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Research Article

Lump Solutions to a (2+1)-Dimensional Fifth-Order KdV-Like Equation

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A (2+1)-dimensional fifth-order KdV-like equation is introduced through a generalized bilinear equation with the prime number $p = 5$. The new equation possesses the same bilinear form as the standard (2+1)-dimensional fifth-order KdV equation. By Maple symbolic computation, classes of lump solutions are constructed from a search for quadratic function solutions to the corresponding generalized bilinear equation. We get a set of free parameters in the resulting lump solutions, of which we can get a nonzero determinant condition ensuring analyticity and rational localization of the solutions. Particular classes of lump solutions with special choices of the free parameters are generated and plotted as illustrative examples.

1. Introduction

The (2+1)-dimensional fifth-order KdV equation [1] is

$$\begin{aligned}36u_t + u_{5x} + 15u_xu_{xx} + 15uu_{3x} + 45u^2u_x - 5u_{xxy} \\ - 15uu_y - 15u_x\int u_y dx - 5\int u_{yy}dx = 0,\end{aligned} \quad (1)$$

which is the (2+1)-dimensional analogue of the Caudrey-Dodd-Gibbon-Kotera-Sawada (CDGKS) equation [2]. When $u_y = 0$, (1) reduces to the Sawada-Kotera equation

$$u_t + u_{5x} + 15u_xu_{xx} + 15uu_{3x} + 45u^2u_x = 0. \quad (2)$$

Konopelchenko and Dubovsky [3] were the first to come up with (1). Lv *et al.* [4] obtained the symmetry transformations for (1) by using its Lax pair. Lü [5] constructed four sets of bilinear Bäcklund transformations in order to obtain multisoliton solutions. Wazwaz [6] derived multiple soliton solutions and multiple singular soliton solutions for (1). Equation (1) has a widespread adoption in many physical branches, such as conserved current of Liouville equation, two-dimensional quantum gravity gauge field, and conformal field theory [7–13].

In recent years, there has been a growing interest in finding exact solutions of nonlinear evolution equations, such as the rational solutions and the rogue wave solutions, which are exponentially localized in certain directions. Lump solutions are a type of rational function solutions, localized in all directions in the space. Lump solutions have been studied for many nonlinear partial differential equations such as the KPI equation [14, 15], the three-dimensional three-wave resonant interaction equation [16], and the B-KP equation [17]. Through Hirota bilinear equations, one of the authors (Ma) [18] introduced a new method to construct lump solutions to the KP equation. Following Ma's method, the lump solutions for more nonlinear evolution equations have been found, for instance, the dimensionally reduced p-gKP and p-gBKP [19], the (2+1)-dimensional Boussinesq equation [20], the BKP equation [21], the (3+1)-dimensional Jambo-Miwa [22], and the KdV equation [23]. In addition to Hirota bilinear forms, generalized bilinear derivatives [24] are used to find rational function solutions to the generalized KdV, KP, and Boussinesq equations [25–27].

In this paper, we investigate lump solutions for a fifth-order KdV-like equation. The organization of the paper is as follows: In Section 2, we formulate a new fifth-order KdV-like equation from generalized bilinear differential equations of KdV type. In Section 3 with the help of Maple, we obtain lump solutions for the constructed equation and analyze their dynamics. Then we draw some figures for a particular classes of lump solutions to show some properties. In the last section, conclusions and some remarks are given.

2. A New (2+1)-Dimensional Fifth-Order KdV-Like Equation

Under the dependent variable transformation

$$u = 2(\ln f)_{xx} \tag{3}$$

with $f = f(x, y, t)$, the (2+1)-dimensional fifth-order KdV equation (1) becomes the following (2+1)-dimensional Hirota bilinear equation:

$$\begin{aligned} B_{5thKdV} &:= \left(D_x^6 - 5D_x^3 D_y + 36 D_x D_t - 5D_y^2\right) f.f \\ &= 72 f_x f_t - 72 f_{xt} f + 2 f_{6x} f - 12 f_{5x} f_x \\ &\quad + 30 f_{4x} f_{xx} - 20 f_{xxx}^2 + 10 f_{xxx} f_y \\ &\quad - 30 f_{xx} f_{xy} + 30 f_{xxy} f_x - 10 f_{xxxy} f \\ &\quad - 10 f_{yy} f + 10 f_y^2 = 0, \end{aligned} \tag{4}$$

where the Hirota derivatives D_x, D_y, and D_t are defined in [28].

