Hindawi
Advances in Mathematical Physics
Volume 2018, Article ID 5971646, 9 pages
https://doi.org/10.1155/2018/5971646

Research Article

Exact Traveling Wave Solutions to the (2 + 1)-Dimensional Jaulent-Miodek Equation

Yongyi Gu,[1] **Bingmao Deng**,[2] **and Jianming Lin**[3]

[1] *School of Statistics and Mathematics, Guangdong University of Finance and Economics, Guangzhou 510320, China*
[2] *Institute of Applied Mathematics, South China Agricultural University, Guangzhou 510642, China*
[3] *School of Economic and Management, Guangzhou University of Chinese Medicine, Guangzhou 510006, China*

Correspondence should be addressed to Yongyi Gu; gdguyongyi@163.com and Jianming Lin; ljmguanli@126.com

Received 9 September 2017; Revised 31 March 2018; Accepted 23 April 2018; Published 19 June 2018

Academic Editor: Stephen C. Anco

We derive exact traveling wave solutions to the (2 + 1)-dimensional Jaulent-Miodek equation by means of the complex method, and then we illustrate our main result by some computer simulations. It has presented that the applied method is very efficient and is practically well suited for the nonlinear differential equations that arise in mathematical physics.

1. Introduction and Main Results

Nonlinear differential equations widely describe many important dynamical systems in various fields of science, especially in nonlinear optics, plasma physics, solid state physics, and fluid mechanics. It has aroused widespread attention in the study of nonlinear differential equations [1–28]. Exact solutions of nonlinear differential equations play an important role in the study of mathematical physics phenomena. Hence, seeking explicit solutions of physics equations is an interesting and significant subject.

In 2001, Geng et al. [29] developed some (2 + 1)-dimensional models from the Jaulent-Miodek hierarchy [30]. Over the past few years, many research results for the (2 + 1)-dimensional Jaulent-Miodek equations have been generated [31–34], such as the algebraic-geometrical solutions, the bifurcation and exact solutions, the N-soliton solution, and Multiple kink solutions for the (2 + 1)-dimensional Jaulent-Miodek equations.

In 2012, Zhang et al. [35] studied the following (2 + 1)-dimensional Jaulent-Miodek equation:

$$a_1 u_{xt} + a_2 u_x^2 u_{xx} - u_{xxxx} - a_3 u_{xx} u_y - a_4 u_x u_{xy} + a_5 u_{yy} = 0, \tag{1}$$

where a_i are constants, $i = 1, 2, \ldots, 5$.

Substituting traveling wave transform

$$u(x, y, t) = v(z), \quad z = x + ly + \lambda t, \tag{2}$$

into (1), and then integrating it we get

$$v''' - \left(a_1\lambda + a_5 l^2\right) v' + \frac{lb}{2}\left(v'\right)^2 - \frac{a_2}{3}\left(v'\right)^3 - \delta = 0, \tag{3}$$

where $b = a_3 + a_4$, l and λ are constants, and δ is the integration constant. Setting $w = v'$, (3) becomes

$$w'' - \left(a_1\lambda + a_5 l^2\right) w + \frac{lb}{2} w^2 - \frac{a_2}{3} w^3 - \delta = 0. \tag{4}$$

We say that a meromorphic function ζ belongs to the class W if ζ is an elliptic function, or a rational function of z, or a rational function of $e^{\mu z}$, $\mu \in \mathbb{C}$. Only these functions can satisfy an algebraic addition theorem which was proved by Weierstrass, so the letter W was utilized [36]. In 2006, Eremenko [36] proved that all meromorphic solutions of the Kuramoto-Sivashinsky algebraic differential equation belong to the class W. Recently, Kudryashov et al. [37, 38] used Laurent series to seek meromorphic exact solutions of some nonlinear differential equations. Following their work, the complex method was proposed by Yuan et al. [39, 40].

