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Research Article

The Approximation of Bivariate Blending Variant Szász Operators Based Brenke Type Polynomials

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We have constructed a new sequence of positive linear operators with two variables by using Szasz-Kantorovich-Chlodowsky operators and Brenke polynomials. We give some inequalities for the operators by means of partial and full modulus of continuity and obtain a Lipschitz type theorem. Furthermore, we study the convergence of Szasz-Kantorovich-Chlodowsky-Brenke operators in weighted space of function with two variables and estimate the rate of approximation in terms of the weighted modulus of continuity.

1. Introduction

The well-known Mirakjan-Favard-Szasz type operators of one variable are defined as

$$S_n(f;x)=\sum_{i=0}^{\infty}\omega_n^i f\left(\frac{i}{n}\right),\quad n\in N,\ x\in[0,\infty),\tag{1}$$

where

$$\omega_n^i(x)=e^{-nx}\frac{(nx)^i}{i!}\tag{2}$$

and $f:[0,\infty)\longrightarrow\mathbb{R}$ is such that the above exist series. For the convergence of $S_n(f;x)$ to $f(x)$, usually f is supposed to be the exponential growth, that is, $|f(x)|\le\alpha e^{\beta x}$, for all $x\in[0,\infty)$, with $\alpha,\beta>0$ (see [1]). Later, in 1969, Jakimovski and Leviatan [2] investigated approximation properties of the generalization of Szász operators by means of the Appell polynomials $p_k(x)=\sum_{i=0}^{k}a_i(x^{k-i}/(k-i)!)$ which satisfy the identity

$$g(t)e^{tx}=\sum_{k=0}^{\infty}p_k(x)t^k\tag{3}$$

where $g(z)=\sum_{k=0}^{\infty}a_kz^k, |z|<R,\ (R>1)$ and $g(1)\neq 0$. Varma and Tasdelen [3] constructed positive linear operators based on orthogonal polynomials, e.g., Brenke polynomials. Suppose that

$$\begin{aligned}\mathcal{K}(t)&=\sum_{k=0}^{\infty}a_kz^k,\quad \mathcal{K}(0)\neq 0,\\ \mathcal{L}(t)&=\sum_{k=0}^{\infty}b_kz^k,\quad \mathcal{L}(0)\neq 0\end{aligned}\tag{4}$$

is analytic functions in the disk $|z|<R,\ (R>1)$, where a_k and b_k are real. The generating function for these polynomials is given by

$$\mathcal{K}(t)\mathcal{L}(tx)=\sum_{k=0}^{\infty}p_k(x)t^k\tag{5}$$

from which the explicit form of $p_k(x)$ is as follows:

$$p_k(x)=\sum_{i=0}^{k}a_{l-k}b_lx^l,\quad k=0,1,2,\ldots.\tag{6}$$

We suppose that

(1) $\mathcal{K}(1)\neq 0, a_{k-l}b_l/\mathcal{K}(1)\ge 1,\ k=0,1,2,\ldots,$

(2) $\mathcal{L}:[0,\infty)\longrightarrow(0,\infty)$,

(3) (5) and (6) converge for $|t|<R,\ (R>1)$,

(4) $\lim_{u\longrightarrow\infty}(\mathcal{L}^{(k)}(u)/\mathcal{L}(u))=1$, for $k\in\{1,2,3,4\}$.

Atakut and Buyukyazici in [4] introduced the Kantorovich-Szász variant based on Brenke type polynomials defined as

$$S_m(f;x) = \frac{c_m}{\mathcal{K}(1)\mathcal{L}(b_m y)}\sum_{j=0}^{\infty} p_j(b_m y)\int_{j/c_m}^{(j+1)/c_m} f(t)\,dt, \quad (7)$$

where $(b_m), (c_m)$ are strictly increasing sequences of positive numbers such that $\lim_{n\longrightarrow\infty}(1/c_m) = 0, b_m/c_m = 1 + O(1/c_m)$. The classical Bernstein-Chlodowsky polynomials are defined by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}\left(\frac{x}{a_n}\right) f\left(\frac{k}{n}a_n\right) \quad (8)$$

where $p_{n,k}(x/a_n) = \binom{n}{k}(x/a_n)^k(1-x/a_n)^{n-k},\ 0 \le x \le a_n$ and (a_n) is a sequence of positive numbers with $\lim_{n\longrightarrow\infty} a_n = \infty$ and $\lim_{n\longrightarrow\infty}(a_n/n) = 0$. In the last few decades the convergence estimation for linear positive operators is an active area of research amongst researchers. Several new operators have been introduced and their convergence behavior has been discussed (see [5–8]). In [9, 10] authors introduced a bivariate blending variant of the Szász type operators and studied local approximation properties for these operators. Also, they estimated the approximation order in terms of Peetre's K-functional and partial moduli of continuity.

In the present paper, we define new bivariate operators associated with a combination of Szasz-Kantorovich-Chlodowsky operators based on Brenke polynomials as follows:

$$T_{n,m,a_n}^{b_m,c_m}(f;x,y) = \frac{nc_m}{a_n\mathcal{K}(1)\mathcal{L}(b_m y)}\sum_{k=0}^{n}\sum_{j=0}^{\infty} p_{n,k}\left(\frac{x}{a_n}\right) \cdot p_j(b_m y)\int_{j/c_m}^{(j+1)/c_m}\int_{(k/n)a_n}^{((k+1)/n)a_n} f(t,s)\,dt\,ds, \quad (9)$$

where the sequences $(a_n), (b_m), (c_m)$ are defined as above and satisfy the following conditions:

$$\lim_{n\longrightarrow\infty}\left(\frac{a_n}{n}\right) = 0,\quad \lim_{m\longrightarrow\infty}\left(\frac{1}{c_m}\right) = 0,\quad \frac{b_m}{c_m} = 1 + O\left(\frac{1}{c_m}\right). \quad (10)$$

For operators defined in (36) we have

$$T_{n,m,a_n}^{b_m,c_m}(f;x,y) = {}_xC_n^*\left({}_y^*S_m^{b_m,c_m}(f;x,y)\right) = {}_y^*S_m^{b_m,c_m}\left({}_xC_n^*(f;x,y)\right) \quad (11)$$

where

$${}_xC_n^*(f;x,y) = \frac{n}{a_n}\sum_{k=0}^{n} p_{n,k}\left(\frac{x}{a_n}\right)\int_{(k/n)a_n}^{((k+1)/n)a_n} f(t,y)\,dt \quad (12)$$

and

$${}_y^*S_m^{b_m,c_m}(f;x,y) = \frac{b_m}{\mathcal{K}(1)\mathcal{L}(b_m y)}\sum_{j=0}^{\infty} p_j(b_m y)\int_{j/c_m}^{(j+1)/c_m} f(x,s)\,ds. \quad (13)$$

In this study, we give some basic convergence properties for the operators defined by (9) and study local approximation properties for these operators. Furthermore, we study the linear positive operators in a weighted space of function with two variables and estimate the rate of approximation of the operators $T_{n,m,a_n}^{b_m,c_m}$ in the terms of the weighted modulus of continuity.

2. Notations and Auxiliary Results

We will subsequently need the following basic results to prove the main results.

In what follows, let $e_{ij}(x,y) = x^i y^j,\ (i,j) \in \mathbb{N}^0 \times \mathbb{N}^0$, where $i + j \le 4$ is the two dimensional test functions.

