Research Article

Hom-Yang-Baxter Equations and Frobenius Monoidal Hom-Algebras

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It is shown that quasi-Frobenius Hom-Lie algebras are connected with a class of solutions of the classical Hom-Yang-Baxter equations. Moreover, a similar relation is discussed on Frobenius (symmetric) monoidal Hom-algebras and solutions of quantum Hom-Yang-Baxter equations. Monoidal Hom-Hopf algebras with Frobenius structures are studied at last.

1. Introduction

Hom-type algebras appeared first in physical contexts, in connection with twisted, discretized, or deformed derivatives and corresponding generalizations, discretizations, and deformations of vector fields and differential calculus. The notion of Hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov in [1–3] as part of a study of deformations of Witt algebras and Virasoro algebras. Because of close relation to discrete and deformed vector fields and differential calculus, Hom-Lie algebras are widely studied recently; see [4–9].

Hom-associative algebras generalize the concept of associative algebras. They were introduced by Makhlouf and Silvestrov in [10]. Hom-associative algebras and their related structures have recently become rather popular, due to the prospect of having a general framework in which one can produce many types of natural deformations of algebras. Among them, there are such structures as Hom-coassociative coalgebras, Hom-Hopf algebras, Hom-alternative algebras, Hom-Jordan algebras, Hom-Poisson algebras, Hom-Leibniz algebras, infinitesimal Hom-bialgebras, Hom-power associative algebras, and quasi-triangular Hom-bialgebras (see [11–16]). Furthermore, some categories of Hom-modules on Hom-Hopf algebras are studied, such as the category of Hom-Hopf modules and the category of Yetter-Drinfel'd Hom-modules in [17].

The concept of Frobenius algebras is very important because of the connections to such diverse areas as group

representations, homology of a compact oriented manifold, topological quantum field theories, quantum cohomology, Gorenstein rings in commutative algebras, Hopf algebras, coding theory, Lie quasi-Frobenius algebras, and the classical (quantum) Yang-Baxter equation (see [18–21]). In addition, there is a “quantum version” of the classical result that any finite dimensional Hopf algebra is Frobenius. The main properties of Frobenius algebras were developed by Nakayama in [22]. Nakayama automorphism is a distinguished k -algebra automorphism of a Frobenius algebra A which measures how far A is from being a symmetric algebra, where k is a fixed field. The automorphism is the identity if and only if A is symmetric. A connection between solutions of the classical Yang-Baxter equation and quasi-Frobenius Lie algebras was studied by Drinfel'd [23]. The authors in [19] gave the relation between Frobenius algebras and solutions of the quantum Yang-Baxter equation.

Naturally, we would like to do some research in Frobenius Hom-algebras. Some preliminary results about monoidal Hom-algebras with Frobenius property have been studied, such as when finite dimensional monoidal Hom-Hopf algebras are Frobenius associated with integral spaces [16], the structures of separable and Frobenius monoidal Hom-algebras related to the Hom-Frobenius-separability equation, as well as the Nakayama automorphism of Frobenius monoidal Hom-algebras [24], and so on.

The main purpose of this paper is to do further research in Frobenius Hom-algebras connected with solutions of the

quantum Hom-Yang-Baxter equation. This paper is organized as follows. In Section 3, we give some preliminary results about surgenerate bilinear forms and the element $Q \in (M \otimes M, \mu \otimes \mu)$, where (M, μ) is an object in the Hom-category $\mathcal{H}(\mathcal{M}_k)$. In addition, a connection between solutions of the classical Hom-Yang-Baxter equation (CHYBE) and quasi-Frobenius Hom-Lie algebras is also discussed. Specifically, for Hom-Lie algebra (L, α) , such an element $Q \in L \otimes L$ satisfies CHYBE if and only if there exists a surgenerate bilinear form on (L, α) such that it is quasi-Frobenius. In Section 4, a similar relationship between Frobenius (symmetric) monoidal Hom-algebras and the quantum Hom-Yang-Baxter equation (QHYBE) is discussed. For any Frobenius monoidal Hom-algebra, there exists a class of solutions of the QHYBE. And a converse result when monoidal Hom-algebras are Frobenius is also presented. It is known that a finite dimensional monoidal Hom-Hopf algebra (H, α) is Frobenius if and only if the space of integrals in H is not zero. Last, the properties of monoidal Hom-Hopf algebras with Frobenius structures are studied in Section 5.

2. Preliminaries

Throughout the paper, all vector spaces, tensor products, and homomorphisms are over k . The symbol τ denotes the transposition map.

Let $\mathcal{H}(\mathcal{M}_k)$ be the Hom-category associated with the category $\mathcal{M}_k$ of k -modules (see [15] cited by [16]). Throughout, we use the conception of [15] for convenience.

Definition 1. A monoidal Hom-algebra ([15, Proposition 2.1]) is a vector space A together with an element $1_A \in A$ and linear maps

$$m : A \otimes A \longrightarrow A; a \otimes b \longmapsto ab, \quad \alpha \in \text{Aut}_k(A) \quad (1)$$

such that

$$\alpha(a)(bc) = (ab)\alpha(c), \quad (2)$$

$$\alpha(ab) = \alpha(a)\alpha(b), \quad (3)$$

$$a1_A = 1_A a = \alpha(a), \quad (4)$$

$$\alpha(1_A) = 1_A, \quad (5)$$

for all $a, b, c \in A$.

The definition of monoidal Hom-algebras is the same to the unital monoidal Hom-associative algebras defined in [16, Definition 2.1]. A monoidal Hom-algebra is indeed a unital and associative algebra in the Hom-category $\mathcal{H}(\mathcal{M}_k)$. Let (A, α) be a monoidal Hom-algebra in the rest of this section.

Right (A, α) -Hom-modules have been introduced in [16, Definition 2.5]. In the following, we give the definition of a left (A, α) -Hom-module similarly.

Definition 2. A left (A, α) -Hom-module consists of (M, μ) in $\mathcal{H}(\mathcal{M}_k)$ together with a morphism $\psi : A \otimes M \longrightarrow M, \psi(a \otimes m) = a \cdot m$ such that

$$\begin{aligned} \alpha(a) \cdot (b \cdot m) &= (ab) \cdot \mu(m), \\ \mu(a \cdot m) &= \alpha(a) \cdot \mu(m), \\ 1_A \cdot m &= \mu(m), \end{aligned} \quad (6)$$

for all $a, b \in A$ and $m \in M$.

A morphism of left (A, α) -Hom-modules is a morphism of left A -modules in the Hom-category $\mathcal{H}(\mathcal{M}_k)$. $\mathcal{H}_{(A, \alpha)}(\mathcal{M})$ will be denoted the category of left (A, α) -Hom-modules and left (A, α) -linear morphisms between them.

If (M, μ) is both a left (A, α) -Hom-module and a right (A, α) -Hom-module such that the following compatibility condition

$$\alpha(a) \cdot (m \cdot b) = (a \cdot m) \cdot \alpha(b) \quad (7)$$

holds, then (M, μ) is called an (A, α) -Hom-bimodule.