They employed the Nevanlinna value distribution theory to investigate the existence of meromorphic solutions to some differential equations and then obtain the representations of meromorphic solutions to these differential equations [41, 42]. It shows that the complex method has a strong theoretical basis which can proof that meromorphic solutions of certain differential equations belong to the class W and obtain exact solutions by the indeterminate forms of the solutions. Besides, this method can be applied to get all traveling wave exact solutions or general solutions of related differential equations [43, 44]. In this article, we would like to use the complex method to obtain exact traveling wave solutions to the $(2+1)$-dimensional Jaulent-Miodek equation.

Theorem 1. *If $a_2 \neq 0$, then the meromorphic solutions w of (4) belong to the class W. In addition, (4) has the following classes of solutions.*

(i) The rational function solutions

$$w_{r,1}(z) = \pm\sqrt{\frac{6}{a_2}}\frac{1}{z-z_0} + \frac{lb}{2a_2},$$

$$w_{r,2}(z) = \pm\sqrt{\frac{6}{a_2}}\left(\frac{1}{z-z_0} - \frac{1}{z-z_0-z_1} - \frac{1}{z_1}\right) + \frac{lb}{2a_2}, \tag{5}$$

where $z_0 \in \mathbb{C}$, $z_1 \neq 0$. $\lambda = -l^2(4a_2a_5 - b^2)/4a_1a_2$, $\delta = -l^3b^3/24a_2^2$ in the former case, or $\lambda = -((4a_2a_5 - b^2)l^2z_1^2 + 24a^2)/4a_1a_2z_1^2$, $\delta = -(l^3b^3z_1^3 - 72lba_2z_1 - 96\sqrt{6}a_2^{3/2})/24a_2^2z_1^3$ in the latter case.

(ii) The simply periodic solutions

$$w_{s,1}(z) = \pm\sqrt{\frac{3}{2a_2}}\mu\coth\frac{\mu}{2}(z-z_0) + \frac{lb}{2a_2},$$

$$w_{s,2}(z) = \pm\sqrt{\frac{3}{2a_2}}\mu\left(\coth\frac{\mu}{2}(z-z_0) - \coth\frac{\mu}{2}(z-z_0-z_1) - \coth\frac{\mu}{2}z_1\right) + \frac{lb}{2a_2}, \tag{6}$$

where $z_0 \in \mathbb{C}$, $z_1 \neq 0$. $\lambda = -((4a_2a_5 - b^2)l^2 + 6a_2\mu^2)/4a_1a_2$, $\delta = -(l^3b^3 + 12\sqrt{6}\mu^3a_2^{3/2} - 18lba_2\mu^2)/24a_2^2$ in the former case, or $\lambda = (3\mu^2/2)\coth^2(\mu/2)z_1 - l^2b^2/4a_2$, $\delta = \sqrt{3/2a_2}\mu^2\coth^2(\mu/2)z_1(\mu\coth(\mu/2)z_1 + \sqrt{3/2a_2}(lb/2)) - l^3b^3/24a_2^2$ in the latter case.

(iii) The elliptic function solutions

$$w_d(z) = \pm\sqrt{\frac{3}{2a_2}}\frac{(-\wp + E)\left(4\wp E^2 + 4\wp^2E + 2\wp' F - \wp g_2 - Eg_2\right)}{\left((12E^2 - g_2)\wp + 4E^3 - 3Eg_2\right)\wp' + \left(4\wp^3 + 12E\wp^2 - 3g_2\wp - Eg_2\right)F} + \frac{lb}{2a_2}, \tag{7}$$

where $g_3 = 0$, $F^2 = 4E^3 - g_2E$, E and g_2 are arbitrary.

