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Research Article

CCNV Space-Times as Potential Supergravity Solutions

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It is of interest to study supergravity solutions preserving a nonminimal fraction of supersymmetries. A necessary condition for supersymmetry to be preserved is that the space-time admits a Killing spinor and hence a null or time-like Killing vector field. Any space-time admitting a covariantly constant null vector (CCNV) field belongs to the Kundt class of metrics and more importantly admits a null Killing vector field. We investigate the existence of additional non-space-like isometries in the class of higher-dimensional CCNV Kundt metrics in order to produce potential solutions that preserve some supersymmetries.

1. Introduction

Supersymmetric supergravity solutions have been studied in the context of the AdS/CFT conjecture, the microscopic properties of black hole entropy, and the interconnection of string theory dualities. For example, in five dimensions, solutions preserving various fractions of supersymmetry of $N = 2$ gauged supergravity have been studied. The Killing spinor equations imply that supersymmetric solutions preserve 2, 4, 6, or 8 of the supersymmetries. The AdS_5 solution with vanishing gauge field strengths and constant scalars preserves all of the supersymmetries. Half supersymmetric solutions in gauged five-dimensional supergravity with vector multiplets possess two Dirac Killing spinors and hence two time-like or null Killing vector fields. These solutions have been fully classified, using the spinorial geometry method, in [1–3]. Indeed, in a number of supergravity theories [4, 5], in order to preserve some supersymmetry, it is necessary that the space-time admits a Killing spinor which then yields a null or time-like Killing vector field (isometry) from its Dirac current. Therefore, a necessary (but not sufficient) condition for supersymmetry to be preserved is that the space-time admits a null or time-like Killing vector field.

In this short communication we study supergravity solutions preserving a nonminimal fraction of supersymmetries by determining the existence of additional non-space-like isometries in the class of higher-dimensional Kundt space-time admitting a covariantly constant null vector field (CCNV) [6, 7]. CCNV space-time belongs to the Kundt class because it contains a null Killing vector field which is geodesic, nonexpanding, shear-free, and nontwisting. The existence of an additional isometry puts constraints on the metric functions and the vector field components. Killing vector fields that are null or time-like locally or globally (for all values of the coordinate v) are of particular importance. As an illustration we present two explicit examples in this paper.

A constant scalar invariant (CSI) space-time is a space-time such that all of the polynomial scalar invariants constructed from the Riemann tensor and its covariant derivatives are constant. In three and four dimensions, it has been proven that all CSI space-times are either locally homogeneous or belongs to the degenerate Kundt class [8–10]; it is conjectured that this is true in higher dimensions as well. The VSI space-times are CSI space-times for which all of these polynomial scalar invariants vanish. The subset of CCNV space-times which are also VSI are of interest. Indeed, it has been shown previously that the higher-dimensional VSI space-time with fluxes and dilaton is solutions of type IIB supergravity [11–13]. The higher-dimensional Weyl type **N** pp-wave space-times (Weyl and Ricci type are defined in terms of the alignment classification [14]) are known to be solutions in type IIB supergravity for an appropriate choice of an R-R five-form or an NS-NS form field [15–19].

In fact, all Ricci type **N** VSI space-times are solutions to supergravity and, moreover, there are VSI space-time solutions of type IIB supergravity which are of Ricci type **III**, including the string gyratons, assuming that appropriate source fields are provided [11–13]. It has been argued that the VSI supergravity space-time is exact string solutions to all orders in the string tension. Those VSI space-time in which supersymmetry is preserved admits a CCNV. Higher-dimensional VSI space-time solutions to type IIB supergravity preserving some supersymmetry are of Ricci type **N**, Weyl type **III**(a), or **N** [11–13].

It is also known that $\mathrm{AdS}_d \times S^{(N-d)}$ space-time is supersymmetric CSI solutions of IIB supergravity. There is an amount of another CSI space-time known to be solution of supergravity admitting supersymmetries [8–10], including generalizations of AdS $\times$ S [20], the chiral null models [15], and the string gyratons [21]. Some explicit examples of CSI CCNV Ricci type **N** supergravity space-time have been constructed [11–13]. The CSI space-times also contain the universal space-times [22]; a space-time is *universal* if every symmetric conserved rank 2 tensor which can be constructed from the metric, the Riemann tensor, and its covariant derivatives is proportional to the metric. This ensures that any quantum correction to such a classical solution is proportional to the metric and hence is a solution to all quantum gravity theories defined in terms of a gravitational Lagrangian. Therefore, a subset of the CSI space-time will be solutions to all quantum gravity theories defined in this manner. This suggests that the CSI CCNV space-times presented in this paper will contain supergravity solutions.