For any object $(M, \mu) \in \mathcal{H}(\mathcal{M}_k)$, let $r : (M \otimes M, \mu \otimes \mu) \longrightarrow (M \otimes M, \mu \otimes \mu)$ be a morphism in $\mathcal{H}(\mathcal{M}_k)$. Consider the maps

$$\begin{aligned} r^{12}, r^{13}, r^{23} : (M \otimes M \otimes M, \mu \otimes \mu \otimes \mu) \\ \longrightarrow (M \otimes M \otimes M, \mu \otimes \mu \otimes \mu) \end{aligned} \quad (8)$$

given by $r^{12} = r \otimes \mu, r^{13} = (\text{id} \otimes \tau)(r \otimes \mu)(\text{id} \otimes \tau), r^{23} = r \otimes \mu$. We say that r is a solution of the quantum Hom-Yang-Baxter equation (or the QHYBE for convenience) if

$$r^{12} r^{13} r^{23} = r^{23} r^{13} r^{12}. \quad (9)$$

For elements $R = \sum_{i=1}^n a_i \otimes b_i \in A \otimes A$, where (A, α) is a monoidal Hom-algebra with unit 1_A . Then R acts on $A \otimes A$ via Hom-multiplication and is a solution of the quantum Hom-Yang-Baxter equation (or the QHYBE for convenience) if

$$\alpha^{\otimes 3}(R^{12})(R^{13}R^{23}) = (R^{23}R^{13})\alpha^{\otimes 3}(R^{12}). \quad (10)$$

where $R^{12} = \sum_{i=1}^n a_i \otimes b_i \otimes 1_A, R^{13} = \sum_{i=1}^n a_i \otimes 1_A \otimes b_i, R^{23} = \sum_{i=1}^n 1_A \otimes a_i \otimes b_i$. In what follows, we often omit the summation symbols for convenience.

3. Hom-Lie Algebras and Classical Hom-Yang-Baxter Equation

In this section, we give some preliminary results about surgenerate bilinear forms and a specific element $Q \in (M \otimes M, \mu \otimes \mu)$, where (M, μ) is an object in $\mathcal{H}(\mathcal{M}_k)$. In addition, an equivalent relation between solutions of the classical Hom-Yang-Baxter equation (CHYBE) and quasi-Frobenius Hom-Lie algebras is also discussed.

Definition 3. A Hom-Lie algebra in [9] is a triple $(L, [-, -], \alpha)$ consisting of vector space L , bilinear map $[-, -] : L^{\otimes 2} \longrightarrow L$

(called the “bracket”), and a linear endomorphism $\alpha : L \rightarrow L$ satisfying

$$\begin{aligned} \alpha[x, y] &= [\alpha(x), \alpha(y)], & (\text{multiplicativity}) \\ [x, y] + [y, x] &= 0, & (\text{anti-symmetric}) \\ \circlearrowleft [[x, y], \alpha(z)] &= 0, & (\text{Hom-Jacobi identity}) \end{aligned} \quad (11)$$

for any $x, y, z \in L$.

Now let $(L, [-, -], \alpha)$ be a Hom-Lie algebra. Element $R = \sum_{i=1}^n a_i \otimes b_i \in L \otimes L$ is called a solution of the classical Hom-Yang-Baxter equation (or the CHYBE for convenience) if

$$\sum_{i=1}^n a_i \otimes b_i = -\sum_{i=1}^n b_i \otimes a_i, \quad (12)$$

and

$$[R^{12}, R^{13}] + [R^{12}, R^{23}] + [R^{13}, R^{23}] = 0. \quad (13)$$

Here

$$\begin{aligned} [R^{12}, R^{13}] &= [a_i, a_j] \otimes \alpha(b_i) \otimes \alpha(b_j), \\ [R^{12}, R^{23}] &= \alpha(a_i) \otimes [b_i, a_k] \otimes \alpha(b_k), \\ [R^{13}, R^{23}] &= \alpha(a_j) \otimes \alpha(a_k) \otimes [b_j, b_k], \end{aligned} \quad (14)$$

where $R^{12} = R \otimes 1 = a_i \otimes b_i \otimes 1$, $R^{13} = (\tau \otimes id)(1 \otimes R) = a_j \otimes 1 \otimes b_j$ and $R^{23} = 1 \otimes R = 1 \otimes a_k \otimes b_k$.

Let $(M, \mu) \in \mathcal{H}(\mathcal{M}_k)$ be finite dimensional. A bilinear form $B : (M \otimes M, \mu \otimes \mu) \rightarrow (k, id)$ in $\mathcal{H}(\mathcal{M}_k)$ is called *surgenerate* if there exists an isomorphism $\phi : (M, \mu) \rightarrow (M^*, (\mu^*)^{-1})$ in $\mathcal{H}(\mathcal{M}_k)$ given by

$$\phi(x)(y) = B(y, x). \quad (15)$$

Here B is a morphism in $\mathcal{H}(\mathcal{M}_k)$ which means that $B(\mu(x), \mu(y)) = B(x, y)$, for all $x, y \in M$. And this makes sure that ϕ is also a morphism in $\mathcal{H}(\mathcal{M}_k)$.

A Hom-Lie algebra $(L, [-, -], \alpha)$ is said to be *quasi-Frobenius* if L is finitely generated projective k -module having a skew-symmetric surgenerate bilinear form $B : L \otimes L \rightarrow k$ satisfying the following equation:

$$B([x, y], \alpha(z)) + B([y, z], \alpha(x)) + B([z, x], \alpha(y)) = 0, \quad (16)$$

for all $x, y, z \in L$.

Lemma 4. Let $(M, \mu) \in \mathcal{H}(\mathcal{M}_k)$ be finite dimensional. Define a homomorphism $\rho_M : (M \otimes M, \mu \otimes \mu) \rightarrow (Hom(M^*, M), \iota)$ by $\rho_M(a \otimes b)(f) = f(a)b$, for all $a, b \in M, f \in M^*$, where $\iota : Hom(M^*, M) \rightarrow Hom(M^*, M)$ is the twisting given by $\iota(\psi) = \mu\psi\mu^*$. Then ρ_M is an isomorphism in $\mathcal{H}(\mathcal{M}_k)$.

Proof. First ρ_m is well defined in $\mathcal{H}(\mathcal{M}_k)$, since $\rho_M(\mu(a) \otimes \mu(b))(f) = f\mu(a)\mu(b) = \mu(\rho_M(a \otimes b)\mu^*(f))$. Define a morphism $\lambda_M : Hom(M^*, M) \rightarrow M \otimes M$ by $\psi \mapsto e_i \otimes \psi(f^i)$, where (e_i, f^i) is a dual basis for M . We can easily check that λ_M is the inverse of ρ_M . Indeed, for any $a, b \in M, f \in M^*$,

$$\begin{aligned} \lambda_M \rho_M(a \otimes b) &= e_i \otimes \rho_M(a \otimes b)(f^i) = e_i \otimes f^i(a)(b) \\ &= a \otimes b, \end{aligned} \quad (17)$$

and

$$\begin{aligned} \rho_M \lambda_M(\psi)(f) &= \rho_M(e_i \otimes \psi(f^i))(f) \\ &= f(e_i)\psi(f^i) = \psi(f). \end{aligned} \quad (18)$$

□

The endomorphism algebra $End_k(M, \mu)$ can be considered as a monoidal Hom-algebra, where the Hom-multiplication is the composite of morphisms, the unit is the identity homomorphism, and the twisting map $\iota : End_k(M, \mu) \rightarrow End_k(M, \mu)$ is given by $\iota(\varphi) = \mu\varphi\mu^{-1}$, for $\varphi \in End_k(M, \mu)$.

