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Research Article

Global Energy Solution to the Schrödinger Equation Coupled with the Chern-Simons Gauge and Neutral Field

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We study the Cauchy problem of the Chern-Simons-Schrödinger equations with a neutral field, under the Coulomb gauge condition, in energy space $H^1(\mathbb{R}^2)$. We prove the uniqueness of a solution by using the Gagliardo-Nirenberg inequality with the specific constant. To obtain a global solution, we show the conservation of total energy and find a bound for the nondefinite term.

1. Introduction

In this paper, we are interested in the Cauchy problem of the Chern-Simons-Schrödinger equations coupled with a neutral field (CSSn) in $\mathbb{R}^{1+2}$:

$$iD_0\psi + D_jD_j\psi = |\psi|^2\psi + 2N\psi, \tag{1}$$

$$\partial_{00}N - \Delta N + N = -2|\psi|^2, \tag{2}$$

$$\partial_0 A_1 - \partial_1 A_0 = 2\,\mathrm{Im}\left(\overline{\psi}D_2\psi\right), \tag{3}$$

$$\partial_0 A_2 - \partial_2 A_0 = -2\,\mathrm{Im}\left(\overline{\psi}D_1\psi\right), \tag{4}$$

$$\partial_1 A_2 - \partial_2 A_1 = |\psi|^2. \tag{5}$$

Here, $\psi(t,x) : \mathbb{R}^{1+2} \to \mathbb{C}$ is the matter field, $N(t,x) : \mathbb{R}^{1+2} \to \mathbb{R}$ is the neutral field, and $A_\mu(t,x) : \mathbb{R}^{1+2} \to \mathbb{R}$ is the gauge field. $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative, $i = \sqrt{-1}$, $\partial_0 = \partial_t$, $\partial_j = \partial_{x_j}$, and $\Delta = \partial_j\partial_j$. We use notation $\mathbf{A} = (A_0, A_j) = (A_0, A_1, A_2)$. From now on, Latin indices are used to denote $1,2$ and the summation convention will be used for summing over repeated indices.

The CSSn system exhibits both conservation of the charge,

$$Q(t) \coloneqq \|\psi(t,\cdot)\|_{L^2} = Q(0), \tag{6}$$

and conservation of the total energy

$$\begin{aligned} E(t) \coloneqq{}& 2\sum_{j=1,2}\left\|D_j\psi(t,\cdot)\right\|_{L^2}^2 + \|\nabla N(t,\cdot)\|_{L^2}^2 \\ &+ \left\|\partial_t N(t,\cdot)\right\|_{L^2}^2 + \|N(t,\cdot)\|_{L^2}^2 + \|\psi(t,\cdot)\|_{L^4}^4 \\ &+ 4\int_{\mathbb{R}^2} N|\psi|^2(t,x)\,dx = E(0). \end{aligned} \tag{7}$$

The CSSn system is invariant under the following gauge transformations:

$$\begin{aligned} \psi &\longrightarrow \psi e^{i\chi}, \\ N &\longrightarrow N, \\ A_\mu &\longrightarrow A_\mu + \partial_\mu\chi, \end{aligned} \tag{8}$$

where $\chi : \mathbb{R}^{1+2} \to \mathbb{R}$ is a smooth function. Therefore, a solution to the CSSn system is formed by a class of gauge equivalent pairs $(\psi, N, \mathbf{A})$. In this paper, we fix the gauge by adopting the Coulomb gauge condition $\partial_j A_j = 0$, which provides elliptic features for gauge fields $\mathbf{A}$. Under the

Coulomb gauge condition, the Cauchy problem of the CSSn system is reformulated as follows:

$$i\partial_t\psi + \Delta\psi = -A_0\psi + A_j^2\psi + 2iA_j\partial_j\psi + |\psi|^2\psi + 2N\psi, \tag{9}$$

$$\partial_{tt}N - \Delta N + N = -2|\psi|^2, \tag{10}$$

$$\Delta A_0 = 2\,\mathrm{Im}\left(\partial_2\overline{\psi}\partial_1\psi - \partial_1\overline{\psi}\partial_2\psi\right) + 2\partial_2\left(A_1|\psi|^2\right) - 2\partial_1\left(A_2|\psi|^2\right), \tag{11}$$

$$\Delta A_1 = -\partial_2\left(|\psi|^2\right), \tag{12}$$

$$\Delta A_2 = \partial_1\left(|\psi|^2\right), \tag{13}$$

with the initial data $\psi(0,x) = \psi_0(x)$, $N(0,x) = n_0(x)$, $\partial_t N(0,x) = n_1(x)$. Note that ψ, N are dynamical variables and $\mathbf{A}$ are determined by ψ through (11)–(13).