1.1. Kundt Metrics and CCNV Space-Times. Any N-dimensional space-time possessing a CCNV, ℓ, must necessarily belong to the class of Kundt metrics. Local coordinates (u, v, x^e) can be chosen, where $\ell = \partial_v$, so that the metric can be written [6, 7]

$$\begin{aligned} ds^2 &= 2du\left[dv + H\left(u, x^e\right)du + \widehat{W}_e\left(u, x^f\right)dx^e\right] \\ &\quad + g_{ef}\left(u, x^g\right)dx^e dx^f, \end{aligned} \tag{1}$$

where the metric functions are independent of the light-cone coordinate v.

A Kundt metric admitting a CCNV is CSI if and only if the transverse metric g_{ef} is locally homogeneous [8–10]. Due to the local homogeneity of g_{ef} a coordinate transformation can be performed so that m_{ie} in (2) are independent of u; this implies that the Riemann tensor is of type **II** or less [14]. If a CSI-CCNV metric satisfies $R_{ab}R^{ab} = 0$, then the metric is VSI, and the Riemann tensor will be of types **III**, **N**, or **O** and the transverse metric is flat (i.e., $g_{ef} = \delta_{ef}$). The constraints on a CSI CCNV space-time to admit an additional Killing vector field are obtained as subcases of the cases analyzed below where the transverse metric is a locally homogeneous Riemannian manifold.

2. CCNV Space-Time with Additional Isometries

Let us choose the coframe $\{m^a\}$

$$m^1 = n = dv + Hdu + \widehat{W}_e dx^e, \quad m^2 = l, \; m^i = m^i{}_e dx^e, \tag{2}$$

where $m^i{}_e m_{if} = g_{ef}$ and $m_{ie} m_j{}^e = \delta_{ij}$. The frame derivatives are given by

$$\begin{aligned} \ell &= D_1 = \partial_v, \\ n &= D_2 = \partial_u - H\partial_v, \\ m_i &= D_i = m_i{}^e\left(\partial_e - \widehat{W}_e\partial_v\right). \end{aligned} \tag{3}$$

The Killing vector field can be written as $X = X_1 n + X_2 \ell + X_i m^i$. A coordinate transformation is made to eliminate $\widehat{W}_3$ in (1) and we rotate the frame in order to set $X_3 \neq 0$ and $X_m = 0$ [6]. X is now given by

$$X = X_1 n + X_2 \ell + \chi m^3. \tag{4}$$

Without loss of generality, we assume that the matrix m_{ie} is upper-triangular. As a note, the indices $e, f, g, \ldots$, range from 3 to N while the indices $m, n, r, p, \ldots$ range from 4 to N.

A subset of the Killing equation can then be written as follows:

$$\begin{aligned} X_{1,v} &= 0, \\ X_{1,u} + X_{2,v} &= 0, \\ m_3{}^e X_{1,e} + X_{3,v} &= 0, \\ m_n{}^e X_{1,e} &= 0, \end{aligned} \tag{5}$$

which imply

$$\begin{aligned} X_1 &= F_1\left(u, x^e\right), \\ X_2 &= -D_2\left(X_1\right)v + F_2\left(u, x^e\right), \\ X_3 &= -D_3\left(X_1\right)v + F_3\left(u, x^e\right), \end{aligned} \tag{6}$$

and the remaining Killing equations are

$$D_2 X_2 + \sum_i J_i X_i = 0, \tag{7}$$

$$D_i X_2 + D_2 X_i - J_i X_1 - \sum_j \left(A_{ji} + B_{ij}\right) X_j = 0, \tag{8}$$