Proposition 5. Let $(M, \mu) \in \mathcal{H}(\mathcal{M}_k)$ be finite dimensional with a surgenerate bilinear form $B : (M \otimes M, \mu \otimes \mu) \rightarrow (k, id)$ in $\mathcal{H}(\mathcal{M}_k)$. Further let $e_i \in M, f^i \in M^*, i = 1, 2, 3, \dots, n$, be a dual basis of M , and $e^i = \phi^{-1}(f^i), i = 1, 2, 3, \dots, n$. We set $Q = e_i \otimes e^i \in M \otimes M$, and define a homomorphism in $\mathcal{H}(\mathcal{M}_k)$

$$\chi_M : (M \otimes M, \mu \otimes \mu) \rightarrow (End_k(M, \mu), \iota) \quad (19)$$

by

$$\begin{aligned} \chi_M(a \otimes b)(x) &= \langle \phi(b), \mu^{-1}(x) \rangle \mu(a) \\ &= B(x, \mu(b)) \mu(a). \end{aligned} \quad (20)$$

Further we define the mapping $Tr : End_k(M, \mu) \rightarrow k$ as follows:

$$\begin{aligned} Tr(\varphi) &= \langle f^i, \varphi(e_i) \rangle = \langle \phi(e^i), \varphi(e_i) \rangle \\ &= B(\varphi(e_i), e^i), \end{aligned} \quad (21)$$

for any $\varphi \in End_k(M, \mu)$.

Then

- (1) $x = B(x, e^i)e_i = B(e_i, x)e^i$ for all $x \in M$.
- (2) χ_M is an isomorphism in $\mathcal{H}(\mathcal{M}_k)$.
- (3) $Q = \rho_M^{-1}(\phi^{-1}) = \chi_M^{-1}(id_M)$.
- (4) The element Q does not depend on the choice of the dual bases $e_i, e^i, i = 1, 2, \dots, n$, of M .
- (5) The mapping Tr does not depend on the choice of the dual bases $e_i, e^i, i = 1, 2, \dots, n$, of M .

Proof. (1) Since $f^i(x) = \phi(e^i)(x) = B(x, e^i)$, $x = f^i(x)e_i = B(x, e^i)e_i$, for all $x \in M$. Given $x \in M$, set $y = B(e_i, x)e^i$. Then

$$\begin{aligned} B(z, y) &= B(z, B(e_i, x)e^i) = B(e_i, x)B(z, e^i) \\ &= B(B(z, e^i)e_i, x) = B(z, x), \end{aligned} \quad (22)$$

for all $z \in M$. Since B is surgenerate, $y = x$.

(2) Define a homomorphism $\varsigma : (M \otimes M^*, \mu \otimes (\mu^*)^{-1}) \longrightarrow (\text{End}_k(M, \mu), \iota)$ in $\mathcal{H}(\mathcal{M}_k)$ by

$$\varsigma(a \otimes f)(x) = f(\mu^{-1}(x))\mu(a), a, x \in M, f \in M^*. \quad (23)$$

It is not difficult to show that ς is an isomorphism with an inverse $\varsigma^{-1} : f \longmapsto f(e_i) \otimes f^i$. Since $\chi_M = \varsigma(\text{id} \otimes \phi)$, χ_M is an isomorphism in $\mathcal{H}(\mathcal{M}_k)$.

(3) For all $k = 1, 2, \dots, n$, we have

$$\begin{aligned} \rho_M(Q)(f^k) &= f^k(e_i)e^i = B(e_i, e^k)e^i = B(e_i, e^k)e^i \\ &= e^k \end{aligned} \quad (24)$$

by (1). On the other hand, $\phi^{-1}(f^k) = e^k$, for all $k = 1, 2, \dots, n$, so $\rho_M(Q) = \phi^{-1}$, and hence $Q = \rho_M^{-1}(\phi^{-1})$. Further, $\chi_M(Q)(x) = B(\mu^{-1}(x), e^i)\mu(e_i) = x$ for all $x \in M$ by (1). Therefore $\chi_M(Q) = \text{id}_M$, which implies $Q = \chi_M^{-1}(\text{id}_M)$.

(4) follows directly from (3).

(5) For $\varphi \in \text{End}(M, \mu)$, define the mapping $\Lambda : (M \otimes M, \mu \otimes \mu) \longrightarrow (k, \text{id}_k)$ by $\Lambda(a \otimes b) = B(\varphi(a), b)$ in $\mathcal{H}(\mathcal{M}_k)$. Obviously, $\text{Tr}(\varphi) = \Lambda(Q) = B(\varphi(e_i), e^i)$. It follows from (4) that $\text{Tr}(\varphi)$ does not depend on the choice of the dual bases $e_i, e^i, i = 1, 2, \dots, n$, of M . $\square$

Lemma 6. Let $(M, \mu) \in \mathcal{H}(\mathcal{M}_k)$ be finite dimensional with a surgenerate bilinear form $B : (M \otimes M, \mu \otimes \mu) \longrightarrow (k, \text{id})$ in $\mathcal{H}(\mathcal{M}_k)$. Further let $e_i \in M, f^i \in M^*, i = 1, 2, 3, \dots, n$ be a dual basis of M . Choose $e^1, e^2, \dots, e^n \in M$ such that $B(e_i, e^j) = \delta_{ij}$ for all $i, j = 1, 2, 3, \dots, n$. We set $Q = e_i \otimes e^i \in M \otimes M$, and write $e^i = \sum_{j=1}^n l^{ji}e_j$, where $l^{ji} \in k$. Next we set $n_{ij} = \phi(e_j)(e_i)$, $N = (n_{ij})_{i,j=1}^n \in M_n(k)$, and $L = (l^{ji})_{i,j=1}^n$.

Then

(1) $LN = E = NL$, where E is the identity of $M_n(k)$;

(2) $Q = \sum_{i=1}^n l^{ji}e_i \otimes e_j$.

Proof. Since $\delta_{ji} = B(e_j, e^i) = B(e_j, \sum_{k=1}^n l^{ki}e_k) = \sum_{k=1}^n B(e_j, e_k)l^{ki} = \sum_{k=1}^n b_{jk}l^{ki}$, the first assertion is true.

The second statement is obvious. $\square$

Theorem 7. Let $(L, [-, -], \alpha)$ be a Hom-Lie algebra with a skew-symmetric surgenerate bilinear form $B : L \otimes L \longrightarrow k$. Suppose that L is finite dimensional and $\{e_i, e^i\}, i = 1, 2, \dots, n$ is a dual basis of L . We set $Q = e_i \otimes e^i \in L \otimes L$. Then

(1) $Q = \rho_L^{-1}(M)(\phi^{-1}) = \chi_L^{-1}(\text{id}_L)$.

(2) The element Q does not depend on the choice of dual basis of L .

(3) The element Q satisfies CHYBE if and only if B satisfies (16).

Proof. The first two statements follow from (3) and (4) of Proposition 5.

Next, if Q satisfies the CHYBE, then by (12) we have $e^i \otimes e_j = \tau(e_j \otimes e^i) = -e_j \otimes e^i$. Choose $w^i \in L$ such that $B(e^j, w^i) = \delta_{ij}$. Then from (4) of Proposition 5,

$$e^i \otimes w^i = e_i \otimes e^i = -e^i \otimes e_i, \quad (25)$$

so $w^i = -e_i$. Therefore we have

$$B(e_i, e^j) = B(e^j, w^i) = B(e^j, -e_i) = -B(e^j, e_i), \quad (26)$$

which proves that B is skew-symmetric.