By simple calculations we get the following lemma.

Lemma 1. *Let $T_{n,m,a_n}^{b_m,c_m}$ be the bivariate of Szasz-Kantorovich-Chlodowsky-Brenke operators defined by (9). For all $m, n \in N$, $T_{n,m,a_n}^{b_m,c_m}$ satisfy the following results:*

(i)

$$T_{n,m,a_n}^{b_m,c_m}(e_{00};x,y) = 1; \quad (14)$$

(ii)

$$T_{n,m,a_n}^{b_m,c_m}(e_{10};x,y) = x + \frac{a_n}{2n}; \quad (15)$$

(iii)

$$T_{n,m,a_n}^{b_m,c_m}(e_{01};x,y) = \frac{b_m y\mathcal{L}'(b_m y)}{c_m\mathcal{L}(b_m y)} + \frac{2\mathcal{K}'(1)+1}{2\mathcal{K}(1)c_m}; \quad (16)$$

(iv)

$$T_{n,m,a_n}^{b_m,c_m}(e_{20};x,y) = \left(1-\frac{1}{n}\right)x^2 + 2\frac{a_n}{n}x + \frac{a_n^2}{3n^2}; \quad (17)$$

(v)

$$T_{n,m,a_n}^{b_m,c_m}(e_{02};x,y) = \frac{b_m^2\mathcal{L}''(b_m y)y^2}{c_m^2\mathcal{L}(b_m y)} + \frac{2b_m\left(\mathcal{K}'(1)+\mathcal{K}(1)\right)\mathcal{L}'(b_m y)y}{c_m^2\mathcal{K}(1)\mathcal{L}(b_m y)} + \frac{3\mathcal{K}''(1)+6\mathcal{K}'(1)+\mathcal{K}(1)}{3c_m^2\mathcal{K}(1)}; \quad (18)$$

(vi)

$$T^{b_m,c_m}_{n,m,a_n}(e_{03};x,y) = \frac{b_m^3\mathcal{L}'''(b_my)y^3}{c_m^3\mathcal{L}(b_my)} + \frac{3b_m^2(2\mathcal{K}'(1)+3\mathcal{K}(1))\mathcal{L}''(b_my)y^2}{c_m^3\mathcal{K}(1)\mathcal{L}(b_my)} + \frac{b_m(6\mathcal{K}''(1)+18\mathcal{K}'(1)+7\mathcal{K}(1))\mathcal{L}'(b_my)y}{2c_m^3\mathcal{K}(1)\mathcal{L}(b_my)} + \frac{4\mathcal{K}'''(1)+16\mathcal{K}'(1)+12\mathcal{K}'(1)+7\mathcal{K}(1)}{4c_m^3\mathcal{K}(1)}; \quad (19)$$

(vii)

$$T^{b_m,c_m}_{n,m,a_n}(e_{04};x,y) = \frac{b_m^4\mathcal{L}^{(4)}(b_my)y^4}{c_m^4} + \frac{4b_m^3(\mathcal{K}'(1)+2\mathcal{K}(1))\mathcal{L}'''(b_my)y^3}{c_m^4\mathcal{K}(1)\mathcal{L}(b_my)} + \frac{3b_m^2(2\mathcal{K}''(1)+8\mathcal{K}'(1)+5\mathcal{K}(1))\mathcal{L}''(b_my)y^2}{2c_m^4\mathcal{K}(1)\mathcal{L}(b_my)} + \frac{b_m(4\mathcal{K}'''(1)+24\mathcal{K}''(1)+30\mathcal{K}'(1)+6\mathcal{K}(1))\mathcal{L}'(b_my)y}{c_m^4\mathcal{K}(1)\mathcal{L}(b_my)} + \frac{5\mathcal{K}^{(4)}(1)+40\mathcal{K}'''(1)+75\mathcal{K}''(1)+30\mathcal{K}'(1)+\mathcal{K}(1)}{c_m^4\mathcal{K}(1)}. \quad (20)$$

Proof. In view of definition of operators defined by (9) we have

$$\begin{aligned}
T^{b_m,c_m}_{n,m,a_n}(e_{00};x,y) &= {}_xC_n^*(e_0;x,y)_y^* S_m^{b_m,c_m}(e_0;x,y)\\
T^{b_m,c_m}_{n,m,a_n}(e_{10};x,y) &= {}_xC_n^*(e_1;x,y)_y^* S_m^{b_m,c_m}(e_0;x,y)\\
T^{b_m,c_m}_{n,m,a_n}(e_{01};x,y) &= {}_xC_n^*(e_0;x,y)_y^* S_m^{b_m,c_m}(e_1;x,y)\\
T^{b_m,c_m}_{n,m,a_n}(e_{20};x,y) &= {}_xC_n^*(e_2;x,y)_y^* S_m^{b_m,c_m}(e_0;x,y)\\
T^{b_m,c_m}_{n,m,a_n}(e_{02};x,y) &= {}_xC_n^*(e_0;x,y)_y^* S_m^{b_m,c_m}(e_2;x,y)\\
T^{b_m,c_m}_{n,m,a_n}(e_{03};x,y) &= {}_xC_n^*(e_0;x,y)_y^* S_m^{b_m,c_m}(e_3;x,y)\\
T^{b_m,c_m}_{n,m,a_n}(e_{04};x,y) &= {}_xC_n^*(e_0;x,y)_y^* S_m^{b_m,c_m}(e_4;x,y)
\end{aligned} \quad (21)$$

with the help of these equalities, we can easily prove required results. ☐

Lemma 2. *It follows from Lemma 1 that*

$$T^{b_m,c_m}_{n,m,a_n}(e_{10}-x;x,y) = \frac{a_n}{2n};$$

$$T^{b_m,c_m}_{n,m,a_n}(e_{10}-x;x,y) = \left(\frac{b_m^2\mathcal{L}''(b_my)}{c_m^2\mathcal{L}(b_my)}-1\right)y + \frac{2\mathcal{K}'(1)+1}{2c_m\mathcal{K}(1)};$$

$$T^{b_m,c_m}_{n,m,a_n}((e_{10}-x)^2;x,y) = \frac{x(a_n-x)}{n} + \frac{a_n^2}{3n^2};$$

$$T^{b_m,c_m}_{n,m,a_n}((e_{01}-y)^2;x,y) = \left(\frac{b_m^2\mathcal{L}''(b_my)}{c_m^2\mathcal{L}(b_my)} - \frac{2b_m\mathcal{L}'(b_my)}{c_m\mathcal{L}(b_my)} + 1\right)y^2 + \left(\frac{2b_m(\mathcal{K}'(1)+\mathcal{K}(1))\mathcal{L}'(b_my)}{c_m^2\mathcal{K}(1)\mathcal{L}(b_my)} - \frac{2\mathcal{K}'(1)+1}{c_m\mathcal{K}(1)}\right)y + \frac{3\mathcal{K}''(1)+6\mathcal{K}'(1)+\mathcal{K}(1)}{3c_m^2\mathcal{K}(1)}. \quad (22)$$

Proof. The results follow from linearity of the operators T^{b_m,c_m}_{n,m,a_n} and Lemma 1.