$$D_j X_i + D_i X_j + 2B_{(ij)} X_1 - 2\sum_k \Gamma_{k(ij)} X_k = 0, \tag{9}$$

where

$$\begin{aligned} B_{ij} &= m_{ie,u} m_j{}^e, \\ W_i &= m_i{}^e \widehat{W}_e, \\ D_{ijk} &\equiv 2 m_{ie,f} m_{[j}{}^e m_{k]}{}^f, \\ J_i &\equiv \Gamma_{2i2} = D_i H - D_2 W_i - B_{ji} W^j, \\ A_{ij} &\equiv D_{[j} W_{i]} + D_{k[ij]} W^k, \\ \Gamma_{ikj} &= -\frac{1}{2}\left(D_{ijk} + D_{jki} - D_{kij}\right) = {}^S\Gamma_{ikj}. \end{aligned} \tag{10}$$

Further information can be found by taking the Killing equations and applying the commutation relations. This produces two cases: (1) $D_3 X_1 = 0$ or (2) $\Gamma_{3n2} = \Gamma_{3n3} = \Gamma_{3nm} = 0$.

Case 1 ($D_3 X_1 = 0$). Using (7) and the definition of F_2 from (6), we have the fact that $X_1 = c_1 u + c_2$. If $c_1 \neq 0$ we may always choose coordinates to set $X_1 = u$; while if $c_1 = 0$ we may choose $c_2 = 1$.

Subcase 1 ($F_3 = 0$). (i) $c_1 \neq 0$, $X_1 = u$; F_2 must be of the form

$$F_2 = \frac{f_2(x^e)}{u} + \frac{g_2(u)}{u}. \tag{11}$$

H and W_m are given in terms of these two functions (where $g' \equiv dg/du$)

$$\begin{aligned} H &= \frac{f_2(x^e)}{u^2} - \frac{g_2'(u)}{u} + \frac{g_2(u)}{u^2}, \\ W_m &= \frac{B_m(x^e)}{u}. \end{aligned} \tag{12}$$

(ii) $c_1 = 0$, $X_1 = 1$; $F_{2,u} = 0$, and H and W_n are

$$\begin{aligned} H &= F_2(x^e) + A_0(u, x^r), \\ W_n &= \int D_n A_0 \, du + C_n(x^e). \end{aligned} \tag{13}$$

In either case, the only requirement on the transverse metric is that it has to be independent of u. The arbitrary functions in this case are F_2 and the functions arising from integration.

Subcase 2 ($F_3 \neq 0$). The transverse metric is now determined by

$$m_{33} = -\int \frac{1}{X_1} F_{3,3} \, du + A_1\left(x^3, x^r\right), \tag{14}$$

$$\begin{aligned} m_{nr,u} &= -m_{nr,3} \frac{F_3}{m_{33} X_1}, \\ m_{3r,u} &= -\frac{F_{3,r}}{X_1} - \frac{m_{3[r,3]} m_3{}^3 F_3}{X_1}. \end{aligned} \tag{15}$$

(i) $c_1 \neq 0$, $X_1 = u$; $F_i(u, x^e)$ $(i = 1, 2)$ are arbitrary functions, H is given by

$$H = -D_2 F_2 - \frac{D_2\left(F_3^2\right)}{2u} - \frac{F_3 D_3 F_2}{u} - \frac{F_3 D_3\left(F_3^2\right)}{2u^2}, \tag{16}$$

and W_n is determined by

$$D_2\left(u W_n\right) + F_3 D_3 W_n + D_n\left(F_2 - uH\right) = 0. \tag{17}$$

(ii) $c_1 = 0$, $c_2 \neq 0$, and $X_1 = 1$; F_2 and F_3 satisfy

$$D_2 F_2 + F_3 D_3 F_2 + \frac{1}{2} D_2\left(F_3^2\right) + \frac{1}{2} F_3 D_3\left(F_3^2\right) = 0. \tag{18}$$

H may be written as

$$H = \int m_{33} D_2 F_3 \, dx^3 + F_2 + \frac{1}{2} F_3^2 + A_2(u, x^r). \tag{19}$$

The only equation for W_n is

$$F_3 D_3 W_n + D_2 W_n = D_n(H). \tag{20}$$

(iii) $X_1 = 0$.