Based on a prime number p, a kind of generalized bilinear operators is introduced as [24, 29]

$$\begin{aligned} \left(D_{p,x}^m D_{p,t}^n\right) f.f &= \left(\partial_x + \alpha_p \partial_{x'}\right)^m \left(\partial_t + \alpha_p \partial_{t'}\right)^n f(x,t) \\ &\cdot f\left(x', t'\right)\Big|_{x'=x, t'=t} = \sum_{i=0}^{m}\sum_{j=0}^{n} \binom{m}{i}\binom{n}{j} \alpha_p^i \alpha_p^j \frac{\partial^{m-i}}{\partial x^{m-i}} \\ &\cdot \frac{\partial^i}{\partial x'^{(i)}} \frac{\partial^{n-j}}{\partial t^{n-j}} \frac{\partial^j}{\partial t'^{(j)}} f(x,t) f\left(x', t'\right)\Bigg|_{x'=x, t'=t} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} \binom{m}{i}\binom{n}{j} \alpha_p^i \alpha_p^j \frac{\partial^{m+n-i-j} f(x,t)}{\partial x^{m-i} t^{n-j}} \frac{\partial^{i+j} f(x,t)}{\partial x^i t^j} \end{aligned} \tag{5}$$

where $m, n \geq 0$, $\alpha_p^s = (-1)^{r_p(s)}$, $s \equiv r_p(s) \bmod p$.

For example, if we assume $p = 5$, we have

$$\begin{aligned} \alpha_5 &= -1, \\ \alpha_5^2 &= 1, \\ \alpha_5^3 &= -1, \\ \alpha_5^4 &= \alpha_5^5 = 1, \\ \alpha_5^6 &= -1, \ldots \end{aligned} \tag{6}$$

With $p = 5$, we can generalize (4) into

$$\begin{aligned} &GB_{5thKdV} \\ &\quad := \left(D_{5,x}^6 - 5D_{5,x}^3 D_{5,y} + 36 D_{5,x} D_{5,t} - 5D_{5,y}^2\right) f.f \\ &\quad = 30 f_{4x} f_{xx} - 20 f_{xxx}^2 - 10 f_{xxxy} f + 10 f_{xxx} f_y \\ &\qquad - 30 f_{xx} f_{xy} + 30 f_{xxy} f_x + 72 f_{xt} f - 72 f_x f_t \\ &\qquad - 10 f_{yy} f + 10 f_y^2 = 0. \end{aligned} \tag{7}$$

Equation (7) is a generalized bilinear fifth-order KdV equation. Under the transformations

$$\begin{aligned} u &= 6(\ln f)_x, \\ v &= 6(\ln f)_y \end{aligned} \tag{8}$$

which were suggested by the Bell polynomial theories [29–31], (7) is transformed into the following fifth-order KdV-like nonlinear differential equation:

$$\begin{aligned} GP_{5thKdV}(u) &= u_t + \frac{11}{279936} u^6 + \frac{25}{15552} u^4 u_x \\ &\quad + \frac{5}{972} u^3 u_{xx} + \frac{5}{288} u^2 u_x^2 - \frac{5}{2592} u^3 v \\ &\quad + \frac{5}{54} u u_{xx} u_x + \frac{5}{432} u_x^3 - \frac{5}{432} u v u_x \\ &\quad - \frac{5}{432} u_y u^2 + \frac{5}{432} u^2 u_{xxx} + \frac{5}{54} u_{xx}^2 \\ &\quad - \frac{5}{72} u_y u_x + \frac{5}{72} u_x u_{xxx} + \frac{1}{36} u u_{4x} \\ &\quad + \frac{1}{36} u_{5x} - \frac{5}{36} v_y = 0 \end{aligned} \tag{9}$$

Therefore, if f solves the bilinear equation (4) or (7), then $u = 6(\ln f)_{xx}$ or $u = 6(\ln f)_x$ will solve the nonlinear equation (1) or (9).

3. Lump Solutions to the Fifth-Order KdV-Like Equation

In this section, we are going to generate lump solutions to (9) by searching for quadratic function solutions to (7) with the assumption

$$
\begin{aligned}
f &= g^2 + h^2 + a_9,\\
g &= a_1 x + a_2 y + a_3 t + a_4,\\
h &= a_5 x + a_6 y + a_7 t + a_8
\end{aligned}
\tag{10}
$$

where a_i, $1 \le i \le 9$, are real constants to be determined later. Note that using a sum involving one square, in the two-dimensional space, will not generate exact solutions which are rationally localized in all directions in the space.