Last, we only need to show that Q satisfies the equation $[Q^{12}, Q^{13}] + [Q^{12}, Q^{23}] + [Q^{13}, Q^{23}] = 0$ if and only if B satisfies (16). Since $e_i \otimes e^i = -e^i \otimes e_i$, we have

$$\begin{aligned} [Q^{12}, Q^{13}] &= [e_i, e_j] \otimes \alpha(e^i) \otimes \alpha(e^j) \\ &= B([e_i, e_j], e^k) e_k \otimes \alpha(e^i) \otimes \alpha(e^j) \\ &= B(\alpha^{-1}([e_i, e_j]), e^k) \alpha(e_k) \alpha(e^i) \\ &\quad \otimes \alpha(e^j) \\ [Q^{12}, Q^{23}] &= \alpha(e_k) \otimes [e^k, e_j] \otimes \alpha(e^i) \\ &= \alpha(e_k) \otimes B(e_i, [e^k, e_j]) e^i \otimes \alpha(e^j) \\ &= B(e_i, [e^k, e_j]) \alpha(e_k) \otimes e^i \otimes \alpha(e^j) \\ &= B([e_j, e^k], e_i) \alpha(e_k) \otimes e^i \otimes \alpha(e^j) \\ &= B(\alpha^{-1}([e_j, e^k]), e_i) \alpha(e_k) \otimes \alpha(e^i) \\ &\quad \otimes \alpha(e^j) \end{aligned} \quad (27)$$

and

$$\begin{aligned} [Q^{13}, Q^{23}] &= -\alpha(e_k) \otimes \alpha(e^i) \otimes [e^k, e_i] \\ &= -\alpha(e_k) \otimes \alpha(e^i) \otimes B(e_j, [e^k, e_i]) e^j \\ &= -B(e_j, \alpha^{-1}([e^k, e_i])) \alpha(e_k) \otimes \alpha(e^i) \\ &\quad \otimes \alpha(e^j) \\ &= B(\alpha^{-1}([e^k, e_i]), e_j) \alpha(e_k) \otimes \alpha(e^i) \\ &\quad \otimes \alpha(e^j). \end{aligned} \quad (28)$$

Then

$$\begin{aligned} & [Q^{12}, Q^{13}] + [Q^{12}, Q^{23}] + [Q^{13}, Q^{23}] \\ &= (B(\alpha^{-1}([e_i, e_j]), e^k) + B(\alpha^{-1}([e_j, e^k]), e_i) \\ &+ B(\alpha^{-1}([e^k, e_i]), e_j)) \alpha(e_k) \otimes \alpha(e^i) \otimes \alpha(e^j). \end{aligned} \quad (29)$$

So

$$[Q^{12}, Q^{13}] + [Q^{12}, Q^{23}] + [Q^{13}, Q^{23}] = 0 \quad (30)$$

if and only if

$$\begin{aligned} & B(\alpha^{-1}([e_i, e_j]), e^k) + B(\alpha^{-1}([e_j, e^k]), e_i) \\ &+ B(\alpha^{-1}([e^k, e_i]), e_j) = 0 \end{aligned} \quad (31)$$

which is equivalent to

$$\begin{aligned} & B([e_i, e_j], \alpha(e^k)) + B([e_j, e^k], \alpha(e_i)) \\ &+ B([e^k, e_i], \alpha(e_j)) = 0. \end{aligned} \quad (32)$$

□

Note that if $(L, [-, -], \alpha)$ is a quasi-Frobenius Hom-Lie algebra, then Lemma 6 implies that $r^{ij}e_i \otimes e_j = e^j \otimes e_j = -Q$ is skew-symmetric.

4. Frobenius Monoidal Hom-Algebras and Quantum Hom-Yang-Baxter Equations

A relationship between Frobenius (symmetric) monoidal Hom-algebras and QHYBE is mainly discussed in this section.

For a finite dimensional monoidal Hom-algebra (A, α) , we can check that $(A^*, (\alpha^*)^{-1})$ is a right (A, α) -Hom-module via the action — given by

$$\theta \leftarrow a = \theta \alpha^{-2}(\alpha(a) -) = \theta_1 \alpha^{-1}(a) (\alpha^*)^{-2}(\theta_2), \quad (33)$$

where $\Delta(\theta) = \theta_1 \otimes \theta_2$ and Δ is the dual Hom-coalgebra structure on $(A^*, (\alpha^*)^{-1})$, for all $a \in A, \theta \in A^*$.

Definition 8 (see [16, Definition 5.1]). A finite dimensional monoidal Hom-algebra (A, α) is called *Frobenius* if

$$\phi : (A, \alpha) \longrightarrow (A^*, (\alpha^*)^{-1}) \quad (34)$$

is an isomorphism as right (A, α) -Hom-modules.

Recall from [16, Proposition 5.2] that a finite dimensional monoidal Hom-algebra being Frobenius has several equivalent characterization as follows.

Lemma 9. For a finite dimensional monoidal Hom-algebra (A, α) , the following assertions are equivalent.

(1) (A, α) is Frobenius.

(2) $(A, \alpha) \cong (A^*, (\alpha^*)^{-1})$ as left (A, α) -Hom-modules, which is called *Frobenius isomorphism*.

(3) There exists a Frobenius structure of (A, α) . Equivalently, there exist elements $\varphi \in A^*$ (called *Frobenius homomorphism*), and $a_i \otimes b_i \in A \otimes A$ such that $\varphi \alpha = \varphi, \alpha(a_i) \otimes \alpha(b_i) = a_i \otimes b_i$ and $\varphi(x a_i) b_i = a_i \varphi(b_i x) = x$, for any $x \in A$.

(4) There exists a Hom-associative, nondegenerate bilinear form for (A, α) . That is, there exists a bilinear map $B : A \times A \longrightarrow k$ in $\mathcal{H}(\mathcal{M}_k)$, such that $B(\alpha(a), bc) = B(ab, \alpha(c))$, and if $B(x, a) = 0$ or $B(a, x) = 0$ for any $x \in A$, then $a = 0$.

Let (A, α) be a finite dimensional monoidal Hom-algebra. And further we assume that (A, α) is Frobenius with Frobenius isomorphism $\phi : (A, \alpha) \longrightarrow (A^*, (\alpha^*)^{-1})$, Frobenius structure $\varphi \in A^*$ and $a_i \otimes b_i \in A \otimes A$, and the corresponding bilinear form $B : A \otimes A \longrightarrow k$. By the proof of the above lemma, we know that

$$B(x, y) = \varphi(xy) = \phi(y)(x), \quad (35)$$

for all $x, y \in A$. Then there exists a dual basis $e_1, e_2, \dots, e_n \in A$ and $f^1, f^2, \dots, f^n \in A^*$ such that $x = f^i(x)e_i = \sum_{i=1}^n \langle f^i, x \rangle e_i$. Setting $e^i = \phi^{-1}(f^i), i = 1, 2, \dots, n$, we note that $f^i(x) = \phi(e^i)(x) = \varphi(xe^i)$ for all $x \in A, i = 1, 2, \dots, n$. So $x = \varphi(xe^i)e_i$ for all $x \in A$. Thus we can also refer to $e_1, e_2, \dots, e_n$ and $e^1, e^2, \dots, e^n$ as dual basis of A .