We have the following constraints on the functions $m^i{}_e$:

$$\begin{aligned} F_{3,3} &= 0, \\ m_{nr,3} &= 0, \\ D_2 \log\left(m_{33}\right) &= -\frac{D_3 F_2}{F_3} - D_2 \log\left(F_3\right), \end{aligned} \tag{21}$$

while the metric functions must be of the form

$$\begin{aligned} W_n &= -\int \frac{m_{33} D_n F_2}{F_3} dx^3 + E_n(u, x^r), \\ H &- \int \frac{m_{33} D_2 F_2}{F_3} dx^3 + A_3(u, x^r). \end{aligned} \tag{22}$$

There are two further subcases depending upon whether $m_{33,r} = 0$ or not; whence we may further integrate to determine the transverse metric.

Case 2 ($\Gamma_{3ia} = 0$). This implies that the upper-triangular matrix m_{ie} takes the form

$$\begin{aligned} m_{33} &= M_{,3}\left(u, x^3\right), \\ m_{3r} &= 0, \\ m_{nr} &= m_{nr}(u, x^r), \end{aligned} \tag{23}$$

while W_n must satisfy $D_3(W_n) = 0$. The remaining Killing equations simplify; in particular, $B_{(mn)} X_1 = 0$, which leads to two subcases: (1) $X_1 = 0$ or (2) $B_{(mn)} = 0$.

Subcase 1 ($X_1 = 0$, $B_{(mn)} \neq 0$). $F_{2,r} = 0$, $F_{3,e} = 0$, with m_{ie}, H, and W_n given by (21) and (22).

Subcase 2 ($B_{(mn)} = 0$, $X_1 \neq 0$). This case is similar to the subcases dealt with in Subcase 1 (see (11)–(14), (20)–(22)). For $n < p$, the vanishing of $B_{(np)}$ implies that $m_{nr,u} = 0$, the special form of m_{ie} implies that $m_r{}^3 = 0$, and the only nonzero component of the tensor B is B_{33}.

If we assume that $F_{1,3} \neq 0$ and F_1 is independent of x^r

$$\frac{m_{33,3}}{m_{33}} = \frac{F_{1,33}}{F_{1,3}},\qquad \frac{m_{33,u}}{m_{33}} = \frac{F_{1,3u}}{F_{1,3}}, \tag{24}$$

thus $m_{33}(u, x^3)$ is entirely defined by F_1. We may now solve for H and W_n

$$H = \frac{D_3 D_2 F_1}{D_3 (F_1)^2} F_3 - \frac{D_2^2 F_1}{D_3 (F_1)^2} F_1 - \frac{2D_{(2}F_{3)}}{D_3 F_1},\qquad W_n = -\frac{D_n F_3}{D_3 F_1}. \tag{25}$$

F_3 is of the form

$$F_3 = \int \frac{m_{33} F_1 D_3 D_2 F_1}{D_3 F_1} dx^3 + A_6 (u, x^r). \tag{26}$$

There are differential equations for F_2 in terms of the arbitrary functions $F_1(u, x^3)$ and $A_6(u, x^r)$. These solutions are summarized in Table 5.2 in [7].

3. Killing Lie Algebras

There are three particular forms for the Killing vector fields in CCNV space-times admitting an additional isometry:

(A) $X_A = cn + F_2(u, x^e)\ell + F_3(u, x^e)m^3$

(B) $X_B = un + [F_2(u, x^e) - v]\ell + F_3(u, x^e)m^3$

(C) $X_C = F_1(u, x^3)n + [F_2(u, x^e) - D_2F_1 v]\ell + [F_3(u, x^e) - D_3F_1 v]m^3$.

To determine if this space-time admits even more isometries we examine the commutator of X with ℓ in each case. In case (A) $[X_A, \ell] = 0$ and in case (B) $[X_B, \ell] = -\ell$, implying that there are no additional Killing vector fields.

In the most general case $Y_C \equiv [X_C, \ell]$ can yield a new Killing vector field; $Y_C = D_2F_1\ell + D_3F_1 m_3$. However, this will always be space-like since $(D_3F_1)^2 > 0$. Note that $[Y_C, \ell] = 0$, while, in general, $[Y_C, X_C] \neq 0$.