Substituting (10) into (7) and equating all the coefficients of different polynomials of x, y, and t to zero using Maple symbolic computation, we obtain a set of algebraic equations in a_i $(1 \le i \le 9)$; solving the set of algebraic equations with the aid of Maple, we attain the following two classes of solutions.

Case 1.

$$
\begin{aligned}
a_1 &= 0,\\
a_3 &= -\frac{5a_2^3 a_9}{54a_5^4},\\
a_6 &= -\frac{a_2^2 a_9}{3a_5^3},\\
a_7 &= \frac{5a_2^2\left(-9a_5^6 + a_2^2 a_9^2\right)}{324a_5^7},
\end{aligned}
\tag{11}
$$

and $a_i = a_i$ $(i = 2, 4, 5, 8, 9)$ are real free parameters which need to satisfy $a_5 \neq 0$ to make the corresponding solutions f to be well defined and $a_9 > 0$ to guarantee the positiveness of f.

The parameters in the sets (11) generate a class of positive quadratic function solutions to (7):

$$
\begin{aligned}
f = {} & \left(-\frac{5a_2^3 a_9 t}{54a_5^4} + a_2 y + a_4\right)^2\\
& + \left(\frac{5a_2^2\left(-9a_5^6 + a_2^2 a_9^2\right)t}{324a_5^7} + a_5 x - \frac{a_2^2 a_9 y}{3a_5^3} + a_8\right)^2\\
& + a_9
\end{aligned}
\tag{12}
$$

and the resulting class of quadratic function solutions, in turn, yields a few classes of lump solutions to the (2+1)-dimensional fifth-order KdV-like equation (9) through the dependent variable transformation:

$$
\begin{aligned}
u &= 6(\ln f)_{xx} = \frac{6\left(f_{xx} f - f_x^2\right)}{f^2}\\
&= \frac{12\left(a_1^2 - a_5^2\right)\left(-g^2 + h^2\right) - 48a_1 a_5 gh + 12\left(a_1^2 + a_5^2\right)a_9}{\left(g^2 + h^2 + a_9\right)^2}
\end{aligned}
\tag{13}
$$

where the function f is defined by (10), and the functions g and h are given as follows:

$$
\begin{aligned}
g &= -\frac{5a_2^3 a_9}{54a_5^4} t + a_2 y + a_4,\\
h &= \frac{5a_2^2\left(-9a_5^6 + a_2^2 a_9^2\right)}{324a_5^7} t + a_5 x - \frac{a_2^2 a_9}{3a_5^3} y + a_8.
\end{aligned}
\tag{14}
$$

Case 2.

$$
\begin{aligned}
a_3 &= \frac{5\left(a_1 a_2^2 - a_1 a_6^2 + 2a_2 a_5 a_6\right)}{36\left(a_1^2 + a_5^2\right)},\\
a_7 &= \frac{5\left(2a_1 a_2 a_6 - a_2^2 a_5 + a_5 a_6^2\right)}{36\left(a_1^2 + a_5^2\right)},\\
a_9 &= \frac{-3\left(a_1 a_2 + a_5 a_6\right)\left(a_1^2 + a_5^2\right)^2}{\left(a_1 a_6 - a_2 a_5\right)^2},
\end{aligned}
\tag{15}
$$

where a_1, a_2, a_4, a_5, a_6, a_8 are arbitrary constants to be determined with the following restricted conditions:

$$
\begin{aligned}
\Delta_1 &:= a_1^2 + a_5^2 = \begin{vmatrix} a_1 & -a_5 \\ a_5 & a_1 \end{vmatrix} \neq 0,\\
\Delta_2 &:= a_1 a_2 + a_5 a_6 = \begin{vmatrix} a_1 & -a_5 \\ a_6 & a_2 \end{vmatrix} < 0,\\
\Delta_3 &:= a_1 a_6 - a_2 a_5 = \begin{vmatrix} a_1 & a_2 \\ a_5 & a_6 \end{vmatrix} \neq 0.
\end{aligned}
\tag{16}
$$

Δ_1 makes the corresponding solutions f well defined and Δ_2 assures that the solution f is positive, while Δ_3 guarantees the localization of the solutions u in all directions in the (x, y)-plane.