Note that there is an automorphism $\sigma : (A, \alpha) \longrightarrow (A, \alpha)$ called the *Nakayama automorphism* of (A, α) defined in [16, Section 5]. It is easy to check that σ is a Hom-algebra automorphism satisfying $B(x, y) = B(y, \sigma(x))$, that is, $\varphi(xy) = \varphi(y\sigma(x))$, for all $x, y \in A$.

If a Frobenius monoidal Hom-algebra (A, α) has an augmentation ε , then the element N in A is called a *left norm* of (A, α) if $\varphi(aN) = \varepsilon(a)$, for any $a \in A$.

Given a monoidal Hom-algebra (A, α) , we denote

$$\begin{aligned} Q(A) &= \{Q \in A \otimes A \mid (1 \otimes a)Q = Q(a \otimes 1), \forall a \in A\} \\ &= \{Q = a_i \otimes b_i \in A \otimes A \mid \alpha(a_i) \otimes ab_i = a_i a \\ &\quad \otimes \alpha(b_i), \forall a \in A\}. \end{aligned} \quad (36)$$

We consider $(A \otimes A, \alpha \otimes \alpha)$ as an (A, α) -Hom-bimodule under the actions given by

$$a \cdot (x \otimes y) = \alpha^{-1}(a) x \otimes \alpha(y) \quad (37)$$

and

$$(x \otimes y) \cdot b = \alpha(x) \otimes y \alpha^{-1}(b), \quad (38)$$

for all $x, y, a, b \in A$.

Lemma 10. Let (A, α) be a monoidal Hom-algebra and $Q = a_i \otimes b_i \in Q(A)$. Then

- (1) $Q^{12}Q^{13} = Q^{23}Q^{12} = Q^{13}Q^{23}$;
- (2) $\alpha^{\otimes 3}(Q^{12})(Q^{23}Q^{12}) = (Q^{23}Q^{12})\alpha^{\otimes 3}(Q^{23})$.

Proof. (1)

$$\begin{aligned} Q^{12}Q^{13} &= a_i a_j \otimes \alpha(b_i) \otimes \alpha(b_j) \\ &= \alpha(a_i) \otimes a_j b_i \otimes \alpha(b_j) = Q^{23}Q^{12} \\ &= \alpha(a_i) \otimes \alpha(a_j) \otimes b_i b_j = Q^{13}Q^{23}. \end{aligned} \quad (39)$$

(2) From (1), we have

$$\begin{aligned} \alpha^{\otimes 3}(Q^{12})(Q^{23}Q^{12}) &= \alpha^{\otimes 3}(Q^{12})(Q^{13}Q^{23}) \\ &= (Q^{12}Q^{13})\alpha^{\otimes 3}(Q^{23}) \\ &= (Q^{23}Q^{12})\alpha^{\otimes 3}(Q^{23}). \end{aligned} \quad (40)$$

□

Theorem 11. Let (A, α) be a Frobenius monoidal Hom-algebra with Nakayama automorphism σ and let $e_i, e^i, i = 1, 2, \dots, n$, be dual bases of A . We set $Q = e_i \otimes e^i \in A \otimes A$. Then

- (1) $Q = \rho_A^{-1}(\phi^{-1}) = \chi_A^{-1}(\text{id}_A)$.
- (2) The element Q does not depend on the choice of the dual bases of e_i, e^i of A .
- (3) $(a \otimes b)Q = Q(b \otimes \sigma(a))$, for all $a, b \in A$.
- (4) $\alpha(x)((e_i z)e^i) = ((e_i z)e^i)\sigma(x)$.
- (5) $Q^{12}Q^{13} = Q^{23}Q^{12} = Q^{13}Q^{23}$.
- (6) The element Q satisfies the braid relation

$$\alpha^{\otimes 3}(Q^{12})(Q^{23}Q^{12}) = (Q^{23}Q^{12})\alpha^{\otimes 3}(Q^{23}). \quad (41)$$

Proof. The first two statements follow directly from (3) and (4) of Proposition 5.

(3) According to Proposition 5 (1), for any $a, b \in A$, we have

$$\begin{aligned} (a \otimes b)Q &= a e_i \otimes b e^i = a e_i \otimes B(e_j, b e^i) e^j \\ &= a e_i \otimes \varphi(e_j(b e^i)) e^j \\ &= a \varphi((\alpha^{-1}(e_j) b) \alpha(e^i)) e_i \otimes e^j \\ &= a \varphi((\alpha^{-2}(e_j) \alpha^{-1}(b)) e^i) e_i \otimes e^j \\ &= a(\alpha^{-2}(e_j) \alpha^{-1}(b)) \otimes e^j \\ &= \alpha^{-1}(\alpha \alpha^{-1}(e_j)) b \otimes e^j \\ &= \varphi(\alpha^{-1}(\alpha \alpha^{-1}(e_j)) e^i) e_i b \otimes e^j \\ &= e_i b \otimes \varphi(\alpha \alpha^{-1}(\alpha^{-1}(e_j) e^i)) e^j \\ &= e_i b \otimes \varphi(\alpha^{-1}(\alpha^{-1}(e_j) e^i) \sigma(a)) e_j \\ &= e_i b \otimes \varphi(\alpha^{-1}(e_j) \alpha^{-1}(e_i \sigma(a))) e^j \\ &= e_i b \otimes \varphi(e_j(e_i \sigma(a))) e^j = e_i b \otimes e^i \sigma(a) \\ &= Q(b \otimes \sigma(a)), \end{aligned} \quad (42)$$

which proves (3).

(4) For any given $z \in A$, define a map $\lambda_z : A \otimes A \rightarrow A$ given by $\lambda_z(a \otimes b) = (az)b$. λ_z is a homomorphism of (A, α) -Hom-bimodules, where the actions on $A \otimes A$ are given by (37) and (38). Then (3) implies that

$$\begin{aligned} \alpha^2(x)((e_i \alpha^{-1}(z)) e^i) &= (\alpha(x)(e_i \alpha^{-1}(z))) \alpha(e^i) \\ &= ((x e_i) z) \alpha(e^i) = \lambda_z(x e_i \otimes \alpha(e^i)) \\ &= \lambda_z((x \otimes 1)(e_i \otimes e^i)) \\ &= \lambda_z((e_i \otimes e^i)(1 \otimes \sigma(x))) = \lambda_z(\alpha(e_i) \otimes e^i \sigma(x)) \\ &= (\alpha(e_i) z) (e^i \sigma(x)) = ((e_i \alpha^{-1}(z)) e^i) \sigma(\alpha(x)), \end{aligned} \quad (43)$$

so $\alpha(x)((e_i z)e^i) = ((e_i z)e^i)\sigma(x)$, for any $x, z \in A$.

It is obvious that $Q \in Q(A)$ by setting $a = 1$ in the equality of (3). The statements (5) and (6) follow from Lemma 10. □

Theorem 12. Let (A, α) be a finite dimensional monoidal Hom-algebra with two bases $e_1, e_2, \dots, e_n$ and $e^1, e^2, \dots, e^n$, and $Q = e_i \otimes e^i \in A \otimes A$. Then the following conditions are equivalent:

- (1) $Q^{12}Q^{13} = Q^{23}Q^{12}$;
- (2) $Q^{12}Q^{13} = Q^{13}Q^{23}$;
- (3) $Q^{23}Q^{12} = Q^{13}Q^{23}$;
- (4) $(1 \otimes a)Q = Q(a \otimes 1)$;
- (5) A is a Frobenius algebra with Frobenius homomorphism $\varphi : A \rightarrow k$ such that $\varphi(e_i e^j) = \delta_{ij}$, for all $i, j = 1, 2, \dots, n$.