The CSSn system is derived from the nonrelativistic Maxwell-Chern-Simons model in [1] by regarding Maxwell term in the Lagrangian as zero. Compared with the Chern-Simons-Schrödinger (CSS) system which comes from the nonrelativistic Maxwell-Chern-Simons model by taking the Chern-Simons limit in [1], the CSSn system has the interaction between the matter field ψ and the neutral field N. The CSS system reads as

$$\begin{aligned} iD_0\psi + D_jD_j\psi &= -|\psi|^2\psi, \\ \partial_0A_1 - \partial_1A_0 &= 2\,\mathrm{Im}\left(\overline{\psi}D_2\psi\right), \\ \partial_0A_2 - \partial_2A_0 &= -2\,\mathrm{Im}\left(\overline{\psi}D_1\psi\right), \\ \partial_1A_2 - \partial_2A_1 &= |\psi|^2 \end{aligned} \tag{14}$$

and has conservation of the total energy

$$E(t) \coloneqq 2\sum_{j=1,2}\left\|D_j\psi(t,\cdot)\right\|_{L^2}^2 - \left\|\psi(t,\cdot)\right\|_{L^4}^4 = E(0). \tag{14}$$

We remark that $\|\psi(t,\cdot)\|_{L^4}^4$ has opposite sign in (7) compared with (14). In fact, this difference causes different global behavior of solution. The local well-posedness of the CSS system in H^2, H^1 was shown in [2, 3], respectively. We can prove the existence of a local solution of the CSSn system by applying similar argument. On the other hands, due to the nondefiniteness of total energy, the CSS system has a finite-time blow-up solution constructed in [2, 4]. The CSSn system also has difficulty with nondefiniteness of $N|\psi|^2$ in the total energy, but we could obtain a global solution by controlling it with H^1-norm.

Considering conservation of the energy (7), it is natural to study the Cauchy problem with the initial data $\psi_0, n_0, n_1 \in H^1\times H^1\times L^2$. Our first result is concerned with a local solution in energy space.

Theorem 1. *For the initial data $(\psi_0, n_0, n_1) \in H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)$, there are $T>0$ and a unique local-in-time solution $(\psi, N, \mathbf{A})$ to (9)–(13) such that*

$$\begin{aligned} \psi &\in L^\infty\left([0,T);H^1\left(\mathbb{R}^2\right)\right)\cap C\left([0,T);L^2\left(\mathbb{R}^2\right)\right), \\ N &\in L^\infty\left([0,T);H^1\left(\mathbb{R}^2\right)\right)\cap C\left([0,T);L^2\left(\mathbb{R}^2\right)\right), \\ \partial_tN &\in L^\infty\left([0,T);L^2\left(\mathbb{R}^2\right)\right), \\ A_0 &\in L^\infty\left([0,T);L^q\left(\mathbb{R}^2\right)\cap L^\infty\left(\mathbb{R}^2\right)\cap \dot{H}^1\left(\mathbb{R}^2\right)\right), \\ A_j &\in L^\infty\left([0,T);L^q\left(\mathbb{R}^2\right)\cap \dot{H}^1\left(\mathbb{R}^2\right)\right), \end{aligned} \tag{15}$$

where $2<q<\infty$. Moreover, the solution has continuous dependence on initial data.

Our second result is concerned with a global solution in energy space.

Theorem 2. *For the initial data $(\psi_0, n_0, n_1) \in H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)$, there exists a unique global solution $(\psi, N, \mathbf{A})$ to (9)–(13) such that*

$$\begin{aligned} \psi &\in L^\infty\left([0,\infty);H^1\left(\mathbb{R}^2\right)\right)\cap C\left([0,\infty);L^2\left(\mathbb{R}^2\right)\right), \\ N &\in L^\infty\left([0,\infty);H^1\left(\mathbb{R}^2\right)\right)\cap C\left([0,\infty);L^2\left(\mathbb{R}^2\right)\right), \\ \partial_tN &\in L^\infty\left([0,\infty);L^2\left(\mathbb{R}^2\right)\right), \\ A_0 &\in L^\infty\left([0,\infty);L^q\left(\mathbb{R}^2\right)\cap L^\infty\left(\mathbb{R}^2\right)\cap \dot{H}^1\left(\mathbb{R}^2\right)\right), \\ A_j &\in L^\infty\left([0,\infty);L^q\left(\mathbb{R}^2\right)\cap \dot{H}^1\left(\mathbb{R}^2\right)\right), \end{aligned} \tag{16}$$

where $2<q<\infty$. Moreover, the solution has continuous dependence on initial data.

Note that, considering (11)–(13), A_j can be determined by ψ as

$$A_j = \frac{(-1)^{j+1}}{2\pi}\left(\frac{x_{j'}}{|x|^2} * |\psi|^2\right), \tag{17}$$

and then A_0 can be determined as

$$A_0 = \sum_{j=1}^{2}\frac{(-1)^{j+1}}{\pi}\left(\frac{x_j}{|x|^2} * \mathrm{Im}\left(\overline{\psi}\partial_{j'}\psi\right)\right) + \sum_{j=1}^{2}\frac{(-1)^{j+1}}{\pi}\left(\frac{x_j}{|x|^2} * \left(A_{j'}|\psi|^2\right)\right), \tag{18}$$

where $j'=2$ if $j=1$, and $j'=1$ if $j=2$. We present estimates for $\mathbf{A}$ and refer to [3, 5] for proof.