3.1. Globally Non-Space-Like Killing Vector Fields. Let us consider the set of CCNV space-times admitting an additional non-space-like isometry. Equation (6) implies that the norm of this vector field must satisfy

$$D_3 (X_1)^2 v^2 + 2 (D_2 (X_1) X_1 - D_3 (X_1) F_3) v + F_3{}^2 - 2X_1 F_2 \leq 0. \tag{27}$$

If the Killing vector field is non-space-like for all values of v, then $D_3(X_1)$ must vanish and X_1 is constant. Therefore, those subcases with X_1 nonconstant are excluded.

We need only consider the Killing vector field of the form X_A. In the time-like case, the subcases with $X_1 = 0$ are no longer valid as this would imply that $F_3{}^2 < 0$.

In the case that X_A is null, if $c = 0$, F_3 must vanish and F_2 must be constant, implying that X is a scalar multiple of ℓ. If $c \neq 0$ we can rescale n so that $2F_2 = F_3{}^2$; we can then integrate out the various cases:

(i) If $F_3 = 0$, F_2 must vanish as well and the Killing vector field, X, is proportional to n. The remaining metric functions are now $H = A_0(u, x^r)$ and $W_n = \int D_n(A_0)du + C_n(x^e)$. The transverse metric is unaffected.

(ii) If $F_3 \neq 0$, the metric functions must satisfy

$$H = A_2 (u, x^r),\qquad D_2 (W_n) + D_3 (W_n) F_3 = D_n (A_2),\qquad (\log m_{33})_{,u} = D_2 (\log F_3). \tag{28}$$

In this section we have shown that if we require that the Killing vector field is non-space-like for all values of v, this puts strong conditions on the permitted form of the Killing vector field. If we only require that the Killing vector field is non-space-like in a subset of the space-time there is greater diversity in the choice of Killing vector fields permitted. We present two explicit examples in the following subsections, one which is non-space-like in a region of the space-time and the other which is globally non-space-like. The CSI CCNV space-times, admitting globally non-space-like Killing vector fields, are the above cases where the transverse space is locally homogeneous.

3.2. Example 1. We first present an explicit example for the case where $X_1 = u$ and $F_3 \neq 0$. Assuming that $F_3(u, x^i) = \epsilon u m_{33}$ and ϵ is a nonzero constant, we obtain

$$m_{is,u} + \epsilon m_{is,3} = 0 \tag{29}$$

and the transverse metric is thus given by

$$m_{is} = m_{is} \left(x^3 - \epsilon u, x^n\right). \tag{30}$$

We have the algebraic solution

$$\widehat{W}_3 = -\frac{1}{\epsilon} (H + F_{2,u}) - F_{2,3} - \epsilon m_{33}{}^2, \tag{31}$$

where $F_2(u, x^i)$ is an arbitrary function and H is given by

$$H \left(u, x^i\right) = \frac{1}{u} \left[- \int S \left(z, x^3 - \epsilon u + \epsilon z, x^n\right) dz + A \left(x^3 - \epsilon u, x^n\right) \right], \tag{32}$$

where A is an arbitrary function and S is given by

$$S\left(u,x^3,x^n\right) = \left(uF_{2,u}\right)_u + \epsilon u F_{2,3u} + \epsilon^2 u \left({m_{33}}^2\right)_u. \quad (33)$$

Furthermore, the solution for $\widehat{W}_n$, $n = 4,\ldots,N$, is

$$\begin{aligned}\widehat{W}_n\left(u,x^i\right) = \frac{1}{u}\Big[-\int T_n\left(z,x^3-\epsilon u+\epsilon z,x^m\right)dz \\ + B_n\left(x^3-\epsilon u,x^m\right)\Big],\end{aligned} \quad (34)$$

where B_n are arbitrary functions and T_n is given by

$$\begin{aligned}T_n\left(u,x^3,x^m\right) = \left[\left(uF_2\right)_u + \epsilon u F_{2,3} + \epsilon^2 u {m_{33}}^2\right]_{,n} \\ + \epsilon m_{3n} m_{33}.\end{aligned} \quad (35)$$

In this example, the Killing vector field and its magnitude are given by

$$\begin{aligned}X &= u\mathbf{n} + \left(-v+F_2\right)\boldsymbol{\ell} + \epsilon u m_{33}\mathbf{m}^3, \\ X_a X^a &= -2uv + 2uF_2 + \left(\epsilon u m_{33}\right)^2.\end{aligned} \quad (36)$$

Clearly, the causal character of X will depend on the choice of $F_2(u,x^i)$ and for any fixed (u,x^i) X is time-like or null for appropriately chosen values of v. Moreover, (36) is an example of case (B); therefore the commutator of X and $\boldsymbol{\ell}$ gives rise to a constant rescaling of $\boldsymbol{\ell}$ and, in general, there are no more Killing vector fields.