Proof. From Lemma 10, it is enough to show that one of the conditions (1), (2), and (3) implies (4). We may as well show that (3) implies (4), and the others can be proved analogously. From (3), it follows that

$$\begin{aligned} Q^{23}Q^{12} &= \alpha(e_i) \otimes e_j e^i \otimes \alpha(e_j) \\ &= \alpha(e_i) \otimes \alpha(e_j) \otimes e^i e^j = Q^{13}Q^{23} \end{aligned} \quad (44)$$

and so

$$e_j e^i \otimes \alpha(e^j) = \alpha(e_j) \otimes e^i e^j, \quad (45)$$

for all $i = 1, 2, \dots, n$, which proves (4).

Due to Theorem 11, it is enough to prove that (4) implies (5). Define a bilinear map $B : (A \otimes A, \alpha \otimes \alpha) \rightarrow (k, \text{id})$ in $\mathcal{H}(\mathcal{M}_k)$ as follows: $B(e_i, e^j) = B(\alpha(e_i), \alpha(e^j)) = \delta_{ij}$, for all $i, j = 1, 2, \dots, n$. Since $e_1, e_2, \dots, e_n$ and $e^1, e^2, \dots, e^n$ are two bases of A , it follows that B is a well-defined surgenerate bilinear form. And we claim that B is Hom-associative. Indeed, it is enough to show that $B(e_j x, \alpha(e^i)) = B(\alpha(e_j), x e^i)$, for all $i, j = 1, 2, \dots, n$. Since for any $a \in A$, $a = B(e_i, a) e^i = B(a, e^i) e_i$, then equality (45) implies that

$$\alpha(e_i) \otimes x e^i = e_j x \otimes \alpha(e^j) \quad (46)$$

and hence

$$\begin{aligned} \alpha(e_i) \otimes B(e_j, \alpha^{-1}(xe^i)) \alpha(e^j) \\ = B(\alpha^{-1}(e_j x), e^i) \alpha(e_i) \otimes \alpha(e^j), \end{aligned} \quad (47)$$

which implies that

$$B(e_j, \alpha^{-1}(xe^i)) = B(\alpha^{-1}(e_j x), e^i). \quad (48)$$

Thus $B(\alpha(e_j), xe^i) = B(e_j x, \alpha(e^i))$, for all $i, j = 1, 2, \dots, n$ and $x \in A$, by the α -invariance of B . Setting $\varphi(x) = B(1, \alpha^{-1}(x))$, we conclude that φ is a Frobenius homomorphism and

$$\varphi(e_i e^j) = B(1, \alpha^{-1}(e_i e^j)) = B(e_i, e^j) = \delta_{ij}, \quad (49)$$

for all $i, j = 1, 2, \dots, n$. $\square$

A Frobenius monoidal Hom-algebra with Frobenius homomorphism $\varphi : A \rightarrow k$ is called *symmetric* if $\varphi(xy) = \varphi(yx)$, for all $x, y \in A$. In this case, the Nakayama automorphism σ is the identity map of A .

Lemma 13. *Let (A, α) be a monoidal Hom-algebra with two bases $e_1, e_2, \dots, e_n$ and $e^1, e^2, \dots, e^n$, and $Q = e_i \otimes e^i \in A \otimes A$. Suppose that $(a \otimes b)Q = Q(b \otimes a)$. Then*

- (1) $Q^{12}Q^{13} = Q^{23}Q^{12} = Q^{13}Q^{23}$ and $Q^{12}Q^{23} = Q^{23}Q^{13} = Q^{13}Q^{12}$;
- (2) $\alpha^{\otimes 3}(Q^{12})(Q^{13}Q^{23}) = (Q^{23}Q^{13})\alpha^{\otimes 3}(Q^{12})$.

Proof. The first assertion of (1) is proved in Theorem 12, and the second one can be showed similarly.

(2) From (1), we have

$$\begin{aligned} \alpha^{\otimes 3}(Q^{12})(Q^{13}Q^{23}) &= (Q^{12}Q^{13})\alpha^{\otimes 3}(Q^{23}) \\ &= (Q^{23}Q^{12})\alpha^{\otimes 3}(Q^{23}) \\ &= \alpha^{\otimes 3}(Q^{23})(Q^{12}Q^{23}) \\ &= \alpha^{\otimes 3}(Q^{23})(Q^{13}Q^{12}) \\ &= (Q^{23}Q^{13})\alpha^{\otimes 3}(Q^{12}). \end{aligned} \quad (50)$$

$\square$

Theorem 14. *Let (A, α) be a symmetric monoidal Hom-algebra with Frobenius homomorphism $\varphi : A \rightarrow k$, and $\{e_i\}, \{e^j\}, i, j = 1, 2, \dots, n$, be dual bases of A . We set $Q = e_i \otimes e^i \in A \otimes A$. Then*

- (1) $(a \otimes b)Q = Q(b \otimes a)$, for any $a, b \in A$;
- (2) $Q^{12}Q^{13} = Q^{23}Q^{12} = Q^{13}Q^{23}$ and $Q^{12}Q^{23} = Q^{23}Q^{13} = Q^{13}Q^{12}$;
- (3) the element Q is a solution of QHYBE, and $\tau(Q) = Q$.

Proof. Since (A, α) is symmetric, the corresponding Nakayama automorphism is the identity. It follows from Theorem 11 (3) that $(a \otimes b)Q = Q(b \otimes a)$, for all $a, b \in A$. The last two statements are true from Lemma 13. $\square$

Next we consider the converse of the above theorem.

Theorem 15. *Let (A, α) be a finite dimensional monoidal Hom-algebra with two bases $e_1, e_2, \dots, e_n$ and $e^1, e^2, \dots, e^n$. Further let $Q = e_i \otimes e^i \in A \otimes A$. Then the following conditions are equivalent:*

- (1) $Q^{12}Q^{13} = Q^{23}Q^{12}$ and $Q^{12}Q^{23} = Q^{23}Q^{13}$;
- (2) $Q^{12}Q^{13} = Q^{13}Q^{23}$ and $Q^{12}Q^{23} = Q^{13}Q^{12}$;
- (3) $Q^{23}Q^{12} = Q^{13}Q^{23}$ and $Q^{23}Q^{13} = Q^{13}Q^{12}$;
- (4) $(a \otimes b)Q = Q(b \otimes a)$, for all $a \in A$;
- (5) A is a symmetric monoidal Hom-algebra with Frobenius homomorphism $\varphi : A \rightarrow k$ such that $\varphi(e^i e^j) = \delta_{ij}$, for all $i, j = 1, 2, \dots, n$.