Proposition 3. *Let $\psi \in H^1(\mathbb{R}^2)$ and let $\mathbf{A}$ be the solution of (11)-(13). Then, we have, for $2 < q < \infty$,*

$$\begin{aligned}\|A_j\|_{L^q} &\lesssim \|\psi\|_{L^2}^{1+2/q} \|\nabla\psi\|_{L^2}^{1-2/q},\\ \|\nabla A_j\|_{L^2} &\lesssim \|\psi\|_{L^2} \|\nabla\psi\|_{L^2},\\ \|A_0\|_{L^q} &\lesssim \left(1+\|\psi\|_{L^2}^2\right)\|\psi\|_{L^2}^{2/q}\|\nabla\psi\|_{L^2}^{2-2/q},\\ \|A_0\|_{L^\infty}+\|\nabla A_0\|_{L^2} &\lesssim \left(1+\|\psi\|_{L^2}^2\right)\|\nabla\psi\|_{L^2}^2.\end{aligned} \tag{19}$$

We will prove Theorems 1 and 2 in Sections 2 and 3, respectively. We conclude this section by giving a few notations. We use the standard Sobolev spaces $H^s(\mathbb{R}^2)$ with the norm $\|f\|_{H^s} = \|(1-\Delta)^{s/2} f\|_{L^2}$. We will use c, C to denote various constants. When we are interested in local solutions, we may assume that $T \leq 1$. Thus we shall replace smooth function of $T, C(T)$ by C. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$.

2. Proof of Theorem 1

In this section we address the local well-posedness of solution to (9)–(13). We note that if we remove the gauge fields and the term $|\psi|^2\psi$ from the CSSn system, it is the same as the Klein-Gordon-Schödinger system with Yukawa coupling (KGS). There are many studies on the Cauchy problem of the KGS system in the Sobolev spaces H^s [6–9]. Moreover, if we ignore the interaction with the neutral field N which does not cause any difficulty in obtaining a local solution, a local solution for the CSSn system can be obtained in a similar way to the CSS system. We could obtain a local regular solution by referring to [2, 8] and then construct a local energy solution by using the compactness argument introduced in [2, 3, 5, 6]. In other words, a local H^1-solution is constructed by the limit of a sequence of more smooth solutions and it satisfies CSSn system in the distribution sense. For the proof, we follow the same argument as in [2]. So we omit the detail of the local existence here. Since the compactness argument does not guarantee the uniqueness and the continuous dependence on initial data of a local solution, we would rather contribute this section to show the uniqueness and the continuous dependence on initial data of a local solution.

Theorem 4. *Let $(\psi, N, \mathbf{A})$ and $(\widetilde{\psi}, \widetilde{N}, \widetilde{\mathbf{A}})$ be solutions to (9)–(13) on $(0,T)\times\mathbb{R}^2$ in the distribution sense with the same initial data $(\psi_0, n_0, n_1) \in H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)\times L^2(\mathbb{R}^2)$ satisfying*

$$\begin{aligned}&\psi, \widetilde{\psi}, N, \widetilde{N} \in L^\infty\left([0,T); H^1\left(\mathbb{R}^2\right)\right)\\ &\quad \cap C\left([0,T); L^2\left(\mathbb{R}^2\right)\right),\\ &\partial_t N, \partial_t \widetilde{N} \in L^\infty\left([0,T); L^2\left(\mathbb{R}^2\right)\right),\\ &\|\psi\|_{L_T^\infty H^1}, \|\widetilde{\psi}\|_{L_T^\infty H^1}, \|N\|_{L_T^\infty H^1}, \left\|\widetilde{N}\right\|_{L_T^\infty H^1}, \|\partial_t N\|_{L_T^\infty L^2},\\ &\quad \left\|\partial_t \widetilde{N}\right\|_{L_T^\infty L^2} \leq M,\end{aligned} \tag{20}$$

for some $M > 0$. Then, we have

$$\begin{aligned}\left\|\left(\psi-\widetilde{\psi}\right)(t,\cdot)\right\|_{L^2} &= 0,\\ \left\|\left(N-\widetilde{N}\right)(t,\cdot)\right\|_{H^1} &= 0,\end{aligned} \tag{21}$$

for $0 \leq t \leq T$. Moreover, the solution depends on initial data continuously.

Before beginning the proof, we gather lemmas used for the proof of Theorem 4. We use the following L^p–$L^{p'}$ estimate proved in [10] which plays an important role to control the difference of solutions. It was used in [6] for the uniqueness of the KGS system.

Lemma 5. *Let $f(t,x) : \mathbb{R}^{1+2} \to \mathbb{R}$ be a solution to*

$$\begin{aligned}\partial_{tt} f - \Delta f + f = F, \quad (t,x) \in \mathbb{R}^{1+2},\\ f(0,x) = 0,\end{aligned} \tag{22}$$

$$\partial_t f(0,x) = 0, \tag{23}$$

and $T(t)$ be the Klein-Gordon propagator. Then, we have

$$f(t,x) = \int_0^t T(t-s) F(s)\, ds, \tag{24}$$

and

$$\|T(t) F\|_{L^6(\mathbb{R}^2)} \lesssim |t|^{-1/3} \|F\|_{L^{6/5}(\mathbb{R}^2)}. \tag{25}$$

The Hardy-Littlewood-Sobolev inequality is also used to control the difference of solutions. For the proof, we refer to Theorem 6.1.3 in [11].

Lemma 6. *Let I_1 be the operator defined by*

$$I_1 f(x) = \int_{\mathbb{R}^2} \frac{f(y)}{|x-y|}\, dy. \tag{26}$$

If $1/q = 1/p - 1/2$, $1 < p < 2$, then we have

$$\|I_1 f\|_{L^q(\mathbb{R}^2)} \leq \sqrt{2}\pi q^{1/2} \|f\|_{L^p(\mathbb{R}^2)}. \tag{27}$$

The following Gagliardo-Nirenberg inequality with the explicit constant depending on q is used to show the uniqueness. It was proved in [12, 13] and used in [3, 5, 12, 13] to show the uniqueness of the nonlinear Schrödinger equations.