The additional isometry is only time-like or null locally (for a restricted range of coordinate values). However, the solutions can be extended smoothly so that the vector field is time-like or null on a physically interesting part of space-time. For example, a solution valid on $u > 0$, $v > 0$ (with $F_2 < 0$), can be smoothly matched across $u = v = 0$ to a solution valid on $u < 0$, $v < 0$ (with $F_2 > 0$), so that the Killing vector field is time-like on the resulting coordinate patch.

As an illustration, suppose that m_{3s} are separable as follows:

$$m_{3s} = \left(x^3-\epsilon u\right)^{p_s} h_s\left(x^n\right) \quad (37)$$

and F_2 has the form

$$F_2 = -\frac{\epsilon}{2p_3+1}\left(x^3-\epsilon u\right)^{2p_3+1} {h_3}^2 + g\left(u,x^n\right), \quad (38)$$

where p_s are constants and h_s are g arbitrary functions. Thus, from (32)

$$\begin{aligned}H = -\epsilon^2\left(x^3-\epsilon u\right)^{2p_3-1}\left[x^3-\epsilon\left(p_3+1\right)u\right]{h_3}^2 - g_{,u} \\ + u^{-1}A\left(x^3-\epsilon u,x^n\right),\end{aligned} \quad (39)$$

and hence from (31)

$$\begin{aligned}\widehat{W}_3 = -\epsilon^2 p_3 u\left(x^3-\epsilon u\right)^{2p_3-1}{h_3}^2 \\ - (\epsilon u)^{-1} A\left(x^3-\epsilon u,x^n\right).\end{aligned} \quad (40)$$

Lastly, (34) gives

$$\begin{aligned}\widehat{W}_n &= \epsilon\left(x^3-\epsilon u\right)^{p_3} \\ &\cdot h_3\left\{\frac{2\left(x^3-\epsilon u\right)^{p_3}}{2p_3+1}\left[x^3-\epsilon\left(p_3+\frac{3}{2}\right)u\right]h_{3,n}\right. \\ &\left.-\left(x^3-\epsilon u\right)^{p_n}h_n\right\} - g_{,n} + u^{-1}B_n\left(x^3-\epsilon u,x^m\right).\end{aligned} \quad (41)$$

3.3. *Example 2.*

A second example belonging to Case 1, that is, with $X_1 = 1$, gives the same solutions to (30) for the transverse metric by assuming that $F_3(u,x^i) = \epsilon m_{33}$ (although, in this case, the additional isometry is globally time-like or null). In addition, we have

$$\widehat{W}_3 = \int H_{,3}du + \epsilon^{-1}\left(F_2+f\right), \quad (42)$$

where (u,x^i), $F_2(x^3-\epsilon u,x^n)$, and $f(x^i)$ are arbitrary functions. Last, the metric functions $\widehat{W}_n$ are

$$\begin{aligned}\widehat{W}_n\left(u,x^i\right) = \int^u L_n\left(z,x^3-\epsilon u+\epsilon z,x^m\right)dz \\ + E_n\left(x^3-\epsilon u,x^m\right),\end{aligned} \quad (43)$$

with E_n arbitrary and L_n given by

$$L_n\left(u,x^3,x^m\right) = H_{,n} + \epsilon\int H_{,3n}du + f_{,n}. \quad (44)$$

The Killing vector field and its magnitude are

$$\begin{aligned}X &= \mathbf{n} + F_2\boldsymbol{\ell} + \epsilon m_{33}\mathbf{m}^3, \\ X_a X^a &= 2F_2 + \left(\epsilon m_{33}\right)^2.\end{aligned} \quad (45)$$