Theorem 2. *If $a_2 \neq 0$, then traveling wave exact solutions of (3) have the following forms.*

(i) The rational function solutions

$$v_{r,1}(z) = \pm\sqrt{\frac{6}{a_2}}\ln(z-z_0) + \frac{lb}{2a_2}(z-z_0) + c_1,$$

$$v_{r,2}(z) = \pm\sqrt{\frac{6}{a_2}}\ln\frac{z-z_0}{z-z_1-z_0} + \left(\frac{lb}{2a_2} - \sqrt{\frac{6}{a_2}\frac{1}{z_1}}\right)(z-z_0) + c_2, \tag{8}$$

where $z_0 \in \mathbb{C}$, $z_1 \neq 0$, c_1 and c_2 are integral constants.

(ii) The simply periodic solutions

$$v_{s,1}(z) = \mp\sqrt{\frac{3}{2a_2}}\mu\ln\left(\coth^2\frac{\mu}{2}(z-z_0) - 1\right) + \frac{lb}{2a_2}(z-z_0) + c_3,$$

$$v_{s,2}(z) = \mp\sqrt{\frac{3}{2a_2}}\mu\ln\left(\coth^2\frac{\mu}{2}(z-z_0) - 1\right) \pm \sqrt{\frac{3}{2a_2}}\mu\ln\left(\coth^2\frac{\mu}{2}(z-z_0-z_1) - 1\right) + \left(\frac{lb}{2a_2} \mp \sqrt{\frac{3}{2a_2}}\mu\coth\frac{\mu}{2}z_1\right)(z-z_0) + c_4, \tag{9}$$

where $z_0 \in \mathbb{C}$, $z_1 \neq 0$, c_3 and c_4 are integral constants.

(iii) The elliptic function solutions

$$v_d(z) = \pm\sqrt{\frac{3}{2a_2}}\ln\left(-\wp(z) + \frac{1}{4}\left(\frac{\wp'(z) + G}{\wp(z) - H}\right)^2 - H\right) + \frac{lb}{2a_2}(z-z_0) + c_5, \tag{10}$$

where c_5 is the integral constant, $G^2 = 4H^3 - g_2H$, $g_3 = 0$.

The rest of this paper is organized as follows. Section 2 introduces some preliminary theory and the complex method. In Section 3, we will give the proof of Theorems 1 and 2. Some computer simulations will be given to illustrate our main results in Section 4. Conclusions are presented at the end of the paper.

2. Preliminary Theory and the Complex Method

At first, we give some notations and definitions, and then we introduce some lemmas and the complex method.

Let $m \in \mathbb{N} := \{1, 2, 3, \ldots\}$, $r_j \in \mathbb{Z}$, $j = 0, 1, \ldots, m$, $r = (r_0, r_1, \ldots, r_m)$, and

$$K_r[w](z) := [w(z)]^{r_0}[w'(z)]^{r_1}[w''(z)]^{r_2}\cdots[w^{(m)}(z)]^{r_m}, \tag{11}$$

then $d(r) := \sum_{j=0}^{m} r_j$ is the degree of $K_r[w]$. Let the differential polynomial be defined by

$$F(w, w', \ldots, w^{(m)}) := \sum_{r\in J} a_r K_r[w], \tag{12}$$

where J is a finite index set, and a_r are constants, then $\deg F(w, w', \ldots, w^{(m)}) := \max_{r\in J}\{d(r)\}$ is the degree of $F(w, w', \ldots, w^{(m)})$.

Consider the following differential equation:

$$F(w, w', \ldots, w^{(m)}) = cw^n + d, \tag{13}$$

where $n \in \mathbb{N}$, $c \neq 0$, d are constants.

Set $p, q \in \mathbb{N}$, and meromorphic solutions w of (13) have at least one pole. If (13) has exactly p distinct meromorphic solutions, and their multiplicity of the pole at $z = 0$ is q, then (13) is said to satisfy the $\langle p, q\rangle$ condition. It could be not easy to show that the $\langle p, q\rangle$ condition of (13) holds, so we need the weak $\langle p, q\rangle$ condition as follows.