Since these parameters are arbitrary, the solutions of (9) are more general. The parameters a_1, a_5 indicate that the wave velocity in the x direction is arbitrary and a_2, a_6 illustrate the arbitrariness of the wave velocity in the y direction. The parameters a_4, a_8 represent the invariance of variables and a_3, a_7 show the wave frequency which are represented by other quantities.

This set of parameters, in turn, generates positive quadratic function solutions to (7):

$$f=\left(a_1x+a_2y+\frac{5\left(a_1a_2^2-a_1a_6^2+2a_2a_5a_6\right)}{36\left(a_1^2+a_5^2\right)}t+a_4\right)^2+\left(a_5x+a_6y+\frac{5\left(2a_1a_2a_6-a_2^2a_5+a_5a_6^2\right)}{36\left(a_1^2+a_5^2\right)}t+a_8\right)^2+\frac{-3\left(a_1a_2+a_5a_6\right)\left(a_1^2+a_5^2\right)^2}{\left(a_1a_6-a_2a_5\right)^2}. \tag{17}$$

Consequently, a kind of lump solutions to (9) through the transformation $u=6(\ln f)_{xx}$ and (10) is achieved as follows:

$$u=6\left(\ln f\right)_{xx}=\frac{6\left(f_{xx}f-f_x^2\right)}{f^2}=\frac{12\left(a_1^2-a_5^2\right)\left(-g^2+h^2\right)-48a_1a_5gh+12\left(a_1^2+a_5^2\right)a_9}{\left(g^2+h^2+a_9\right)^2} \tag{18}$$

where the functions g and h are given by

$$g=a_1x+a_2y+\frac{5\left(a_1a_2^2-a_1a_6^2+2a_2a_5a_6\right)}{36\left(a_1^2+a_5^2\right)}t+a_4,$$
$$h=a_5x+a_6y+\frac{5\left(2a_1a_2a_6-a_2^2a_5+a_5a_6^2\right)}{36\left(a_1^2+a_5^2\right)}t+a_8. \tag{19}$$

Choosing a special value for the free parameters in Cases 1 and 2, we construct specific lump solutions u of (9). One special pair of positive quadratic function solutions and lump solutions with specific values of the parameters in Case 1 is given as follows. First, the selection of the parameters,

$$a_2=4,\quad a_4=0,\quad a_5=2,\quad a_8=0,\quad a_9=1, \tag{20}$$

leads to

$$f=\frac{34225}{26244}t^2-\frac{370}{243}ty+\frac{148}{9}y^2-\frac{350}{81}tx+4x^2-\frac{8}{3}xy+1 \tag{21}$$

and lump solution

$$u=-\frac{1259712\left(27025t^2-113400tx+115560ty+104976x^2-69984xy-408240y^2-26244\right)}{\left(34225t^2-113400tx-39960ty+104976x^2-69984xy+431568y^2+26244\right)^2} \tag{22}$$

If we take a particular choice of the parameters in Case 2 as

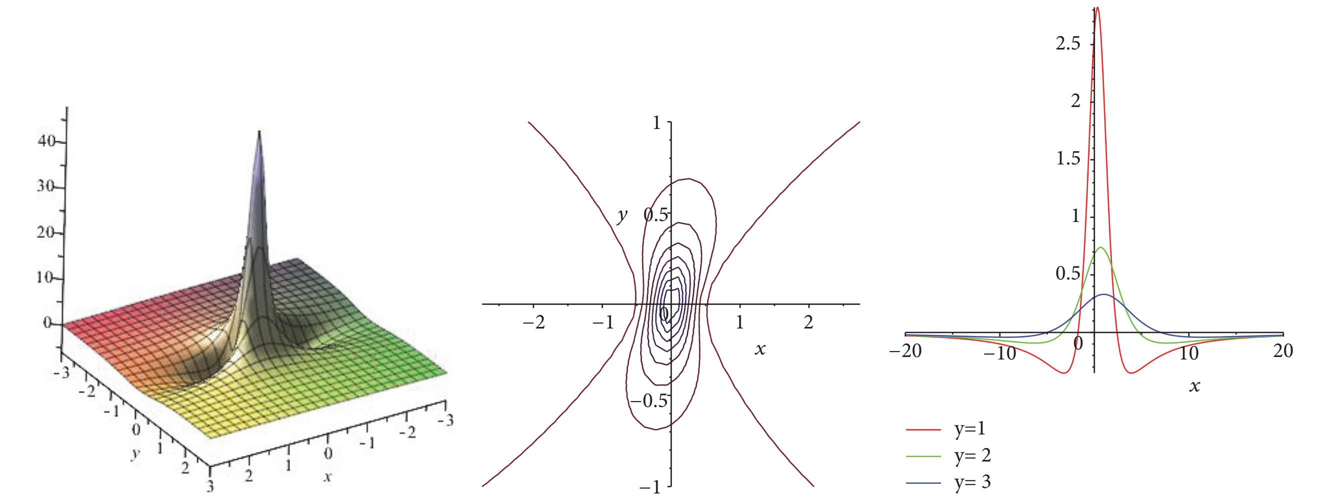
$$a_1=1,\quad a_2=-\frac{1}{2},\quad a_4=0,\quad a_5=0,\quad a_6=3,\quad a_8=0, \tag{23}$$