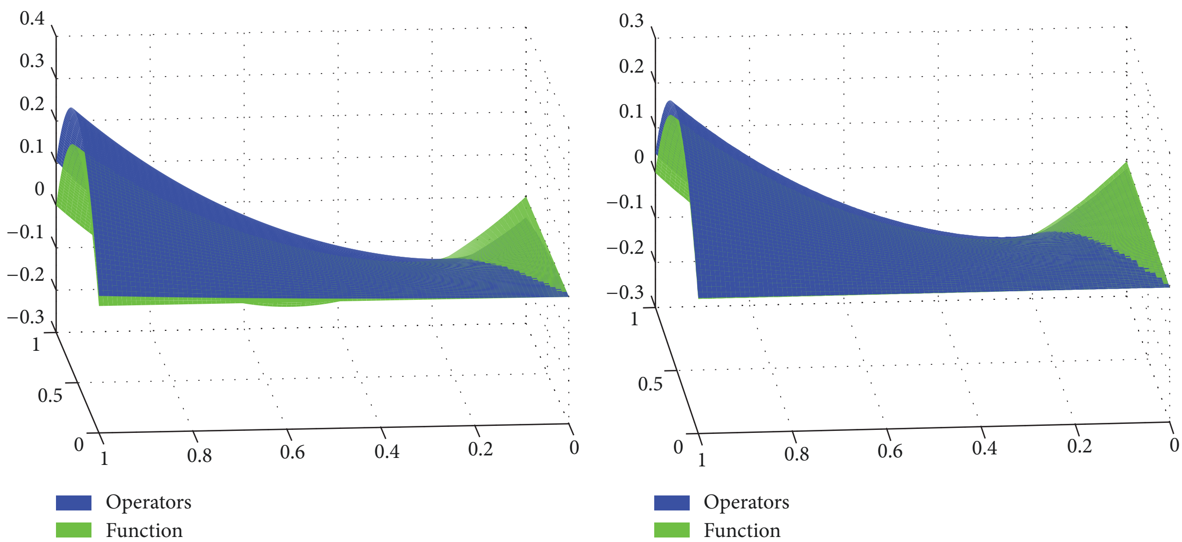
For sufficiently large n, m, for all $(x,y) \in I_{a_n}$, by taking into consideration Lemma 1, and condition (10), we have the following equalities:

$$\Delta_1 = T^{b_m,c_m}_{n,m,a_n}((e_{10}-x)^2;x,y) = {}_xC_n^*((e_{10}-x)^2;x,y) = O\left(\frac{a_n}{n}\right)\sum_{i=1}^{2}x^i; \quad (23)$$

$$\Delta_2 = T^{b_m,c_m}_{n,m,a_n}((e_{10}-x)^4;x,y) = {}_xC_n^*((e_{10}-x)^4;x,y) = O\left(\frac{a_n}{n}\right)\sum_{i=1}^{4}x^i; \quad (24)$$

Table 1: Error estimation for operator (9) to the function $f(x,y)=x^2y-xy^2$ for $n=m=100$.

x	(0.01,0.5)	(0.1,0.5)	(0.4,0.5)	(0.5,0.5)	(0.6,0.5)	(0.8,0.5)	(0.9,0.5)
n=m= 1000	0.0037	0.0011	0.0076	0.0105	0.0135	0.0196	0.0226
n = m=100	0.0118	0.0052	0.0187	0.0273	0.0361	0.0548	0.0645



(a) (b)

Figure 1

and

$$\Delta_1^* = T_{n,m,a_n}^{b_m,c_m}\left(\left(e_{10}-y\right)^2; x, y\right) =_y^* S_m^{b_m,c_m}\left(\left(e_{10}-y\right)^2; x,y\right) = O\left(\frac{1}{c_m}\right)\sum_{i=1}^{2} y^i; \tag{25}$$

$$\Delta_2^* = T_{n,m,a_n}^{b_m,c_m}\left(\left(e_{10}-y\right)^4; x, y\right) =_y^* S_m^{b_m,c_m}\left(\left(e_{10}-y\right)^4; x,y\right) = O\left(\frac{1}{c_m}\right)\sum_{i=0}^{4} y^i. \tag{26}$$

Further, let $\delta_n(x)=\Delta_1$, $\delta_m(y)=\Delta_1^*$, and $\delta_m(x,y)=(\Delta_1+\Delta_1^*)^{1/2}$. □

3. Main Results

To study the convergence of the sequence $\{T_{n,m,a_n}^{b_m,c_m}\}$ we shall use the following Korovkin type theorem, established by Volkov [11]. Next, the degree of approximation of the operator $\{T_{n,m,a_n}^{b_m,c_m}\}$ given by (36) will be established in the space of continuous function on compact set $I_{ab}=[0,a]\times[0,b]\subset I_{a_n}$. For $I_{ab}=[0,a]\times[0,b]$, let $C(I_{ab})$, denote the space of all real valued continuous functions on I_{ab}, endowed with the norm $\|f\|_{C(I_{ab})}=\sup_{(x,y)\in I_{ab}}|f(x,y)|$.

Theorem 3. *Let $\{T_{n,m,a_n}^{b_m,c_m}\}$ be the sequences of linear positive operators defined by (36). Then for each $f\in C(I_{ab})$, we have*

$\lim_{n,m\to\infty} T_{n,m,a_n}^{b_m,c_m} = f(x,y)$ uniformly on the compact set I_{ab}.

Proof. From Lemma 1, we have

$$\lim_{n,m\to\infty} T_{n,m,a_n}^{b_m,c_m}\left(e_{ij}; x, y\right) = e_{ij} = 0, \quad (i,j)\in\{(0,0),(1,1),(2,2)\} \tag{27}$$

and

$$\lim_{n,m\to\infty} T_{n,m,a_n}^{b_m,c_m}\left(e_{20}+e_{02}; x,y\right) = e_{20}+e_{02}, \tag{28}$$

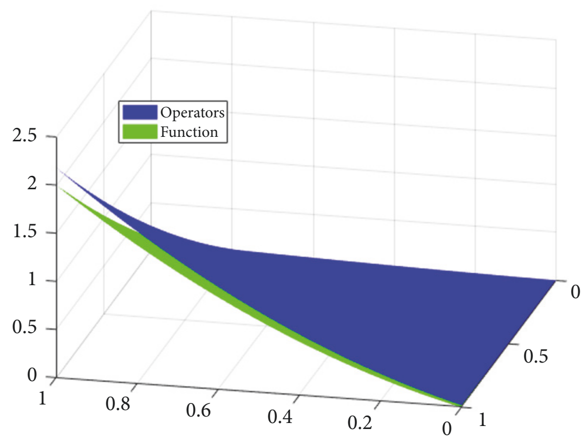
uniformly on I_{ab}. The result follows from the well-known Volkov theorem. □

Example 4. Let us consider the function $f(x,y)=x^2y-xy^2$. For $n=m=100,1000$, $\mathscr{K}(t)=e^{2t}$; $\mathscr{L}(t)=e^t$ and $a_n=\sqrt{n}$; $b_m=m$; $c_m=m+1/\sqrt{m}$ the convergence of $T_{n,m,a_n}^{b_m,c_m}$ *to* $f(x,y)$ is illustrated in Figures 1(a) and 1(b), respectively. Further, in Table 1 we compute error estimation for operator (9) to the function f.