Lemma 7. *For $2 \leq q < \infty$, we have*

$$\|f\|_{L^q(\mathbb{R}^2)} \leq (4\pi)^{1/q-1/2}\left(\frac{q}{2}\right)^{1/2} \|f\|_{L^2(\mathbb{R}^2)}^{2/q} \|\nabla f\|_{L^2(\mathbb{R}^2)}^{1-2/q}. \tag{28}$$

We need the following Grönwall type inequality.

Lemma 8. *Let $f(t)$ be a continuous nonnegative function defined on $I = [0,a)$ and has zero only at 0. Suppose that f satisfies*

$$f(t) \leq \alpha\left(\int_0^t f(s)\, ds\right)^{1-2/q} + \beta\int_0^t f(s)\, ds \qquad \text{for } t \in I, \tag{29}$$

where α, $\beta > 0$ and $q > 2$. Then we have

$$\int_0^t f(s)\,ds \le \left(\frac{\alpha e^{2\beta t/q} - \alpha}{\beta}\right)^{q/2} \quad \text{for } t \in I. \tag{30}$$

Proof. Define

$$h(t) = \frac{q}{2}\left(\int_0^t f(s)\,ds\right)^{2/q} + \frac{q\alpha}{2\beta}. \tag{31}$$

Then, the assumption (29) implies

$$\begin{aligned} h'(t) &= \left(\int_0^t f(s)\,ds\right)^{2/q-1} f(t) \\ &\le \alpha + \beta\left(\int_0^t f(s)\,ds\right)^{2/q} = \frac{2\beta}{q} h(t), \end{aligned} \tag{32}$$

and the standard Grönwall' inequality gives

$$h(t) \le h(0)\, e^{2\beta t/q} = \frac{q\alpha}{2\beta} e^{2\beta t/q}. \tag{33}$$

Considering the definition of $h(t)$ in the above inequality, we have (30). □

We also need the following inequality to show that the solution is continuously dependent on initial data. We refer to [14].

Lemma 9. *Let $q > 1$ and $a, b > 0$. Let $f : [0,\infty) \to [0,\infty)$ satisfy*

$$f(t) \le a + b\int_0^t f^{1-1/q}(s)\,ds \tag{34}$$

for all $t \ge 0$. Then, $f(t) \le (a^{1/q} + bq^{-1}t)^q$ for all $t \ge 0$.

Now we are ready to prove Theorem 4. The basic rationale is borrowed from [3, 5, 15]. Let $(\psi, N, \mathbf{A})$ and $(\widetilde{\psi}, \widetilde{N}, \widetilde{\mathbf{A}})$ be solutions of (9)–(13) with the same initial data. If we set

$$u = \psi - \widetilde{\psi} \quad \text{and} \quad v = N - \widetilde{N}, \tag{35}$$

then the equations for u and v satisfy

$$\begin{aligned} i\partial_t u + \Delta u &= \left(\widetilde{A}_0 - A_0\right)\psi - \widetilde{A}_0 u \\ &\quad + 2i\left(A_j - \widetilde{A}_j\right)\partial_j\psi + 2i\widetilde{A}_j\partial_j u \\ &\quad + \left(A_j^2 - \widetilde{A}_j^2\right)\psi + \widetilde{A}_j^2 u \\ &\quad + \left(|\psi|^2 - |\widetilde{\psi}|^2\right)\psi + |\widetilde{\psi}|^2 u + 2v\psi \\ &\quad + 2\widetilde{N}u, \end{aligned} \tag{36}$$

$$\partial_{tt} v - \Delta v + v = -2\left(|\psi|^2 - |\widetilde{\psi}|^2\right), \tag{37}$$

where

$$\begin{aligned} &u, v \in L^\infty\left([0,T); H^1\left(\mathbb{R}^2\right)\right) \cap C\left([0,T); L^2\left(\mathbb{R}^2\right)\right), \\ &\partial_t v \in L^\infty\left([0,T); L^2\left(\mathbb{R}^2\right)\right). \end{aligned} \tag{38}$$

First of all, we will derive, for $q > 2$,

$$\begin{aligned} \sup_{[0,t]} \|u\|_{L^2}^2 &\le \alpha\left(\int_0^t \sup_{[0,s]} \|u\|_{L^2}^2\,ds\right)^{1-2/q} \\ &\quad + \beta\int_0^t \sup_{[0,s]} \|u\|_{L^2}^2\,ds, \end{aligned} \tag{39}$$

where

$$\alpha = T^{2/q} q M^2\left(1 + M^{4/q} + M^{2+4/q}\right) \quad \text{and} \quad \beta = M^2. \tag{40}$$

Once we obtain (39), considering $\|u(0,\cdot)\|_{L^2} = 0$, Lemma 8 gives

$$\begin{aligned} &\int_0^t \sup_{[0,s]} \|u\|_{L^2}^2\,ds \\ &\quad \le T\left(1 + M^{4/q} + M^{2+4/q}\right)^{q/2}\left[q\left(e^{2M^2T/q} - 1\right)\right]^{q/2}, \end{aligned} \tag{41}$$

for $0 \le t \le T$. We note that

$$\lim_{t\to 0^+} \frac{e^{bt} - 1}{t} = b. \tag{42}$$

Let us take the time interval $T' \le T$ satisfying $(2 + M^2)(2M^2T') < 1/2$. Letting $q \to \infty$ we have that $\|u(t,\cdot)\|_{L^2} = 0$ for $0 \le t \le T'$. Using this argument repeatedly, we conclude that $\|u(t,\cdot)\|_{L^2} = 0$ for $0 \le t \le T$.