Since F_2 and m_{33} have the same functional dependence there always exists an F_2 such that X is everywhere time-like or null. The Killing vector field (45) is an example of case (A) and thus X and $\boldsymbol{\ell}$ commute and hence no additional isometries arise. For instance, suppose that $H = H(x^3-\epsilon u,x^n)$ and f is analytic at $x^3 = 0$ then (42) and (43) simplify to give

$$\begin{aligned}\widehat{W}_3 &= -\epsilon^{-1}\left(H-F_2-f\right), \\ \widehat{W}_n &= \epsilon^{-1}\sum_{p=0}^{\infty}\partial_n{\partial_3}^p f\left(0,x^m\right)\frac{\left(x^3\right)^{p+1}}{(p+1)!} \\ &+ E_n\left(x^3-\epsilon u,x^m\right).\end{aligned} \quad (46)$$

This explicit solution is an example of a space-time admitting 2 global null or time-like Killing vector fields, and thus it preserves a nonminimal fraction of supersymmetries.

4. Discussion

The CCNV space-times discussed in this paper will preserve a nonminimal fraction of supersymmetries, if they are solutions of some supergravity theory. To show that there are indeed CCNV space-times that are solutions to a supergravity theory, we note that there exist VSI and CSI space-times which are solutions to supergravity theories [11–13]. The CSI and VSI solutions admit a covariantly constant null vector (i.e., they are CCNV-CSI space-times) or are constructed from the warped product of a CCNV-VSI space-time and a locally homogeneous Riemannian manifold.

The construction of CSI solutions for supergravity theories was motivated by the observation that $\mathrm{AdS}_d \times S^{(D-d)}$ is a supersymmetric exact solution of supergravity (for certain values of (D, d) and for particular ratios of the radii of curvature of the two space forms; in particular, $d = 5$, $D = 10$, and $\mathrm{AdS}_5 \times S^5$). The more general D-dimensional product space-time $M_d \times K^{(D-d)}$ (in brief $M \times K$) can be considered as a Freund–Rubin background. For example, for $(D, d) = (11, 4), (11, 7)$, and $(5, 5)$ it is sufficient that M and K are Einstein. Since $M \times K$ is a Freund–Rubin background, if M is any Lorentzian Einstein manifold and K is any Riemannian Einstein manifold (with the same ratio of the radii of curvature as in the AdS $\times$ S case), then $M \times K$ will be a solution of some supergravity theory without any consideration of preservation of supersymmetry, where the fluxes are given purely in terms of the volume forms of the relevant factor(s). In general, M must have negative scalar curvature and K will have positive scalar curvature in order to satisfy the supergravity equations of motion.

There are many examples of CSI space-times in the Freund–Rubin $M \times K$ supergravity set. K could be a homogeneous space or a space of constant curvature. One must ask whether these CSI solutions preserve any supersymmetry. The condition for preservation of supersymmetry demands that M and K admit Killing spinors which imply the existence of Killing vectors. In this paper we have examined the class of space-time that will potentially admit more than one Killing spinor without requiring that it is solution to some supergravity theory. Noting that a CCNV space-time will be CSI if the transverse metric is locally homogeneous we can relate particular instances of the two examples presented in Section 3 to subcases of known CCNV-CSI supergravity solutions.

For example, requiring that the transverse space is flat leads to the condition that $m_{33} = 1$ and the two CCNV examples in Section 3 will contain the subclass of CCNV-VSI $\epsilon = 0$ space-time admitting two time-like or null Killing vectors [11]. The Ricci type **N** CCNV-VSI $\epsilon = 0$ space-time has been shown to be solution to superstring and heterotic string theory [15]. Similarly, imposing the condition that the transverse metric is locally homogeneous will yield CCNV-CSI space-time admitting two non-space-like Killing vectors which can be related to subcases of known CCNV-CSI solutions of supergravity theories. As a simple example in five dimensions, we may choose the transverse space to be S^3 with unit radius; then for an appropriate choice of metric functions, the example in Section 3.3 corresponds to the metric given by (10) and (11) in [12] admitting an additional null or time-like Killing vector field. The metric, together with a constant dilaton and appropriate antisymmetric field, is an exact solution of bosonic string theory.