Inserting the Laurent series

$$w(z) = \sum_{\tau=-q}^{\infty} \beta_\tau z^\tau, \quad \beta_{-q} \neq 0,\ q > 0, \tag{14}$$

into (13), we can determine exactly p different Laurent singular parts:

$$\sum_{\tau=-q}^{-1} \beta_\tau z^\tau; \tag{15}$$

then (13) is said to satisfy the weak $\langle p, q\rangle$ condition.

Given two complex numbers ν_1, ν_2, $\operatorname{Im}(\nu_1/\nu_2) > 0$, and let L be the discrete subset $L[2\nu_1, 2\nu_2] := \{\nu \mid \nu = 2c_1\nu_1 + 2c_2\nu_2,\ c_1, c_2 \in \mathbb{Z}\}$, and L is isomorphic to $\mathbb{Z}\times\mathbb{Z}$. Let the discriminant $\Delta = \Delta(b_1, b_2) := b_1^3 - 27b_2^2$ and

$$h_n = h_n(L) := \sum_{\nu\in L\setminus\{0\}} \frac{1}{\nu^n}. \tag{16}$$

A meromorphic function $\wp(z) := \wp(z, g_2, g_3)$ with double periods $2\nu_1$, $2\nu_2$, which satisfies the equation

$$(\wp'(z))^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \tag{17}$$

in which $g_2 = 60h_4$, $g_3 = 140h_6$, and $\Delta(g_2, g_3) \neq 0$, is called the Weierstrass elliptic function.

In 2009, Eremenko et al. [45] studied the m-order Briot-Bouquet equation (BBEq)

$$F(w, w^{(m)}) = \sum_{j=0}^{n} F_j(w)(w^{(m)})^j = 0, \tag{18}$$

where $F_j(w)$ are constant coefficient polynomials, $m \in \mathbb{N}$. For the m-order BBEq, we have the following lemma.

Lemma 3 (see [37, 40, 46]). *Let $m, n, p, s \in \mathbb{N}$, $\deg F(w, w^{(m)}) < n$, and a m-order BBEq*

$$F(w, w^{(m)}) = cw^n + d \tag{19}$$

satisfies the weak $\langle p, q\rangle$ condition; then the meromorphic solutions w belong to the class W. Suppose for some values of parameters such solution w exists; then other meromorphic solutions form a one-parametric family $(z - z_0)$, $z_0 \in \mathbb{C}$. Furthermore, each elliptic solution with pole at $z = 0$ can be written as

$$\begin{aligned} w(z) = {}& \sum_{i=1}^{s-1}\sum_{j=2}^{q} \frac{(-1)^j\beta_{-ij}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\left(\frac{1}{4}\left[\frac{\wp'(z)+C_i}{\wp(z)-D_i}\right]^2 - \wp(z)\right) + \sum_{i=1}^{s-1}\frac{\beta_{-i1}}{2}\frac{\wp'(z)+C_i}{\wp(z)-D_i} \\ & + \sum_{j=2}^{q}\frac{(-1)^j\beta_{-sj}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z) + \beta_0, \end{aligned} \tag{20}$$

where β_{-ij} are determined by (14), $\sum_{i=1}^{s}\beta_{-i1} = 0$, and $C_i^2 = 4D_i^3 - g_2D_i - g_3$.

Each rational function solution $w := R(z)$ is expressed as

$$R(z) = \sum_{i=1}^{s}\sum_{j=1}^{q}\frac{\beta_{ij}}{(z - z_i)^j} + \beta_0, \tag{21}$$

which has $s(\leq p)$ distinct poles of multiplicity q.

Each simply periodic solution $w := R(\eta)$ is a rational function of $\eta = e^{\mu z}$ $(\mu \in \mathbb{C})$ and is expressed as

$$R(\eta) = \sum_{i=1}^{s}\sum_{j=1}^{q}\frac{\beta_{ij}}{(\eta - \eta_i)^j} + \beta_0, \tag{22}$$

which has $s(\leq p)$ distinct poles of multiplicity q.