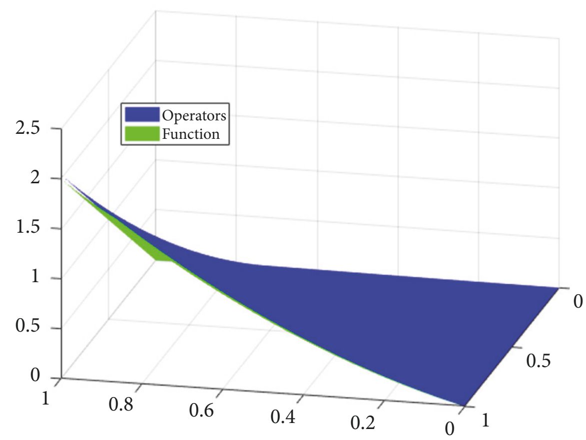
Example 5. For $\mathscr{K}(t)=e^{2t}$, $\mathscr{L}(t)=e^t$ the convergence of operators $T_{n,m,a_n}^{b_m,c_m}$ to function $f(x,y)$ is illustrated in Figures 2(a) and 2(b), respectively, where $(x,y)=xy+xy^2$, $n=m=100,1000$, and $a_n=\sqrt{n}$; $b_m=m$; $c_m=m+1/\sqrt{m}$. In Table 2 there are are compute error estimations for operator (9) to the function f.

TABLE 2: Error estimation for operator (9) to the function $f(x, y) = xy + xy^2$ for $n = m = 1000$.

x	(0.01,0.5)	(0.1,0.5)	(0.3,0.5)	(0.4,0.5)	(0.6,0.5)	(0.8,0.5)	(0.9,0.5)
n =m= 1000	0.0116	0.0121	0.0131	0.0136	0.0151	0.0156	0.0161
n = m=100	0.0390	0.0435	0.0535	0.0584	0.0734	0.0784	0.0834



(a)



(b)

FIGURE 2

An estimation of the rate of convergence can be obtained using the modulus of continuity for two dimensional real valued functions. Let $f \in C(I_{ab})$ and $\delta > 0$. In what follows, we shall use the following modulus of continuity for bivariate real functions:

$$\omega(f; \delta_n, \delta_m) = \sup \{|f(t,s) - f(x,y)| : (t,s), (x,y) \in I_{ab}, \ |t - x| \le \delta_n, \ |s - y| \le \delta_m\}. \tag{29}$$

Alternately, the complete modulus of continuity of f which we denote by $\omega(f;\delta)$ is defined as

$$\omega(f;\delta) = \sup_{\sqrt{(t-x)^2+(s-y)^2} \le \delta} \{|f(t,s) - f(x,y)| : (t,s), (x,y) \in I_{ab}, \}. \tag{30}$$

Theorem 6. *For any $f \in C(I_{ab})$, then we have estimated*

$$\left|T^{b_m,c_m}_{n,m,a_n}(f;x,y) - f(x,y)\right| \le 2\omega(f;\delta_{n,m}) \tag{31}$$

where $\delta_{n,m} = \delta_{n,m}(x,y)$.

Proof. From (9) and by definition of $\omega(f;\delta)$, we can write

$$\begin{aligned}
&\left|T^{b_m,c_m}_{n,m,a_n}(f;x,y) - f(x,y)\right| \le \frac{nc_m}{a_n \mathcal{K}(1)\mathcal{L}(b_m y)} \\
&\cdot \sum_{k=0}^{n}\sum_{j=0}^{\infty} p_{n,k}\left(\frac{x}{a_n}\right) p_j(b_m y) \\
&\cdot \int_{j/c_m}^{(j+1)/c_m} \int_{(k/n)a_n}^{((k+1)/n)a_n} \left(\omega\left(f; \sqrt{(t-x)^2 + (s-y)^2}; x, y\right)\right) dt\, ds \\
&\le \omega(f;\delta_{n,m}) \left\{1 + \frac{1}{\delta_{m,n}} \frac{nc_m}{a_n \mathcal{K}(1)\mathcal{L}(b_m y)}\right. \\
&\cdot \sum_{k=0}^{n}\sum_{j=0}^{\infty} p_{n,k}\left(\frac{x}{a_n}\right) p_j(b_m y) \\
&\left.\cdot \int_{j/c_m}^{(j+1)/c_m} \int_{(k/n)a_n}^{((k+1)/n)a_n} \sqrt{(t-x)^2 + (s-y)^2} dt\, ds\right\}
\end{aligned} \tag{32}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\left|T^{b_m,c_m}_{n,m,a_n}(f;x,y) - f(x,y)\right| \le \omega(f;\delta_{n,m}) \left(1 \right. \\
&\left. + \frac{1}{\delta_{n,m}} \left(T^{b_m,c_m}_{n,m,a_n}\left((t-x)^2 + (s-y)^2; x, y\right)\right)^{1/2}\right) \\
&\le \omega(f;\delta_{n,m}) \left(1 \right. \\
&\left. + \frac{1}{\delta_{n,m}} \left(O\left(\frac{a_n}{n}\right)\sum_{i=1}^{2} x^i + O\left(\frac{1}{c_m}\right)\sum_{i=1}^{2} y^i\right)^{1/2}\right).
\end{aligned} \tag{33}$$

Taking $\delta_{n,m} = \delta_{n,m}(x,y)$, we obtain the desired result.

The partial modulus of continuity with respect to x and y is given by

$$\begin{aligned}\omega^{(1)}(f;\delta) &= \sup_{0\le y\le b}\ \sup_{|x_1-x_2|\le\delta}\{|f(x_1,y)-f(x_2,y)|\},\\ \omega^{(2)}(f;\delta) &= \sup_{0\le x\le a}\ \sup_{|y_1-y_2|\le\delta}\{|f(x,y_1)-f(x,y_2)|\},\end{aligned} \tag{34}$$

□

Theorem 7. *For any $f \in C(I_{ab})$, then the inequalities satisfy*

$$\begin{aligned}&\left|T^{b_m,c_m}_{n,m,a_n}(f;x,y)-f(x,y)\right|\\ &\quad\le 2\left(\omega^{(1)}(f;\delta_n)+\omega^{(2)}(f;\delta_m)\right)\end{aligned} \tag{35}$$

where $\delta_n = \delta_n(x)$, $\delta_m = \delta_m(y)$.

Proof. Using the definition of partial modulus of continuity $\omega^{(i)}(f;\delta)$, $i = 1, 2$, we may write

$$\begin{aligned}\left|T^{b_m,c_m}_{n,m,a_n}(f;x,y)-f(x,y)\right| &\le \frac{nc_m}{a_n\mathcal{K}(1)\mathcal{L}(b_my)}\\ &\cdot\sum_{k=0}^{n}\sum_{j=0}^{\infty}p_{n,k}\left(\frac{x}{a_n}\right)p_j(b_my)\\ &\cdot\int_{j/c_m}^{(j+1)/c_m}\int_{(k/n)a_n}^{((k+1)/n)a_n}|f(t,s)-f(t,y)|\,dt\,ds\\ &+\frac{nc_m}{a_n\mathcal{K}(1)\mathcal{L}(b_my)}\sum_{k=0}^{n}\sum_{j=0}^{\infty}p_{n,k}\left(\frac{x}{a_n}\right)p_j(b_my)\\ &\cdot\int_{j/c_m}^{(j+1)/c_m}\int_{(k/n)a_n}^{((k+1)/n)a_n}|f(t,y)-f(x,y)|\\ &\le\frac{nc_m}{a_n\mathcal{K}(1)\mathcal{L}(b_my)}\sum_{k=0}^{n}\sum_{j=0}^{\infty}p_{n,k}\left(\frac{x}{a_n}\right)p_j(b_my)\\ &\cdot\int_{j/c_m}^{(j+1)/c_m}\int_{(k/n)a_n}^{((k+1)/n)a_n}\omega^{(2)}(f;|s-y|)\,dt\,ds\\ &+\frac{nc_m}{a_n\mathcal{K}(1)\mathcal{L}(b_my)}\sum_{k=0}^{n}\sum_{j=0}^{\infty}p_{n,k}\left(\frac{x}{a_n}\right)p_j(b_my)\\ &\cdot\int_{j/c_m}^{(j+1)/c_m}\int_{(k/n)a_n}^{((k+1)/n)a_n}\omega^{(1)}(f;|t-x|)\,dt\,ds\\ &=\eta_1(x,y)+\eta_2(x,y).\end{aligned} \tag{36}$$

