\tau} \frac{d\theta^\sigma}{d\tau}. \end{aligned} \quad (\text{C.6})$$

Combining (C.5) and (C.6), after some tensor algebra manipulations, we obtain

$$\begin{aligned} \frac{D^2(\delta\theta^\alpha)}{D\tau^2} \\ = (\partial_\rho \Gamma_{\eta\sigma}^\alpha - \partial_\eta \Gamma_{\rho\sigma}^\alpha + \Gamma_{\lambda\sigma}^\alpha \Gamma_{\eta\rho}^\lambda - \Gamma_{\eta\lambda}^\alpha \Gamma_{\rho\sigma}^\lambda) \delta\theta^\eta \frac{d\theta^\rho}{d\tau} \frac{d\theta^\sigma}{d\tau}. \end{aligned} \quad (\text{C.7})$$

Finally, observing that the Riemannian curvature tensor components  $\mathcal{R}_{\rho\eta\sigma}^\alpha$  are given by

$$\mathcal{R}_{\rho\eta\sigma}^\alpha \stackrel{\text{def}}{=} \partial_\eta \Gamma_{\rho\sigma}^\alpha - \partial_\rho \Gamma_{\eta\sigma}^\alpha + \Gamma_{\eta\lambda}^\alpha \Gamma_{\rho\sigma}^\lambda - \Gamma_{\lambda\sigma}^\alpha \Gamma_{\eta\rho}^\lambda, \quad (\text{C.8})$$

the component form of the geodesic deviation equation becomes

$$\frac{D^2 J^\alpha}{D\tau^2} + \mathcal{R}_{\rho\eta\sigma}^\alpha \frac{d\theta^\rho}{d\tau} J^\eta \frac{d\theta^\sigma}{d\tau} = 0, \quad (\text{C.9})$$

where  $J^\alpha \stackrel{\text{def}}{=} \delta\theta^\alpha$  denotes the  $\alpha$ -component of the so-called Jacobi vector field [99]. Equation (C.9) is the so-called JLC equation. From the JLC equation in (C.9), we note that neighboring geodesics accelerate relative to each other with a rate directly measured by the Riemannian curvature tensor  $R_{\alpha\beta\gamma\delta}$ . The quantity  $J^\alpha$  is given by

$$J^\alpha = \delta\theta^\alpha \stackrel{\text{def}}{=} \delta_\phi \theta^\alpha = \left( \frac{\partial\theta^\alpha(\tau; \phi)}{\partial\phi} \right)_{\tau=\text{constant}} \delta\phi, \quad (\text{C.10})$$

with  $\{\theta^\mu(\tau; \phi)\}$  denoting the one-parameter  $\phi$  family of geodesics whose evolution is described in terms of the affine parameter  $\tau$ . The Jacobi vector field intensity  $J(\tau)$  is defined as

$$J(\tau) \stackrel{\text{def}}{=} \left( J^\alpha g_{\alpha\beta} J^\beta \right)^{1/2}. \quad (\text{C.11})$$

Observe that (C.9) yields a system of coupled ordinary differential equations *linear* in the components of the deviation vector field but *nonlinear* in derivatives of the metric. We remark that although the JLC equation already seems intractable at rather small dimensions, in the case of isotropic manifolds it reduces to the following simplified form:

$$\frac{D^2 J^\mu}{D\tau^2} + \mathcal{K} J^\mu = 0, \quad (\text{C.12})$$

where  $\mathcal{K}$  is the constant value assumed throughout the manifold by the sectional curvature. In particular, when  $\mathcal{K} < 0$ , unstable solutions of (C.12) are of the form

$$J^\mu(\tau) = \frac{\omega_0^\mu}{\sqrt{-\mathcal{K}}} \sinh(\sqrt{-\mathcal{K}}\tau), \quad (\text{C.13})$$

assuming that the initial conditions are given by  $J^\mu(0) = 0$  and  $dJ^\mu(0)/d\tau = \omega_0^\mu(0) = \omega_0^\mu \neq 0$ , respectively, for any  $1 \leq \mu \leq n$  with  $n$  denoting the dimensionality of the underlying curved manifold. For further details on the JLC equation, we refer to [99, 101, 102].

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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