Proof. We have proved that (4) implies the first three equalities in Lemma 13. And each of the first three conditions implies the fourth one in the same way as it was done in Theorem 12. Taking into account Theorem 14, we have only to prove that (4) implies (5). Set $B : (A \otimes A, \alpha \otimes \alpha) \rightarrow (k, \text{id})$ in $\mathcal{H}(\mathcal{M}_k)$ as follows: $B(e_i, e^j) = B(\alpha(e_i), \alpha(e^j)) = \delta_{ij}$, for all $i, j = 1, 2, \dots, n$. We have seen that B is a Hom-associative surgenerate bilinear form by Theorem 12, and $\varphi(x) = B(1, \alpha^{-1}(x))$ is a Frobenius homomorphism. To show $\varphi(xy) = \varphi(yx)$, we only need to show $B(1, xy) = B(1, yx)$, for $x, y \in A$, which is equivalent to prove that $B(x, y) = B(y, x)$, for $x, y \in A$. From (4) we see that $ae_i \otimes \alpha(e^i) = \alpha(e_j) \otimes e^j a$, for all $a \in A$, so

$$B(ae_i, e^j) e_j \otimes \alpha(e^i) = \alpha(e_j) \otimes B(e_i, e^j a) e^i \quad (51)$$

by Proposition 5 (1). Hence

$$\begin{aligned} B(\alpha^{-1}(ae_i), e^j) \alpha(e_j) \otimes \alpha(e^i) \\ = \alpha(e_j) \otimes B(e_i, \alpha^{-1}(e^j a)) \alpha(e^i), \end{aligned} \quad (52)$$

which implies

$$B(ae_i, \alpha(e^j)) = B(\alpha(e_i), e^j a), \quad (53)$$

for all $i, j = 1, 2, \dots, n, a \in A$. Thus $B(ax, \alpha(y)) = B(\alpha(x), ya)$, for all $x, y \in A$. Taking $x = 1_A$, we get

$$B(\alpha(a), \alpha(y)) = B(1, ya) = B(\alpha(y), \alpha(a)), \quad (54)$$

so $B(a, y) = B(y, a)$, for all $a, y \in A$. $\square$

5. Frobenius Monoidal Hom-Hopf Algebras

It is known that a finite dimensional monoidal Hom-Hopf algebra (H, α) is Frobenius if and only if the space of integrals in H is not zero. In this section, we do further research on monoidal Hom-Hopf algebras with Frobenius structures.

Definition 16 (see [16, Definition 2.2]). A monoidal Hom-coalgebra is an object (C, γ) in the category $\mathcal{H}(\mathcal{M}_k)$ together

with linear maps $\Delta : C \longrightarrow C \otimes C$, $\Delta(c) = c_1 \otimes c_2$ and $\varepsilon : C \longrightarrow k$ such that

$$\gamma^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \gamma^{-1}(c_2), \quad (55)$$

$$c_1 \varepsilon(c_2) = \gamma^{-1}(c) = \varepsilon(c_1) c_2,$$

$$\begin{aligned} \Delta(\gamma(c)) &= \gamma(c_1) \otimes \gamma(c_2), \\ \varepsilon(\gamma(c)) &= \varepsilon(c), \end{aligned} \quad (56)$$

for all $c \in C$.

Definition 17. A monoidal Hom-bialgebra [16, Definition 2.3] $H = (H, \alpha, m, \eta, \Delta, \varepsilon)$ is a bialgebra in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. This means that (H, α, m, η) is a monoidal Hom-algebra and $(H, \alpha, \Delta, \varepsilon)$ is a monoidal Hom-coalgebra such that Δ and ε are Hom-algebra maps; that is, for any $h, g \in H$,

$$\begin{aligned} \Delta(hg) &= \Delta(h) \Delta(g), \\ \Delta(1_H) &= 1_H \otimes 1_H, \\ \varepsilon(hg) &= \varepsilon(h) \varepsilon(g), \\ \varepsilon(1_H) &= 1_k. \end{aligned} \quad (57)$$

Definition 18 (see [16, Definition 2.5]). A monoidal Hom-bialgebra (H, α) is called a *monoidal Hom-Hopf algebra* if there exists a morphism (called antipode) $S : H \longrightarrow H$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ (i.e., $S \circ \alpha = \alpha \circ S$), such that, for any $h \in H$,

$$S(h_1) h_2 = \varepsilon(h) 1_H = h_1 S(h_2). \quad (58)$$

In fact, monoidal Hom-coalgebras, monoidal Hom-bialgebras, and monoidal Hom-Hopf algebras are coalgebras, bialgebras, and Hopf algebras in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, respectively. Further, the antipodes of monoidal Hom-Hopf algebras have similar properties (see [16]) of those of Hopf algebras. The self-duality of finite dimensional monoidal Hom-Hopf algebras also holds from [16], and the dual structure of monoidal Hom-Hopf algebra $(H, \alpha, m, \eta, \Delta, \varepsilon, S)$ is given as follows: for all $g, h \in H, \phi, \varphi \in H^*$,

$$\begin{aligned} \langle \phi \varphi, h \rangle &= \langle \phi, h_1 \rangle \langle \varphi, h_2 \rangle, \\ 1_{H^*} &= \varepsilon; \\ \langle \Delta(\phi), g \otimes h \rangle &= \langle \phi, gh \rangle, \\ \varepsilon_{H^*} &= \eta; \\ (\alpha^*)^{-1}(\phi) &= \phi \circ \alpha^{-1}, \\ S^*(\phi) &= \phi \circ S^{-1}. \end{aligned} \quad (59)$$

Definition 19 (see [16, Definition 3.1]). A *left integral* in (H, α) is an α -invariant element $t \in H$ (i.e., $\alpha(t) = t$) such that

$$ht = \varepsilon(h) t, \quad (60)$$

for all $h \in H$.

Lemma 20. Let (H, α) be a monoidal Hom-Hopf algebra with antipode S . Further let φ be an α^* -invariant element in H^* . Then for all $h \in H$,

$$h_1 \varphi(h_2) = \varphi(h) 1_H \quad (61)$$

if and only if φ is a left integral in H^* .

Proof. Let $\psi \in H^*$. Since for any $h \in H$,

$$\begin{aligned} (\varepsilon_{H^*}(\psi) \varphi)(h) &= \varphi(h) \psi(1_H) = \psi(\varphi(h) 1_H), \\ (\psi \varphi)(h) &= \psi(h_1) \varphi(h_2) = \psi(h_1 \varphi(h_2)), \end{aligned} \quad (62)$$

we have $h_1 \varphi(h_2) = \varphi(h) 1_H$ if and only if $\psi \varphi = \varepsilon_{H^*}(\psi) \varphi$, which means that φ is a left integral in H^* . $\square$

Lemma 21. Let (H, α) be a monoidal Hom-bialgebra and Frobenius with Frobenius homomorphism $\varphi : (H, \alpha) \longrightarrow k$ such that $h_1 \varphi(h_2) = \varphi(h) 1_H$ and with left norm N , for all $h \in H$. Then (H, α) is a monoidal Hom-Hopf algebra.

Proof. We define $S : (H, \alpha) \longrightarrow (H, \alpha)$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ by

$$S(h) = \alpha(N_1) \varphi(\alpha^{-1}(h) N_2), \quad (63)$$

for all $h \in H$. Then by the properties of φ , we see that

$$\begin{aligned} h_1 S(h_2) &= h_1 \alpha(N_1) \varphi(\alpha^{-1}(h_2) N_2) \\ &= h_1 \alpha(N_1) \varphi(h_2 \alpha(N_2)) = \varphi(h \alpha(N)) 1_H \\ &= \varphi(\alpha^{-1}(h) N) 1_H = \varepsilon(\alpha^{-1}(h)) 1_H \\ &= \eta \varepsilon(h), \end{aligned} \quad (64)$$

for all $h \in H$.