Consider $\eta(x,y)$. Using Lemma 1 and the well-known properties of the modulus of continuity, we have

$$\begin{aligned}\eta_1(x,y) &\le \omega^{(2)}(f;\delta_m)\left[1+\frac{nc_m}{\delta_ma_n\mathcal{K}(1)\mathcal{L}(b_my)}\right.\\ &\cdot\sum_{k=0}^{n}\sum_{j=0}^{\infty}p_{n,k}\left(\frac{x}{a_n}\right)p_j(b_my)\\ &\left.\cdot\int_{j/c_m}^{(j+1)/c_m}\int_{(k/n)a_n}^{((k+1)/n)a_n}|s-y|\,dt\,ds\right]\end{aligned} \tag{37}$$

By using Cauchy-Schwarz inequality, we get

$$\begin{aligned}\eta_1(x,y) &\le \omega^{(2)}(f;\delta_m)\left(1+\frac{1}{\delta_m}\left\{\frac{nc_m}{a_n\mathcal{K}(1)\mathcal{L}(b_my)}\right.\right.\\ &\cdot\sum_{k=0}^{n}\sum_{j=0}^{\infty}p_{n,k}\left(\frac{x}{a_n}\right)p_j(b_my)\\ &\left.\left.\times\int_{j/c_m}^{(j+1)/c_m}\int_{(k/n)a_n}^{((k+1)/n)a_n}(s-y)^2\,dt\,ds\right\}^{1/2}\right).\end{aligned} \tag{38}$$

So, by using (25), we obtain

$$\eta_1(x,y)\le\omega^{(2)}(f;\delta_m)\left(1+\frac{O(1/c_m)\sum_{i=1}^{2}y^i}{\delta_m}\right). \tag{39}$$

In the same way we gain

$$\eta_2(x,y)\le\omega^{(1)}(f;\delta_n)\left(1+\frac{O(a_n/n)\sum_{i=1}^{2}x^i}{\delta_n}\right). \tag{40}$$

Hence from (39), (40), and (32), we arrive at

$$\begin{aligned}&\left|T^{b_m,c_m}_{n,m,a_n}(f;x,y)-f(x,y)\right|\\ &\quad\le\omega^{(2)}(f;\delta_m)\left(1+\frac{O(1/c_m)\sum_{i=1}^{2}y^i}{\delta_m}\right)\\ &\qquad+\omega^{(1)}(f;\delta_n)\left(1+\frac{O(a_n/n)\sum_{i=1}^{2}x^i}{\delta_n}\right).\end{aligned} \tag{41}$$

Finally, choosing $\delta_n = \Delta_1 = \delta_n(x)$ and $\delta_m = \Delta_1^* = \delta_m(y)$, for all $(x, y) \in I_{ab}$, we reach the desired result.

For $0 < \gamma \le 1$, we define the Lipschitz class $Lip_L(\gamma)$ for bivariate case as follows:

$$\begin{aligned}&Lip_L(\gamma)\\ &:=\left\{f:|f(t_1,t_2)-f(x,y)|\le L\frac{\|r-s\|^\gamma}{(\|r\|+x+y)^{\gamma/2}}\right\},\end{aligned} \tag{42}$$

where $r = (t_1, t_2)$, $s = (x, y)$ in I_{ab}, and $\|r - s\| = \{(t_1 - x)^2 + (t_2 - y)^2\}^{1/2}$ is the Euclidean norm. □

Theorem 8. *Suppose that $f \in Lip_L(\gamma)$. Then, for every $(x, y) \in I_{ab}$, we have*

$$\left|T^{b_m,c_m}_{n,m,a_n}(f;x,y)-f(x,y)\right|\le\frac{L\delta_m^{1/2}}{(x+y)^{1/2}}, \tag{43}$$

where $\delta_m = \delta_m(x, y)$.

Proof. First, we prove theorem for case $\gamma = 1$. Then, for $f \in Lip_L(\gamma)$ and for each $x, y \in I_{ab}$, using the monotonicity and linearity of operators, we may write

$$\begin{aligned}
&\left|T_{n,m,a_n}^{b_m,c_m}(f;x,y) - f(x,y)\right| \\
&\quad \leq T_{n,m,a_n}^{b_m,c_m}\left(\left|f(t_1,t_2) - f(x,y)\right|;x,y\right) \\
&\quad \leq LT_{n,m,a_n}^{b_m,c_m}\left(\frac{\|r-s\|}{(\|r\|+x+y)^{1/2}};x,y\right) \\
&\quad \leq \frac{L}{(x+y)^{1/2}}T_{n,m,a_n}^{b_m,c_m}\left(\|r-s\|;x,y\right),
\end{aligned} \tag{44}$$

where $r = (t_1, t_2)$ and $s = (x, y)$.

Using the Cauchy- Shwarz inequality and Lemma 2, the above inequality implies that

$$\begin{aligned}
&\left|T_{n,m,a_n}^{b_m,c_m}(f;x,y) - f(x,y)\right| \\
&\quad \leq \frac{L}{(x+y)^{1/2}}\left\{T_{n,m,a_n}^{b_m,c_m}\left(\|r-s\|^2,x,y\right)\right\}^{1/2} \\
&\quad \leq \frac{L}{(x+y)^{1/2}} \\
&\quad \cdot\left\{{}_xC_n^*\left((t_1-x)^2,x,y\right) +{}_y^* S_m^{b_m,c_m}\left((t_2-y)^2,x,y\right)\right\}^{1/2} \\
&\quad \leq \frac{L\delta_m^{1/2}(x,y)}{(x+y)^{1/2}}.
\end{aligned} \tag{45}$$

Thus, the result holds for $\gamma = 1$. Secondly, let $0 < \gamma < 1$. Then, for $f \in Lip_L(\gamma)$ and for each $x, y \in I_{ab}$, we get

$$\begin{aligned}
&\left|T_{n,m,a_n}^{b_m,c_m}(f;x,y) - f(x,y)\right| \\
&\quad \leq T_{n,m,a_n}^{b_m,c_m}\left(\left|f(t_1,t_2) - f(x,y)\right|;x,y\right) \\
&\quad \leq LT_{n,m,a_n}^{b_m,c_m}\left(\frac{\|r-s\|^\gamma}{(\|r\|+x+y)^{\gamma/2}};x,y\right) \\
&\quad \leq \frac{L}{(x+y)^{\gamma/2}}T_{n,m,a_n}^{b_m,c_m}\left(\|r-s\|^\gamma;x,y\right). \quad ,
\end{aligned} \tag{46}$$