then we have

$$f=\frac{34225}{26244}t^2-\frac{370}{243}ty+\frac{148}{9}y^2-\frac{350}{81}tx+4x^2-\frac{8}{3}xy+1 \tag{24}$$

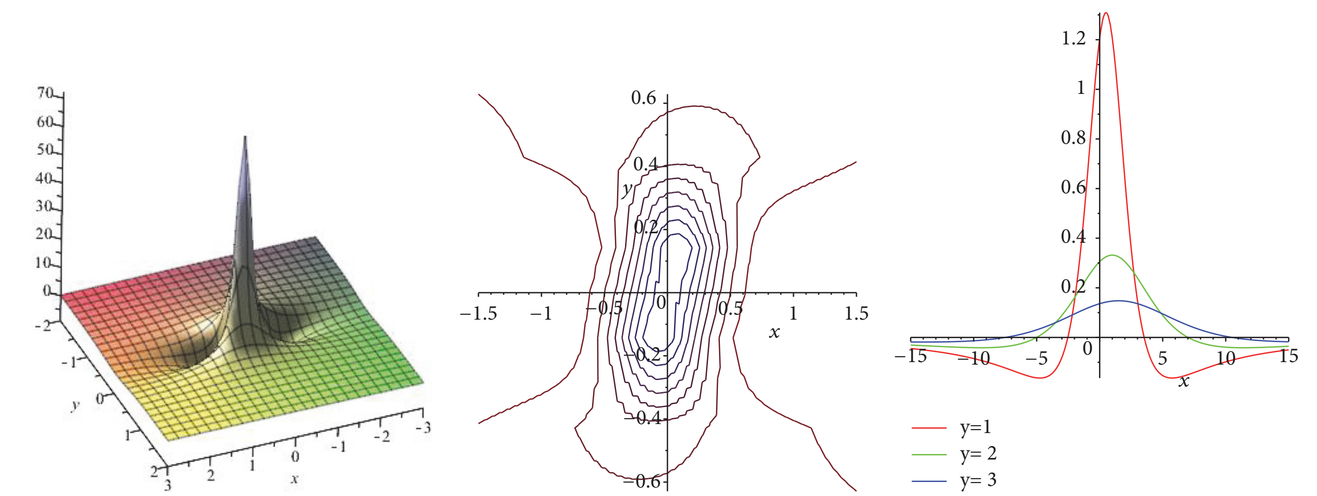
and lump solution

$$u=-\frac{248832\left(27025t^2-113400tx+115560ty+104976x^2-69984xy-408240y^2-26244\right)}{\left(34225t^2-113400tx-39960ty+104976x^2-69984xy+431568y^2+26244\right)^2} \tag{25}$$



(a) (b) (c)

Figure 1: Plots of lump solution in Case 1 with $a_2 = 4$, $a_4 = 0$, $a_5 = 2$, $a_8 = 0$, $a_9 = 1$ at $t = 0$: (a) 3D plot, (b) density plot, and (c) x-curves.



(a) (b) (c)

Figure 2: Plots of lump solution in Case 2 with $a_1 = 1$, $a_2 = -1/2$, $a_4 = 0$, $a_5 = 0$, $a_6 = 3$, $a_8 = 0$ at $t = 0$: (a) 3D plot, (b) density plot, and (c) x-curves.