It is not difficult to check that the endomorphism $\text{End}_k(H, \alpha)$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ forms a monoidal Hom-algebra with Hom-multiplication convolution $*$, twisting map id and unit $\eta \varepsilon$. Indeed, for all $h \in H, f \in \text{End}_k(H, \alpha)$

$$(f * \eta \varepsilon)(h) = f(h_1) \eta \varepsilon(h_2) = f(h). \quad (65)$$

And the Hom-associativity is true by the Hom-coassociativity of (H, α) . The map

$$- * S : \text{End}_k(H, \alpha) \longrightarrow \text{End}_k(H, \alpha); f \longmapsto f * S \quad (66)$$

is epimorphism, since $f = f * \eta \varepsilon = f * (\text{id} * S) = (f * \text{id}) * S$. By Lemma 4 in [25], $- * S$ is an isomorphism with inverse map $- * \text{id}$. So $S * \text{id} = (\eta \varepsilon * S) * \text{id} = \eta \varepsilon$. Thus S is an antipode of (H, α) . $\square$

Theorem 22. Let (H, α) be a monoidal Hom-Hopf algebra with antipode S and also Frobenius with Frobenius homomorphism $\varphi : (H, \alpha) \longrightarrow k$ such that

$$h_1 \varphi(h_2) = \varphi(h) 1_H, \quad (67)$$

for all $h \in H$ and with the Nakayama automorphism σ . Further let $\{e_i\}, \{e^i\}, i = 1, 2, \dots, n$, be dual bases of H . Then

- (1) $N = \varepsilon(e_i)e^i = S(e_i)\varepsilon(e^i)$ is a left norm and a left integral in H . Further $\Delta(N) = S\alpha^{-1}(e_i) \otimes \alpha^{-1}(e^i)$, and $\varepsilon(N)1_H = e^i e_i = S^2(e_i)e^i = \varepsilon(e_i)\varepsilon(e^i)1_H$;
 (2) φ generates the Hom-submodule of left integral in H^* and

$$\text{Tr}(S^2) = \varepsilon(N) \varepsilon_{H^*}(\varphi). \quad (68)$$

Proof. (1) Setting $N = \varepsilon(e_i)e^i$, we have $\varepsilon(N) = \varepsilon(e_i)\varepsilon(e^i)$ and

$$\begin{aligned} \varphi(N) &= \varepsilon(e_i) \varphi(e^i) = \varepsilon(\varphi(e^i)e_i) = \varepsilon(\varphi(1_H e^i)e_i) \\ &= \varepsilon(1_H) = 1 \end{aligned} \quad (69)$$

by the α -invariance of φ and Proposition 5 (1). Further, for all $x \in H$,

$$\begin{aligned} xN &= \varepsilon(e_i) x e^i = \varepsilon(e_i) \varphi(e_j(x e^i)) e^j \\ &= \varepsilon(\varphi(e_j(x e^i)) e_i) e^j \\ &= \varepsilon(\varphi((\alpha^{-1}(e_j)x) \alpha(e^i)) e_i) e^j \\ &= \varepsilon(\varphi(\alpha^{-1}(\alpha^{-1}(e_j)x) e^i) e_i) e^j \\ &= \varepsilon(\alpha^{-1}(\alpha^{-1}(e_j)x)) e^j = \varepsilon(\alpha^{-1}(e_j)x) e^j \\ &= \varepsilon(e_j) \varepsilon(x) e^j = \varepsilon(x) N. \end{aligned} \quad (70)$$

Since $\{\alpha(e_i)\}, \{\alpha^{-1}(e^i)\}, i = 1, 2, \dots, n$, can also be considered as dual bases of H , hence we see that

$$\begin{aligned} \alpha^{-1}(N) &= \varepsilon(e_i) \alpha^{-1}(e^i) = \varepsilon \alpha(e_i) \alpha^{-1}(e^i) = \varepsilon(e_i) e^i \\ &= N, \end{aligned} \quad (71)$$

which implies that N is α -invariant. Thus N is a left integral in H by Definition 19.

Next, $\varphi(xN) = \varphi(\varepsilon(x)N) = \varepsilon(x)\varphi(N) = \varepsilon(x)$, so N is a left norm.

From Lemma 21, we know that the antipode $S : (H, \alpha) \longrightarrow (H, \alpha)$ is given by

$$S(x) = \alpha(N_1) \varphi(\alpha^{-1}(x) N_2). \quad (72)$$

And S is bijective when H is finite dimensional by Theorem 5.3 of [16]. So $\{S\alpha^{-1}(e_i)\}$ forms a basis of H . Therefore $\Delta(N) = S\alpha^{-1}(e_i) \otimes a_i$ for some $a_i \in H$. Then we rewrite the expression of S as

$$S(x) = S(e_i) \varphi(\alpha^{-1}(x) a_i), \quad (73)$$

for all $x \in H$. Applying S^{-1} we get

$$x = e_i \varphi(\alpha^{-1}(x) a_i), \quad (74)$$

for all $x \in H$. On the other hand $x = e_i \varphi(x e^i)$. So $\varphi(x \alpha(a_i)) = \varphi(\alpha^{-1}(x) a_i) = \varphi(x e^i)$. It follows that $\alpha(a_i) = e^i$, so $a_i = \alpha^{-1}(e^i)$, and hence

$$\Delta(N) = S\alpha^{-1}(e_i) \otimes \alpha^{-1}(e^i). \quad (75)$$

By the counitary property of (H, α) we have $N = \alpha S\alpha^{-1}(e_i) \varepsilon\alpha^{-1}(e^i) = S(e_i) \varepsilon(e^i)$. And $N_1 S(N_2) = S\alpha^{-1}(e_i) S\alpha^{-1}(e^i) = \varepsilon(N)1_H$, so $\alpha^{-1}(e^i e_i) = \varepsilon(N) S^{-1}(1_H) = \varepsilon(N)1_H$ by applying S^{-1} . Therefore $e^i e_i = \varepsilon(N)1_H$. Similarly, $S^2\alpha^{-1}(e_i) \alpha^{-1}(e^i) = \varepsilon(N)1_H$, which implies that $S^2(e_i) e^i = \varepsilon(N)1_H$. According to the above, so we obtain

$$e^i e_i = S^2(e_i) e^i = \varepsilon(N)1_H = \varepsilon(e_i) \varepsilon(e^i)1_H. \quad (76)$$

(2) From Proposition 3.4 of [16], we know that the integral in finite dimensional monoidal Hom-Hopf algebras does not necessarily exist. But if it exists, it is unique up to a scalar multiplication. And φ is a left integral in H^* by Lemma 21, so φ generates the Hom-submodule of left integral in H^* . Since $\varepsilon(N)1_H = S^2(e_i) e^i$ by (1), we see that

$$\begin{aligned} \varepsilon(N) \varphi(1_H) &= \varphi(\varepsilon(N)1_H) = \varphi(S^2(e_i) e^i) \\ &= \text{Tr}(S^2) \end{aligned} \quad (77)$$

by applying φ and the definition of Tr in Proposition 5. So $\text{Tr}(S^2) = \varepsilon(N)\varphi(1_H) = \varepsilon(N)\varepsilon_{H^*}(\varphi)$. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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