Now, applying Holder's inequality with $u_1 = 2/\gamma, u_2 = 2/(2-\gamma)$, and Lemma 2, we get

$$\begin{aligned}
&\left|T_{n,m,a_n}^{b_m,c_m}(f;x,y) - f(x,y)\right| \\
&\quad \leq \frac{L}{(x+y)^{\gamma/2}}\left\{T_{n,m,a_n}^{b_m,c_m}\left(\|r-s\|^2,x,y\right)\right\}^{\gamma/2} \\
&\quad \leq \frac{L}{(x+y)^{\gamma/2}}\left\{{}_xC_n^*\left((t_1-x)^2,x,y\right) +{}_y^* S_m^{b_m,c_m}\left((t_2-y)^2,x,y\right)\right\}^{\gamma/2} \\
&\quad \leq \frac{L}{(x+y)^{1/2}}\left\{O\left(\frac{a_n}{n}\right)\sum_{i=1}^{2}x^i + O\left(\frac{1}{c_m}\right)\sum_{i=1}^{2}y^i)\right\}^{\gamma/2}.
\end{aligned} \tag{47}$$

which leads us to the required result. □

Theorem 9. *If $f(x,y)$ has continuous partial derivatives f_x' and f_y' and $\omega^{(1)}(f_x';\delta)$ and $\omega^{(2)}(f_y';\delta)$ denote the partial moduli of continuity of f_x' and f_y' respectively. Then we have estimate*

$$\begin{aligned}
&\left|T_{n,m,a_n}^{b_m,c_m}(f;x,y) - f(x,y)\right| \\
&\quad \leq \Lambda\frac{a_n}{2n} \\
&\qquad + \Upsilon\left(\left|\frac{b_m^2\mathscr{L}''(b_m y)}{c_m^2\mathscr{L}(b_m y)} - 1\right| y + \frac{2\mathscr{K}'(1)+1}{2c_m\mathscr{K}(1)}\right) \\
&\qquad + \omega^{(1)}\left(f_x';\delta_n\right)\left(1+\sqrt{\delta_n}\right) \\
&\qquad + \omega^{(2)}\left(f_x';\delta_m\right)\left(1+\sqrt{\delta_m}\right),
\end{aligned} \tag{48}$$

where Λ, Υ are the positive constants such that $|\partial f/\partial x| \leq \Lambda, |\partial f/\partial y| \leq \Upsilon$, $(x, y \in I_{ab})$.

Proof. From the mean value theorem we have

$$\begin{aligned}
&f(t_1,t_2) - f(x,y) \\
&\quad = f(t_1,y) - f(x,y) + f(t_1,t_2) - f(t_1,y) \\
&\quad = (t_1-x)\frac{\partial f(\eta,y)}{\partial x} + (t_2-y)\frac{\partial f(x,\zeta)}{\partial y} \\
&\quad = (t_1-x)\frac{\partial f(x,y)}{\partial x} \\
&\qquad + (t_1-x)\left(\frac{\partial f(\eta,y)}{\partial x} - \frac{\partial f(x,y)}{\partial x}\right) \\
&\qquad + (t_2-y)\frac{\partial f(x,y)}{\partial y} \\
&\qquad + (t_2-y)\left(\frac{\partial f(x,\zeta)}{\partial y} - \frac{\partial f(x,y)}{\partial y}\right),
\end{aligned} \tag{49}$$

where $x < \eta < t_1$ and $y < \zeta < t_2$. By using the above identity, we get

$$\begin{aligned}
&T_{n,m,a_n}^{b_m,c_m}(f;x,y) - f(x,y) \\
&\quad = T_{n,m,a_n}^{b_m,c_m}\left((t_1-x)\frac{\partial f(x,y)}{\partial x};x,y\right) \\
&\quad + T_{n,m,a_n}^{b_m,c_m}\left((t_1-x)\left(\frac{\partial f(\eta,y)}{\partial x} - \frac{\partial f(x,y)}{\partial x}\right);x,y\right) \\
&\quad + T_{n,m,a_n}^{b_m,c_m}\left((t_2-y)\frac{\partial f(x,y)}{\partial y};x,y\right) \\
&\quad + T_{n,m,a_n}^{b_m,c_m}\left((t_2-y)\left(\frac{\partial f(x,\zeta)}{\partial y} - \frac{\partial f(x,y)}{\partial y}\right);x,y\right).
\end{aligned} \tag{50}$$

Hence,

$$\begin{aligned}\left|T_{n,m,a_n}^{b_m,c_m}(f;x,y)-f(x,y)\right| &\leq \left|\frac{\partial f(x,y)}{\partial x}\right| \\ &\cdot\left|T_{n,m,a_n}^{b_m,c_m}((t_1-x);x,y)\right| + T_{n,m,a_n}^{b_m,c_m}\left(|t_1-x|\right. \\ &\cdot\left.\left|\frac{\partial f(\eta,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right|;x,y\right) + \left|\frac{\partial f(x,y)}{\partial y}\right| \\ &\cdot\left|T_{n,m,a_n}^{b_m,c_m}((t_2-y);x,y)\right| + + T_{n,m,a_n}^{b_m,c_m}\left(|t_2-y|\right. \\ &\cdot\left.\left|\frac{\partial f(x,\zeta)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right|;x,y\right) \\ &\leq \Lambda\left|T_{n,m,a_n}^{b_m,c_m}((t_1-x);x,y)\right| \\ &+\Upsilon\left|T_{n,m,a_n}^{b_m,c_m}((t_2-y);x,y)\right| + T_{n,m,a_n}^{b_m,c_m}\left(|t_1-x|\right. \\ &\cdot\left.\omega^{(1)}(f'_x;\delta_n)\left(\frac{|t_1-x|}{\delta_n}+1\right);x,y\right) \\ &+T_{n,m,a_n}^{b_m,c_m}\left(|t_2-y|\,\omega^{(1)}(f'_y;\delta_m)\right. \\ &\cdot\left.\left(\frac{|t_2-y|}{\delta_m}+1\right);x,y\right),\end{aligned} \tag{51}$$

Since $|\eta-x|<|t_1-x|$ and $|\zeta-y|<|t_2-y|$.