Motivated by the examples of CCNV-CSI space-times given in the literature, it is worthwhile to ask if the CCNV space-time admitting additional Killing vector fields will contain supergravity solutions beyond the CSI or VSI examples. By removing the condition for the transverse space to be locally homogeneous, such a solution would preserve a nonminimal fraction of supersymmetries while reducing the number of space-like symmetries. This will be investigated in future work in the context of five-dimensional bosonic string theory.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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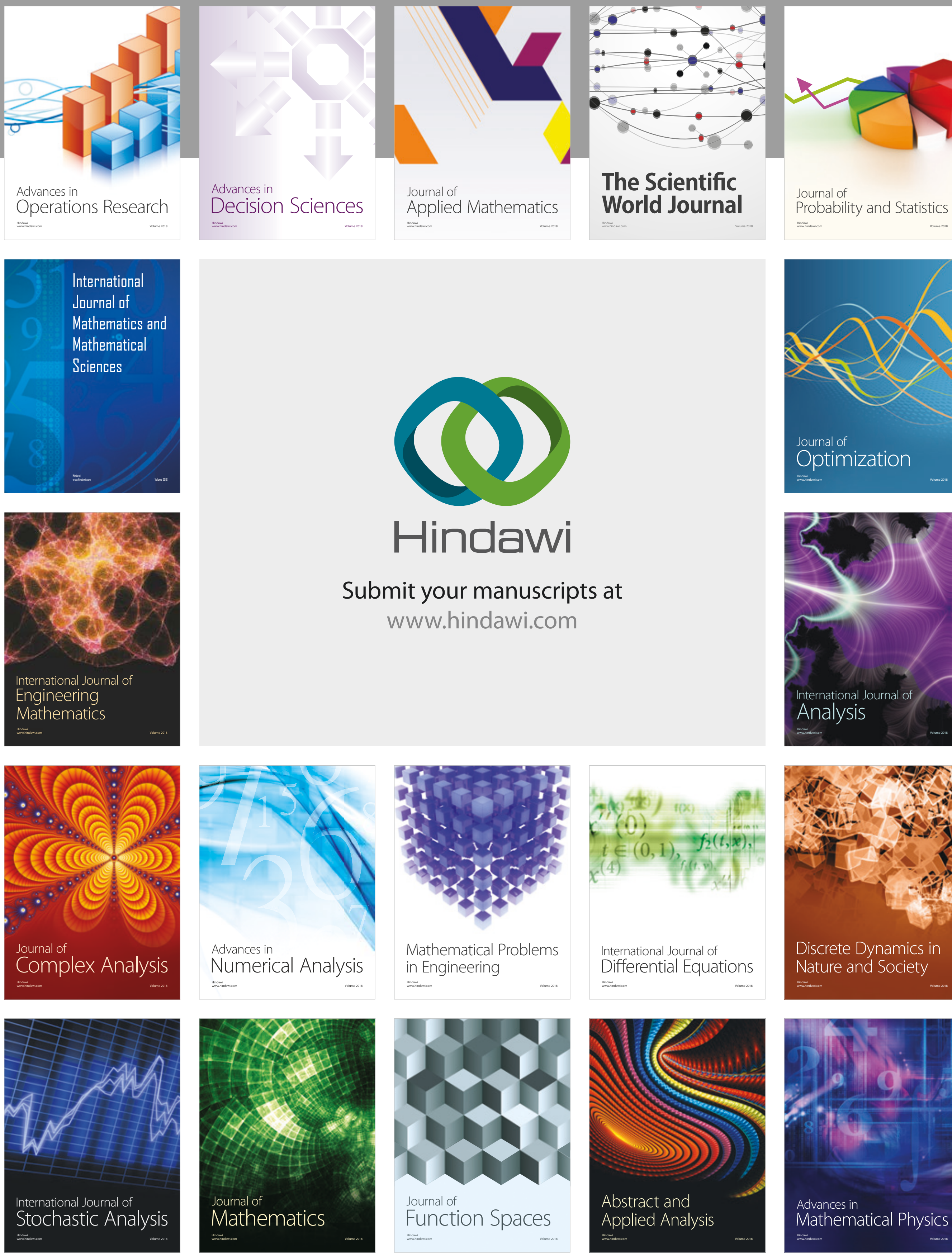
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