To derive the estimate (39), multiplying (36) by $\overline{u}$ and integrating the imaginary part on $[0,t] \times \mathbb{R}^2$, we have

$$\begin{aligned} \|u(t)\|_{L^2}^2 = \int_0^t \int_{\mathbb{R}^2} &\underbrace{2\left(\widetilde{A}_0 - A_0\right)\operatorname{Im}\left(\psi\overline{u}\right)}_{(\mathrm{I})} \\ &+ \underbrace{4\left(A_j - \widetilde{A}_j\right)\operatorname{Re}\left(\partial_j\psi\overline{u}\right)}_{(\mathrm{II})} + \underbrace{2\widetilde{A}_j\partial_j|u|^2}_{(\mathrm{III})} \\ &+ \underbrace{2\left(A_j^{\,2} - \widetilde{A}_j^2\right)\operatorname{Im}\left(\psi\overline{u}\right)}_{(\mathrm{IV})} \\ &+ \underbrace{2\left(|\psi|^2 - |\widetilde{\psi}|^2\right)\operatorname{Im}\left(\psi\overline{u}\right)}_{(\mathrm{V})} \\ &+ \underbrace{4v\operatorname{Im}\left(\psi\overline{u}\right)}_{(\mathrm{VI})}\,dx\,ds. \end{aligned} \tag{43}$$

Considering $\partial_j\widetilde{A}_j = 0$, we have (III) $= 0$. Except for the integral (VI), the right-hand side of (43) is bounded, by adopting the same manner described in [3, 5], as follows.

$$\begin{aligned} &(\mathrm{I}) + (\mathrm{II}) + (\mathrm{IV}) + (\mathrm{V}) \\ &\quad \lesssim T^{2/q} q M^2\left(1 + M^{4/q} + M^{2+4/q}\right) \\ &\quad \cdot \left(\int_0^t \sup_{[0,s]} \|u\|_{L^2}^2\,ds\right)^{1-2/q}. \end{aligned} \tag{44}$$

We will provide, for instance, the bound for (II) and (IV). The rest can be proved in a similar way. Due to (17), Lemma 6 and Lemma 7 lead to

$$\left\|A_j\right\|_{L^6} \lesssim \left\|I_1\left(|\psi|^2\right)\right\|_{L^6} \lesssim \|\psi\|_{L^3}^2 \tag{45}$$

and

$$\begin{aligned}\left\|A_j-\widetilde{A}_j\right\|_{L^q} &\lesssim \left\|I_1\left(|\psi|^2-|\widetilde{\psi}|^2\right)\right\|_{L^q}\\ &\lesssim q^{1/2}\left\|\left(|\psi|+|\widetilde{\psi}|\right)|u|\right\|_{L^p}\\ &\lesssim q^{1/2}\left(\|\psi\|_{L^q}+\|\widetilde{\psi}\|_{L^q}\right)\|u\|_{L^2}\\ &\lesssim qM\|u\|_{L^2}\lesssim qM^{1+2/q}\|u\|_{L^2}^{1-2/q},\end{aligned} \tag{46}$$

where p is determined by $1/q = 1/p - 1/2$. For $1/r + 1/q = 1/2$, the Hölder's inequality and Gagliardo-Nirenberg inequality yield

$$\begin{aligned}&\int_{\mathbb{R}^2}\left|4\left(A_j-\widetilde{A}_j\right)\operatorname{Re}\left(\partial_j\psi\overline{u}\right)\right|dx\\ &\quad\lesssim \left\|A_j-\widetilde{A}_j\right\|_{L^q}\|\nabla\psi\|_{L^2}\|u\|_{L^r}\\ &\quad\lesssim M^{2-2/r}\|u\|_{L^2}^{2/r}\left\|A_j-\widetilde{A}_j\right\|_{L^q}\lesssim qM^{2+4/q}\|u\|_{L^2}^{2-4/q}.\end{aligned} \tag{47}$$

Thus, the Hölder's inequality gives

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2}\left|4\left(A_j-\widetilde{A}_j\right)\operatorname{Re}\left(\partial_j\psi\overline{u}\right)\right|dx\\ &\quad\lesssim qM^{2+4/q}\int_0^t\|u(s)\|_{L^2}^{2-4/q}ds\\ &\quad\lesssim T^{2/q}qM^{2+4/q}\left(\int_0^t\|u(s)\|_{L^2}^2ds\right)^{1-2/q}.\end{aligned} \tag{48}$$