Using last inequalities, we have

$$\begin{aligned}&\left|T_{n,m,a_n}^{b_m,c_m}(f;x,y)-f(x,y)\right| \\ &\quad\leq \Lambda\left|T_{n,m,a_n}^{b_m,c_m}((t_1-x);x,y)\right| \\ &\qquad+\Upsilon\left|T_{n,m,a_n}^{b_m,c_m}((t_2-y);x,y)\right| \\ &\qquad+ + \omega^{(1)}(f'_x;\delta_{n_1})\,T_{n,m,a_n}^{b_m,c_m}(|t_1-x|;x,y) \\ &\qquad+\frac{\omega^{(1)}(f'_x;\delta_{n_1})}{\delta_{n_1}}T_{n,m,a_n}^{b_m,c_m}(|t_1-x|^2;x,y) \\ &\qquad+\omega^{(2)}(f'_y;\delta_m)\,T_{n,m,a_n}^{b_m,c_m}(|t_2-y|;x,y) \\ &\qquad+\frac{\omega^{(2)}(f'_y;\delta_{n_2})}{\delta_m}T_{n,m,a_n}^{b_m,c_m}(|t_2-y|^2;x,y).\end{aligned} \tag{52}$$

Now, applying the Cauchy-Schwarz inequality

$$\begin{aligned}&\left|T_{n,m,a_n}^{b_m,c_m}(f;x,y)-f(x,y)\right| \\ &\quad\leq \Lambda\left|T_{n,m,a_n}^{b_m,c_m}((t_1-x);x,y)\right| \\ &\qquad+\Upsilon\left|T_{n,m,a_n}^{b_m,c_m}((t_2-y);x,y)\right| \\ &\qquad+ + \omega^{(1)}(f'_x;\delta_n)\left\{{}_xC_n^*\left((t_1-x)^2;x,y\right)\right\}^{1/2} \\ &\qquad+\frac{\omega^{(1)}(f'_x;\delta_n)}{\delta_n}\,{}_xC_n^*\left((t_1-x)^2;x,y\right) \\ &\qquad+\omega^{(2)}(f'_y;\delta_m)\left\{{}_y^*S_m^{b_m,c_m}\left((t_2-y)^2;x,y\right)\right\}^{1/2} \\ &\qquad+\frac{\omega^{(2)}(f'_y;\delta_m)}{\delta_m}\,{}_y^*S_m^{b_m,c_m}\left((t_2-y)^2;x,y\right).\end{aligned} \tag{53}$$

Now choosing $\delta_n=\delta_n(x)$ and $\delta_m=\delta_m(y)$, we have

$$\begin{aligned}&\left|T_{n,m,a_n}^{b_m,c_m}(f;x,y)-f(x,y)\right| \\ &\quad\leq \Lambda\frac{a_n}{2n} \\ &\qquad+\Upsilon\left(\left|\frac{b_m^2\mathcal{L}''(b_my)}{c_m^2\mathcal{L}(b_my)}-1\right|y+\frac{2\mathcal{K}'(1)+1}{2c_m\mathcal{K}(1)}\right) \\ &\qquad+\omega^{(1)}(f'_x;\delta_n)\left(1+\sqrt{\delta_n}\right) \\ &\qquad+\omega^{(2)}(f'_y;\delta_m)\left(1+\sqrt{\delta_m}\right).\end{aligned} \tag{54}$$

This completes the proof. □

4. Weighted Approximation Properties

The weighted Korovkin-type theorems are used for the purpose of this study, which are previously proved by Gadjiev [12, 13]. Therefore we need to introduce the notations of [13]. Let $\mathbb{R}_+^2=\{(x,y):x\geq 0,\ y\geq 0\}$ and $B_\rho(\mathbb{R}_+^2)$ be the space of all functions having the property $|f(x,y)|\leq M_f\rho(x,y)$, where $(x,y)\in\mathbb{R}_+^2$ and M_f is a constant depending on function only. By $C_\rho(\mathbb{R}_+^2)$ we denote the subspace of all continuous functions belonging to $B_\rho(\mathbb{R}_+^2)$. It is clear that $C_\rho(\mathbb{R}_+^2)$ is a linear normed space with the norm $\|f\|_\rho=\sup_{(x,y)\in\mathbb{R}_+^2}(|f(x,y)|/\rho(x,y))$. Also, let $C_\rho^*(\mathbb{R}_+^2)$ be the subspace of all functions $f\in C_\rho(\mathbb{R}_+^2)$, for which $\lim_{|(x,y)|\longrightarrow\infty}(f(x,y)/(1+|(x,y)|^2))=k_f<\infty$, where $|(x,y)|=\sqrt{x^2+y^2}$.

Theorem 10. *Let f belong to $C_\rho^*(\mathbb{R}_+^2)$ and $|f(x,y)|\leq M_f\rho(x,y)$. Then*

$$\lim_{n,m\longrightarrow\infty}\left\|T_{n,m,a_n}^{b_m,c_m}(f;x,y)-f\right\|_{C_\rho^*(R_+^2)}=0 \tag{55}$$

if and only if

(i) $\|T_{n,m,a_n}^{b_m,c_m}(1;x,y)-1\|_{C_\rho^*(\mathbb{R}_+^2)}=0$;

(ii) $\|T_{n,m,a_n}^{b_m,c_m}(e_{10};x,y)-x\|_{C_\rho^*(\mathbb{R}_+^2)}=0$;

(iii) $\|T_{n,m,a_n}^{b_m,c_m}(e_{01};x,y)-y\|_{C_\rho^*(\mathbb{R}_+^2)}=0$;

(iv) $\|T_{n,m,a_n}^{b_m,c_m}(e_{20};x,y)-x^2\|_{C_\rho^*(\mathbb{R}_+^2)}=0$;

(v) $\|T^{b_m,c_m}_{n,m,a_n}(e_{02};x,y)-y^2\|_{C^*_\rho(\mathbb{R}^2_+)}=0$;

as $n,m\longrightarrow\infty$ *for* $(x,y)\in I_{ab}$.

Proof. The necessity part is trivial; then we need only to prove sufficiency. Let $(x,y),(s,t)\in I_{ab}$, and $f\in C^*_\rho(\mathbb{R}^2_+)$. Since for each $f\in C^*_\rho(\mathbb{R}^2_+)$ is uniformly on I_{ab}, for each $\epsilon>0$ there exists some $\delta>0$, such that for each $(s,t)\in I_{ab}$ with $\sqrt{(s-x)^2+(t-y)^2}<\delta$ implies $|f(s,t)-f(x,y)|<\epsilon$. Now let $(x,y)\in I_{ab}$ and $(s,t)\in\mathbb{R}^2_+$ and let (x_1,y_1) be an arbitrary boundary point of I_{ab} such that $0\le x_1\le a$, $0\le y_1\le b$. Since f is continuous on the boundary points also, then for each $\epsilon>0$ there exists $\delta>0$ such that $\sqrt{(s-x)^2+(t-y)^2}<\delta$ implies

$$\begin{aligned}|f(s,t)-f(x,y)|&\le|f(s,t)-f(x_1,y_1)|\\&\quad+|f(x_1,y_1)-f(x,y)|<\epsilon.\end{aligned}\tag{56}$$

On the other hand, if $\sqrt{(s-x)^2+(t-y)^2}\ge\delta$, we have

$$|f(s,t)-f(x,y)|\le C\left(\frac{(s-x)^2+(t-y)^2}{\delta^2}\right),\tag{57}$$

where $C>0$ is constant and $(x,y)\in I_{ab}$ and $(s,t)\in\mathbb{R}^2_+$. So, we get the following inequality:

$$|f(s,t)-f(x,y)|\le\epsilon+C\left(\frac{(s-x)^2+(t-y)^2}{\delta^2}\right),\tag{58}$$

for $(x,y)\in I_{ab},(s,t)\in\mathbb{R}^2_+$. Now applying the operators $T^{b_m,c_m}_{n,m,a_n}(f;x,y)$ in the last inequality and taking relations (i)-(v) of Theorem 10, sufficiency is obtained easily.