Figure 1 shows the profile of lump solutions in Case 1 with the special choice of the parameters (20) at $t = 0$, while Figure 2 shows the profile of lump solutions in Case 2 with the special choice of the parameters (23) at $t = 0$.

4. Conclusions

In this paper, we studied a new (2+1)-dimensional fifth-order KdV-like equation, obtained by using the generalized Hirota bilinear formulation with $p = 5$. Through symbolic computation with Maple we constructed a few classes of lump solutions. The analyticity and localization of the resulting lump solutions are guaranteed by a nonzero determinant condition and a positivity condition. A subclass of lump solutions under special choices of the parameters involved covers the lump solutions. Contour plots with small determinant values are sequentially made to exhibit that the corresponding lump solution tends to zero when the determinant tends to zero. Recently, there have been some systematical studies on lump solutions [32] and interaction solutions between lumps and solitons for many integrable equations in (2+1)-dimensions. We refer the reader to [33] for lump-kink interaction solutions and [34, 35] for lump-soliton interaction solutions.

Data Availability

All data are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] X. Meng, "The Periodic Solitary Wave Solutions for the (2+1)-Dimensional Fifth-Order KdV Equation," *Journal of Applied Mathematics and Physics*, vol. 02, no. 07, pp. 639–643, 2014.

[2] X.-G. Xu, X.-H. Meng, C.-Y. Zhang, and Y.-T. Gao, "Analytical investigation of the Caudrey-Dodd-Gibbon-Kotera-Sawada equation using symbolic computation," *International Journal of Modern Physics B*, vol. 27, no. 6, 2013.

[3] B. G. Konopelchenko and V. G. Dubrovsky, "Some new integrable nonlinear evolution equations in (2+1) dimensions," *Physics Letters A*, vol. 102, no. 1-2, pp. 15–17, 1984.

[4] N. Lü, J.-Q. Mei, and H.-Q. Zhang, "Symmetry reductions and group-invariant solutions of (2+1)-dimensional Caudrey–Dodd–Gibbon–Kotera–Sawada equation," *Communications in Theoretical Physics*, vol. 53, no. 4, pp. 591–595, 2010.

[5] X. Lü, "New bilinear Bäcklund transformation with multisoliton solutions for the (2+1)-dimensional Sawada–Kotera model," *Nonlinear Dynamics*, vol. 76, no. 1, pp. 161–168, 2014.

[6] A.-M. Wazwaz, "Multiple soliton solutions for (2+1)-dimensional Sawada–Kotera and Caudrey–Dodd–Gibbon equations," *Mathematical Methods in the Applied Sciences*, vol. 34, no. 13, pp. 1580–1586, 2011.

[7] Z. Xu, H. Chen, and W. Chen, "The multisoliton solutions for the (2+1)-dimensional Sawada-Kotera equation," *Abstract and Applied Analysis*, vol. 2013, 5 pages, 2013.

[8] J. Satsuma and D. J. Kaup, "A Bäcklund transformation for a higher order Korteweg-de Vries equation," *Journal of the Physical Society of Japan*, vol. 43, no. 2, pp. 692–697, 1977.

[9] D. J. Kaup, "On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx} + 6Q\psi_x + 6R_\psi = \lambda\psi$," *Studies in Applied Mathematics*, vol. 62, no. 3, pp. 189–216, 1980.

[10] J. Weiss, "On classes of integrable systems and the Painlevé property," *Journal of Mathematical Physics*, vol. 25, no. 1, pp. 13–24, 1984.

[11] C. Rogers and S. Carillo, "On reciprocal properties of the Caudrey-Dodd-Gibbon and Kaup-KUPershmidt hierarchies," *Physica Scripta. An International Journal for Experimental and Theoretical Physics*, vol. 36, no. 6, pp. 865–869, 1987.

[12] Y. Ma and X. Geng, "Darboux and Bäcklund transformations of the bidirectional Sawada-Kotera equation," *Applied Mathematics and Computation*, vol. 218, no. 12, pp. 6963–6965, 2012.

[13] S. Y. Lou, "Symmetries of the KDV Equation and four hierarchies of the integrodifferential KDV Equations," *Journal of Mathematical Physics*, vol. 35, no. 5, pp. 2390–2396, 1994.