For the integral (IV), similar estimate shows

$$\begin{aligned}&\int_{\mathbb{R}^2}\left|2\left(A_j{}^2-\widetilde{A}_j^2\right)\operatorname{Im}\left(\psi\overline{u}\right)\right|dx\\ &\quad\lesssim \left\|\mathrm{A}_j-\widetilde{A}_j\right\|_{L^q}\left(\left\|A_j\right\|_{L^6}+\left\|\widetilde{A}_j\right\|_{L^6}\right)\|\psi\|_{L^3}\|u\|_{L^r}\\ &\quad\lesssim qM^{1+2/q}\|u\|_{L^2}^{1-2/q}\left(\|\psi\|_{L^3}^2+\|\widetilde{\psi}\|_{L^3}^2\right)\|\psi\|_{L^3}\|u\|_{L^r}\\ &\quad\lesssim qM^{4+4/q}\|u\|_{L^2}^{2-4/q},\end{aligned} \tag{49}$$

which implies

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2}\left|2\left(A_j{}^2-\widetilde{A}_j^2\right)\operatorname{Im}\left(\psi\overline{u}\right)\right|dx\\ &\quad\lesssim T^{2/q}qM^{4+4/q}\left(\int_0^t\|u(s)\|_{L^2}^2ds\right)^{1-2/q}.\end{aligned} \tag{50}$$

For the integral (VI), we first apply Lemma 5 to (37) which leads to

$$\begin{aligned}\|v(s)\|_{L^6} &\lesssim \int_0^s\left\|T(s-\tau)\left(|\psi(\tau)|^2-|\widetilde{\psi}(\tau)|^2\right)\right\|_{L^6}d\tau\\ &\lesssim \int_0^s|s-\tau|^{-1/3}\|u(\tau)\|_{L^2}\left(\|\psi(\tau)\|_{L^3}+\|\widetilde{\psi}(\tau)\|_{L^3}\right)d\tau\\ &\lesssim Ms^{2/3}\sup_{[0,s]}\|u\|_{L^2}.\end{aligned} \tag{51}$$

Then, we have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2}\left|4v\operatorname{Im}\left(\psi\overline{u}\right)\right|dxds\\ &\quad\lesssim \int_0^t\|v(s)\|_{L^6}\|\psi(s)\|_{L^3}\|u(s)\|_{L^2}ds\\ &\quad\lesssim M^2\int_0^t\sup_{[0,s]}\|u\|_{L^2}^2ds.\end{aligned} \tag{52}$$

Collecting these bounds (44), (52), we obtain (39) which implies

$$\|u(t,\cdot)\|_{L^2}=0\quad\text{for } 0\le t\le T. \tag{53}$$

On the other hand, multiplying (37) by $\partial_t v$ and integrating over $[0,t]\times\mathbb{R}^2$, we have

$$\begin{aligned}&\left\|\partial_t v(t)\right\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2+\|v(t)\|_{L^2}^2\\ &\quad=-4\int_0^t\int_{\mathbb{R}^2}\left(|\psi|-|\widetilde{\psi}|\right)\left(|\psi|+|\widetilde{\psi}|\right)\partial_t v\,dxds,\end{aligned} \tag{54}$$

for $0 \le t \le T$. The Hölder's inequality and Gagliardo-Nirenberg inequality give us

$$\begin{aligned}&\left\|\partial_t v(t)\right\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2+\|v(t)\|_{L^2}^2\\ &\quad\lesssim \int_0^t\|u(s)\|_{L^6}\left(\|\psi(s)\|_{L^3}+\|\widetilde{\psi}(s)\|_{L^3}\right)\left\|\partial_t v(s)\right\|_{L^2}ds\\ &\quad\lesssim M^2\int_0^t\|u(s)\|_{L^2}^{1/3}\|\nabla u(s)\|_{L^2}^{2/3}ds=0.\end{aligned} \tag{55}$$

Finally, continuous dependence on initial data follows from the same estimates above and the same argument in [14]. Let $(\psi, N, \mathbf{A})$ and $(\widetilde{\psi}, \widetilde{N}, \widetilde{\mathbf{A}})$ be solutions of (9)–(13) with the initial data (ψ_0, n_0, n_1) and $(\widetilde{\psi}_0, \widetilde{n}_0, \widetilde{n}_1)$, respectively. If we set $u = \psi - \widetilde{\psi}$ and $u_0 = \psi_0 - \widetilde{\psi}_0$, the above estimates show

$$\begin{aligned}\sup_{[0,t]}\|u\|_{L^2}^2 &\le \left\|u_0\right\|_{L^2}^2+qM^2\left(1+M^{4/q}+M^{2+4/q}\right)\\ &\quad\cdot\int_0^t\sup_{[0,s]}\|u(s)\|_{L^2}^{2(1-2/q)}ds.\end{aligned} \tag{56}$$

Applying Lemma 9 to (56), we have

$$\begin{aligned}&\sup_{[0,t]}\|u\|_{L^2}^2\\ &\quad\le\left(\left\|u_0\right\|_{L^2}^{4/q}+M^2\left(1+M^{4/q}+M^{2+4/q}\right)T\right)^{q/2},\end{aligned} \tag{57}$$

and this implies that the solution depends on initial data continuously in L^2 locally uniformly in time.