Now we estimate the rate of approximation of the operators T^{b_m,c_m}_{n,m,a_n} in the terms of the weighted modulus of continuity $\Omega(f;\delta_n,\delta_m)$ (see [14]) defined by

$$\begin{aligned}&\Omega(f;\delta_n,\delta_m)\\&=\sup_{(x,y)\in\mathbb{R}^2_+}\ \sup_{|h_1|\le\delta_1,|h_2|\le\delta_2}\frac{|f(x+h_1,y+h_2)-f(x,y)|}{\rho(x,y)\rho(h1,h2)},\\&\qquad f\in C^*_\rho(\mathbb{R}^+).\end{aligned}\tag{59}$$

and it satisfied the following properties:

$$\begin{aligned}&|f(t,s)-f(x,y)|\\&\quad\le 8\Omega(f;\delta_n,\delta_m)(1+x^2+y^2)\,g(t,x)\,g(t,y)\end{aligned}\tag{60}$$

where $g(t,x)=((1+|t-x|/\delta_n)(1+(t-x)^2))$ and $g(s,y)=((1+|s-y|/\delta_m)(1+(s-y)^2))$. □

Theorem 11. *For each $f\in C^*_\rho(\mathbb{R}^+)$, there exists a positive constant B independent of a_n,b_m, and c_m such that the inequality*

$$\left\|T^{b_m,c_m}_{n,m,a_n}(f;x,y)-f\right\|_{\rho^3}\le B\Omega(f;\delta_n,\delta_m),\tag{61}$$

is satisfied for a sufficiently large n, m, where $\delta_n=a_n/n$ and $\delta_m=1/c_m$.

Proof. By the linearity and monotonicity of T^{b_m,c_m}_{n,m,a_n} applied to inequality (60) we obtain

$$\begin{aligned}&\left|T^{b_m,c_m}_{n,m,a_n}(f;x,y)(f;x,y)-f(x,y)\right|\le 8\\&\quad\cdot\frac{nc_m}{a_n\mathcal{K}(1)\mathcal{L}(b_my)}\Omega(f;\delta_n,\delta_m)(1+x^2+y^2)\\&\quad\times\sum_{k=0}^{n}p_{n,k}\left(\frac{x}{a_n}\right)\int_{(k/n)a_n}^{((k+1)/n)a_n}g(x,t)\,dt\times\sum_{j=0}^{\infty}p_j(b_my)\\&\quad\cdot\int_{j/c_m}^{(j+1)/c_m}g(s,y)\,ds.\end{aligned}\tag{62}$$

$$\begin{aligned}&\le 8\Omega(f;\delta_n,\delta_m)(1+x^2+y^2)\Bigg\{1+\frac{n}{\delta_na_n}\\&\quad\cdot\sum_{k=0}^{n}p_{n,k}\left(\frac{x}{a_n}\right)\int_{(k/n)a_n}^{((k+1)/n)a_n}|t-x|\,dt+\frac{n}{\delta_na_n}\\&\quad\cdot\sum_{k=0}^{n}p_{n,k}\left(\frac{x}{a_n}\right)\int_{(k/n)a_n}^{((k+1)/n)a_n}(t-x)^2\,dt+\frac{n}{\delta_na_n}\\&\quad\cdot\sum_{k=0}^{n}p_{n,k}\left(\frac{x}{a_n}\right)\int_{(k/n)a_n}^{((k+1)/n)a_n}|t-x|(t-x)^2\,dt\Bigg\}\\&\quad\times\Bigg\{1+\frac{c_m}{\delta_m\mathcal{K}(1)\mathcal{L}(b_my)}\sum_{j=0}^{\infty}p_j(b_my)\\&\quad\cdot\int_{j/c_m}^{(j+1)/c_m}|s-y|\,ds+\frac{c_m}{\delta_m\mathcal{K}(1)\mathcal{L}(b_my)}\\&\quad\cdot\sum_{j=0}^{\infty}p_j(b_my)\int_{j/c_m}^{(j+1)/c_m}|s-y|(s-y)^2\,ds\\&\quad+\frac{c_m}{\mathcal{K}(1)\mathcal{L}(b_my)}\sum_{j=0}^{\infty}p_j(b_my)\\&\quad\cdot\int_{j/c_m}^{(j+1)/c_m}(s-y)^2\,ds\Bigg\}\end{aligned}\tag{63}$$

where $g(x,t)=((1+|t-x|/\delta_n)(1+(t-x)^2))$ and $g(s,y)=((1+|s-y|/\delta_m)(1+(s-y)^2))$.

Applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\left|T^{b_m,c_m}_{n,m,a_n}(f;x,y)-f(x,y)\right|\\&\quad\le 8\Omega(f;\delta_n,\delta_m)(1+x^2+y^2)\end{aligned}$$

$$\times\left\{1+\frac{\sqrt{\Delta_1}}{\delta_n}+\delta_1+\frac{\sqrt{\Delta_1\Delta_2}}{\delta_n}\right\}$$
$$\times\left\{1+\frac{\sqrt{\Delta_1^*}}{\delta_n}+\delta_1+\frac{\sqrt{\Delta_1^*\Delta_2^*}}{\delta_n}\right\}. \quad (64)$$

where $\Delta_1, \Delta_1^*\Delta_2$, and Δ_2^* are defined by Lemma 2.

Combining (64) and all identities in Lemma 2, we obtain

$$\begin{aligned}\left|T_{n,m,a_n}^{b_m,c_m}(f;x,y)-f(x,y)\right| &\le 8\Omega(f;\delta_n,\delta_m)\left(1+x^2\right.\\ &\left.+y^2\right)\times\left\{1+\frac{1}{\delta_n}\sqrt{O\left(\frac{a_n}{n}\right)\sum_{i=1}^{2}x^i}\right.\\ &\left.+O\left(\frac{a_n}{n}\right)\sum_{i=1}^{2}x^i\frac{1}{\delta_n}\sqrt{O\left(\frac{a_n}{n}\right)\sum_{i=1}^{2}x^iO\left(\frac{a_n}{n}\right)\sum_{i=1}^{4}x^i}\right\}\\ &\times\left\{1+\frac{1}{\delta_m}\sqrt{O\left(\frac{1}{c_m}\right)\left(\sum_{i=1}^{2}y^i\right)}\right.\\ &+O\left(\frac{1}{c_m}\right)\left(\sum_{i=1}^{2}y^i\right)\\ &\left.+\frac{1}{\delta_m}\sqrt{O\left(\frac{1}{c_m}\right)\left(\sum_{i=1}^{2}y^i\right)O\left(\frac{1}{c_m}\right)\left(\sum_{i=1}^{4}y^i\right)}\right\}\end{aligned} \quad (65)$$

Choosing $\delta_n = a_n/n$ and $\delta_m = 1/c_m$, and for sufficiently large value of n and m, we obtain

$$\left\|T_{n,m,a_n}^{b_m,c_m}(f;x,y)-f\right\|_{\rho^3}\le B\Omega(f;\delta_n,\delta_m), \quad (66)$$

where B is a constant independent of a_n, b_m, and c_m. □

5. Conclusion

We studied a new sequence generalization of the Szasz-Kantorovich-Chlodowsky type operators defined by means of the Brenke type polynomials defined by (9). This type of modification enables better error estimation for a certain function in comparison to the Szász-Kantorovich-Chlodowsky operators and Szasz-Chlodowsky-type operators. We find the rate of convergence using weighted Korovkin-type theorem. We give some inequalities for these operators means of partial and full modulus of continuity and also obtain a Lipschitz type theorem. At the end, we mentioned results on the weighted modulus of continuity due to Ispir for the operators $T_{n,m,a_n}^{b_m,c_m}$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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