[14] S. V. Manakov, V. E. Zakharov, L. A. Bordag, A. R. Its, and V. B. Matveev, "Two-dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction," *Physics Letters A*, vol. 63, no. 3, pp. 205-206, 1977.

[15] J. Satsuma and M. J. Ablowitz, "Two-dimensional lumps in nonlinear dispersive systems," *Journal of Mathematical Physics*, vol. 20, no. 7, pp. 1496–1503, 1979.

[16] D. J. Kaup, "The lump solutions and the Bäcklund transformation for the three-dimensional three-wave resonant interaction," *Physical Review A*, vol. 90, no. 5, 1980.

[17] C. R. Gilson and J. J. Nimmo, "Lump solutions of the BKP equation," *Physics Letters A*, vol. 147, no. 8-9, pp. 472–476, 1990.

[18] W.-X. Ma, "Lump solutions to the Kadomtsev-Petviashvili equation," *Physics Letters A*, vol. 379, no. 36, pp. 1975–1978, 2015.

[19] W. X. Ma, Z. Qin, and X. Lü, "Lump solutions to dimensionally reduced p -gKP and p -gBKP equations," *Nonlinear Dynamics*, vol. 84, no. 2, pp. 923–931, 2016.

[20] H.-C. Ma and A.-P. Deng, "Lump solution of (2+1)-dimensional Boussinesq equation," *Communications in Theoretical Physics*, vol. 65, no. 5, pp. 546–552, 2016.

[21] J.-Y. Yang and W.-X. Ma, "Lump solutions to the BKP equation by symbolic computation," *International Journal of Modern Physics B*, vol. 30, no. 28-29, 2016.

[22] H.-Q. Sun and A.-H. Chen, "Lump and lump-kink solutions of the (3+1)-dimensional Jimbo-Miwa and two extended Jimbo–Miwa equations," *Applied Mathematics Letters*, vol. 68, pp. 55–61, 2017.

[23] C. Wang, "Spatiotemporal deformation of lump solution to (2+1)-dimensional KdV equation," *Nonlinear Dynamics*, vol. 84, no. 2, pp. 697–702, 2016.

[24] W. X. Ma, "Generalized bilinear differential equations," *Studies in Nonlinear Sciences*, vol. 2, no. 4, pp. 140–144, 2011.

[25] Y. Zhang and W.-X. Ma, "Rational solutions to a KdV-like equation," *Applied Mathematics and Computation*, vol. 256, pp. 252–256, 2015.

[26] Y. F. Zhang and W. X. Ma, "A study on rational solutions to a KP-like equation," *Zeitschrift für Naturforschung A*, vol. 70, no. 4, p. 263, 2015.

[27] C.-G. Shi, B.-Z. Zhao, and W.-X. Ma, "Exact rational solutions to a Boussinesq-like equation in 1+1-dimensions," *Applied Mathematics Letters*, vol. 48, pp. 170–176, 2015.

[28] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, Cambridge, UK, 2004.

[29] W. X. Ma, "Bilinear equations, Bell polynomials and linear superposition principal," vol. 411, p. 12021, 2013.

[30] Y.-H. Wang and Y. Chen, "Binary Bell polynomials, bilinear approach to exact periodic wave solutions of (2+1)-dimensional nonlinear evolution equations," *Communications in Theoretical Physics*, vol. 56, no. 4, p. 1, 2011.

[31] C. Gilson, F. Lambert, J. Nimmo, and R. Willox, "On the combinatorics of the Hirota D-Operators," *Proceedings of the Royal Society A Mathematical, Physical and Engineering Sciences*, vol. 452, no. 1945, pp. 223–234, 1996.

[32] W.-X. Ma and Y. Zhou, "Lump solutions to nonlinear partial differential equations via Hirota bilinear forms," *Journal of Differential Equations*, vol. 264, no. 4, pp. 2633–2659, 2018.

[33] J.-b. Zhang and W.-X. Ma, "Mixed lump-kink solutions to the BKP equation," *Computers and Mathematics with Applications*, 2017.

[34] J. Y. Yang, W. X. Ma, and Z. Qin, "Lump and lump-soliton solutions to the (2+1)-dimensional Ito equation," *Analysis and Mathematical Physics*, vol. 95, no. 1, p. 1, 2017.

[35] W.-X. Ma, X. Yong, and H.-Q. Zhang, "Diversity of interaction solutions to the (2+1)-dimensional Ito equation," *Computers and Mathematics with Applications*, 2017.