3. Proof of Theorem 2

In this section we study the existence of a global solution to (9)–(13). Firstly, we derive the conservation laws (6) and (7). Multiplying (1) by $\overline{\psi}$ and taking its conjugate, we have

$$i\partial_t\psi\overline{\psi} + \Delta\psi\overline{\psi} = -A_0|\psi|^2 + A_j^2|\psi|^2 + 2iA_j\partial_j\psi\overline{\psi} + i\partial_jA_j|\psi|^2 + |\psi|^4 + 2N|\psi|^2, \quad (58)$$

$$-i\partial_t\overline{\psi}\psi + \Delta\overline{\psi}\psi = -A_0|\psi|^2 + A_j^2|\psi|^2 - 2iA_j\partial_j\overline{\psi}\psi - i\partial_jA_j|\psi|^2 + |\psi|^4 + 2N|\psi|^2. \quad (59)$$

Subtracting (59) from (58), we obtain

$$i\partial_t|\psi|^2 + \Delta\psi\overline{\psi} - \Delta\overline{\psi}\psi = 2i\partial_j\left(A_j|\psi|^2\right). \quad (60)$$

Then, integration by parts on $\mathbb{R}^2$ gives

$$\frac{d}{dt}\|\psi(t,\cdot)\|_{L^2} = 0, \quad (61)$$

which implies (6).

Multiplying (1) by $\partial_t\overline{\psi}$ and taking its conjugate, we have

$$\begin{aligned} i\partial_t\psi\partial_t\overline{\psi} + \Delta\psi\partial_t\overline{\psi} &= -A_0\psi\partial_t\overline{\psi} + A_j^2\psi\partial_t\overline{\psi} \\ &\quad + 2iA_j\partial_j\psi\partial_t\overline{\psi} + i\partial_jA_j\psi\partial_t\overline{\psi} \\ &\quad + |\psi|^2\psi\partial_t\overline{\psi} + 2N\psi\partial_t\overline{\psi}, \\ &\quad - i\partial_t\overline{\psi}\partial_t\psi + \Delta\overline{\psi}\partial_t\psi \\ &= -A_0\overline{\psi}\partial_t\psi + A_j^2\overline{\psi}\partial_t\psi \\ &\quad - 2iA_j\partial_j\overline{\psi}\partial_t\psi - i\partial_jA_j\overline{\psi}\partial_t\psi \\ &\quad + |\psi|^2\overline{\psi}\partial_t\psi + 2N\overline{\psi}\partial_t\psi. \end{aligned} \quad (62)$$

Summing the both sides and integrating by parts on $\mathbb{R}^2$, we obtain

$$\begin{aligned} &-\int_{\mathbb{R}^2}\partial_t|\nabla\psi|^2dx - \underbrace{\int_{\mathbb{R}^2}A_j^2\partial_t|\psi|^2dx}_{\text{(i)}} \\ &+ \underbrace{\int_{\mathbb{R}^2}4A_j\operatorname{Im}\left(\partial_j\psi\partial_t\overline{\psi}\right) + 2\partial_jA_j\operatorname{Im}\left(\psi\partial_t\overline{\psi}\right)dx}_{\text{(ii)}} \\ &= -\int_{\mathbb{R}^2}A_0\partial_t|\psi|^2dx + \frac{1}{2}\int_{\mathbb{R}^2}\partial_t|\psi|^4dx \\ &\quad + 2\int_{\mathbb{R}^2}N\partial_t|\psi|^2dx. \end{aligned} \quad (63)$$

Considering

$$\begin{aligned} \text{(i)} &= \int_{\mathbb{R}^2}\partial_t\left[A_j^2|\psi|^2\right] - \partial_tA_j^2|\psi|^2dx, \\ \text{(ii)} &= \int_{\mathbb{R}^2}2A_j\operatorname{Im}\left(\partial_j\psi\partial_t\overline{\psi}\right) + 2A_j\operatorname{Im}\left(\overline{\psi}\partial_j\partial_t\psi\right)dx \\ &= \int_{\mathbb{R}^2}2\partial_t\left[A_j\operatorname{Im}\left(\overline{\psi}\partial_j\psi\right)\right] \\ &\quad - 2\partial_tA_j\operatorname{Im}\left(\overline{\psi}\partial_j\psi\right)dx, \end{aligned} \quad (64)$$

and

$$\partial_t\left|D_j\psi\right|^2 = \partial_t|\nabla\psi|^2 - 2\partial_t\left[A_j\operatorname{Im}\left(\overline{\psi}\partial_j\psi\right)\right] + \partial_t\left[A_j^2|\psi|^2\right], \quad (65)$$

the left side of (63) becomes

$$-\frac{d}{dt}\int_{\mathbb{R}^2}\left|D_j\psi\right|^2dx + \underbrace{\int_{\mathbb{R}^2}\partial_tA_j^2|\psi|^2 - 2\partial_tA_j\operatorname{Im}\left(\overline{\psi}\partial_j\psi\right)dx.}_{\text{(iii)}} \quad (66)$$

On the other hands, multiplying (3), (4) by ∂_tA_2, ∂_tA_1, respectively, we have

$$\begin{aligned} &\partial_tA_2\left(\partial_0A_1 - \partial_1A_0\right) \\ &\quad = 2\partial_tA_2\operatorname{Im}\left(\overline{\psi}\partial_2\psi\right) - 2A_2\partial_tA_2|\psi|^2, \\ &-\partial_tA_1\left(\partial_0A_2 - \partial_2A_0\right) \\ &\quad = 2\partial_tA_1\operatorname{Im}\left(\overline{\psi}\partial_1\psi\right) - 2A_1\partial_tA_1|\psi|^2. \end{aligned} \quad (67)$$

Adding the both sides, we have

$$\partial_tA_j^2|\psi|^2 - 2\partial_tA_j\operatorname{Im}\left(\overline{\psi}\partial_j\psi\right) = \partial_tA_2\partial_1A_0 - \partial_tA_1\partial_2A_0. \quad (68)$$

Replacing (iii) with this, integration by parts gives

$$\begin{aligned} \text{(iii)} &= -\int_{\mathbb{R}^2}A_0\partial_t\left(\partial_1A_2 - \partial_2A_1\right)dx \\ &= -\int_{\mathbb{R}^2}A_0\partial_t|\psi|^2dx, \end{aligned} \quad (69)$$

where (5) is used. Inserting (66) and (69) into (63), we have

$$\frac{d}{dt}\int_{\mathbb{R}^2}2\left|D_j\psi\right|^2 + |\psi|^4dx = -4\int_{\mathbb{R}^2}N\partial_t|\psi|^2dx. \quad (70)$$

Now, multiplying (2) by ∂_tN and integrating on $\mathbb{R}^2$ provide

$$\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla N|^2 + \left|\partial_tN\right|^2 + N^2dx = -4\int_{\mathbb{R}^2}\partial_tN|\psi|^2dx. \quad (71)$$

Adding (70) and (71), we finally obtain

$$\frac{d}{dt} E(t) = 0, \tag{72}$$

which leads to (7).

Now we are ready to prove the existence of global solution. By the conservation laws (6) and (7), we have

$$Q(t) = \|\psi(t)\|_{L^2} = Q(0), \tag{73}$$

and

$$\begin{aligned} E(t) = {} & 2 \sum_{j=1,2} \left\| D_j \psi(t) \right\|_{L^2}^2 + \|\nabla N(t)\|_{L^2}^2 + \left\| \partial_t N(t) \right\|_{L^2}^2 \\ & + \|N(t)\|_{L^2}^2 + \|\psi(t)\|_{L^4}^4 \\ & + 4 \int_{\mathbb{R}^2} N |\psi|^2 (t, x)\, dx = E(0). \end{aligned} \tag{74}$$

Because we do not know the sign of the last term $N|\psi|^2$, the energy conservation (74) does not imply a global energy solution directly. Therefore we would find a bound for the last term and then a uniform bound for H^1-norm of solution which leads to global existence. We refer to [8].

Using the Hölder's inequality, Lemma 7, and Young's inequality, we have

$$\begin{aligned} & -4 \int_{\mathbb{R}^2} N |\psi|^2 (t, x)\, dx \\ & \leq 4 \|N(t)\|_{L^4} \|\psi(t)\|_{L^4} \|\psi(t)\|_{L^2} \\ & \leq 4 \|N(t)\|_{L^2}^{1/2} \|\nabla N(t)\|_{L^2}^{1/2} \|\psi(t)\|_{L^4} \|\psi(t)\|_{L^2} \\ & \leq \frac{1}{4} \|\nabla N(t)\|_{L^2}^2 + \frac{1}{4} \|N(t)\|_{L^2}^2 + \frac{1}{4} \|\psi(t)\|_{L^4}^4 \\ & \quad + 2^6 \|\psi(t)\|_{L^2}^4 . \end{aligned} \tag{75}$$

From (75) and (74), it follows that

$$\begin{aligned} & 2 \left\| D_j \psi(t) \right\|_{L^2}^2 + \|\nabla N(t)\|_{L^2}^2 + \left\| \partial_t N(t) \right\|_{L^2}^2 + \|N(t)\|_{L^2}^2 \\ & \quad + \|\psi(t)\|_{L^4}^4 \\ & \quad \leq E(0) + 2^6 Q(0)^4 + \frac{1}{4} \|\nabla N(t)\|_{L^2}^2 + \frac{1}{4} \|N(t)\|_{L^2}^2 \\ & \quad + \frac{1}{4} \|\psi(t)\|_{L^4}^4 \end{aligned} \tag{76}$$

which implies

$$\begin{aligned} & \left\| D_j \psi(t) \right\|_{L^2}^2 + \|\nabla N(t)\|_{L^2}^2 + \left\| \partial_t N(t) \right\|_{L^2}^2 + \|N(t)\|_{L^2}^2 \\ & \quad + \|\psi(t)\|_{L^4}^4 \leq c. \end{aligned} \tag{77}$$

Referring to Proposition 3, the Hölder's inequality and Young's inequality give

$$\begin{aligned} \|\nabla \psi(t)\|_{L^2}^2 & \leq 2 \left\| D_j \psi(t) \right\|_{L^2}^2 + 2 \left\| A_j(t) \right\|_{L^4}^2 \|\psi(t)\|_{L^4}^2 \\ & \leq 2 \left\| D_j \psi(t) \right\|_{L^2}^2 \\ & \quad + 2C \|\nabla \psi(t)\|_{L^2} \|\psi(t)\|_{L^2}^3 \|\psi(t)\|_{L^4}^2 \\ & \leq 2 \left\| D_j \psi(t) \right\|_{L^2}^2 + \frac{1}{2} \|\nabla \psi(t)\|_{L^2}^2 \\ & \quad + 2C^2 \|\psi(t)\|_{L^2}^6 \|\psi(t)\|_{L^4}^4 , \end{aligned} \tag{78}$$

which yields

$$\|\nabla \psi(t)\|_{L^2}^2 \leq 4c \left(1 + C^2 Q(0)^6\right). \tag{79}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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