Lemma 4 (see [46, 47]). *Weierstrass elliptic functions $\wp(z)$ have an addition formula as below:*

$$\wp(z - z_0) = -\wp(z_0) - \wp(z) + \frac{1}{4}\left[\frac{\wp'(z)+\wp'(z_0)}{\wp(z)-\wp(z_0)}\right]^2. \tag{23}$$

When $g_2 = g_3 = 0$, Weierstrass elliptic functions can be degenerated to rational functions according to

$$\wp(z,0,0) = \frac{1}{z^2}. \tag{24}$$

When $\Delta(g_2, g_3) = 0$, it can also be degenerated to simple periodic functions according to

$$\wp\left(z, 3d^2, -d^3\right) = 2d - \frac{3d}{2}\coth^2\sqrt{\frac{3d}{2}}z. \tag{25}$$

By the above definitions and lemmas, we now present the complex method as below for the convenience of readers.

Step 1. Substitute the transformation $T : u(x,t) \to w(z)$ defined by $(x,t) \to z$ into a given partial differential equation (PDE) to yield a nonlinear ordinary differential equation (ODE).

Step 2. Substitute (14) into the ODE to determine whether the weak $\langle p, q\rangle$ condition holds.

Step 3. Find out meromorphic solutions $w(z)$ of the ODE with a pole at $z = 0$, in which we have $m-1$ integral constants.

Step 4. Obtain meromorphic solutions $w(z - z_0)$ by Lemmas 3 and 4.

Step 5. Substituting the inverse transformation T^{-1} into the meromorphic solutions, we get the exact solutions for the original PDE.

3. Proof of Main Results

Proof of Theorem 1. Substituting (14) into (4) we have $p = 2$, $q = 1$, $\beta_{-1} = \pm\sqrt{6/a_2}$, $\beta_0 = lb/2a_2$, $\beta_1 = -\sqrt{6}((4a_2a_5 - b^2)l^2 + 4\lambda a_1a_2)/24a_2^{3/2}$, $\beta_2 = -((6ba_2a_5 - b^3)l^3 + 6lb\lambda a_1a_2 + 12\delta a_2^2)/48a_2^2$ and β_3 is an arbitrary constant.

Therefore, (4) is a second-order BBEq and satisfies weak $\langle 2, 1\rangle$ condition. Hence, by Lemma 3, we obtain that meromorphic solutions of (4) belong to W. We will show meromorphic solutions of (4) in the following.

By (21), we infer that the indeterminate rational solutions of (4) are

$$R_1(z) = \frac{\beta_{11}}{z} + \frac{\beta_{12}}{z - z_1} + \beta_{10}, \tag{26}$$

with pole at $z = 0$.

Substituting $R_1(z)$ into (4), we have

$$R_{1,1}(z) = \pm\sqrt{\frac{6}{a_2}}\frac{1}{z} + \frac{lb}{2a_2}, \tag{27}$$

where $\lambda = -l^2(4a_2a_5 - b^2)/4a_1a_2$ and $\delta = -l^3b^3/24a_2^2$.

$$R_{1,2}(z) = \pm\sqrt{\frac{6}{a_2}}\left(\frac{1}{z} - \frac{1}{z - z_1} - \frac{1}{z_1}\right) + \frac{lb}{2a_2}, \tag{28}$$

where $\lambda = -((4a_2a_5 - b^2)l^2z_1^2 + 24a^2)/4a_1a_2z_1^2$ and $\delta = -(l^3b^3z_1^3 - 72lba_2z_1 - 96\sqrt{6}a_2^{3/2})/24a_2^2z_1^3$.

So the rational solutions of (4) are

$$\begin{aligned} w_{r,1}(z) &= \pm\sqrt{\frac{6}{a_2}}\frac{1}{z - z_0} + \frac{lb}{2a_2}, \\ w_{r,2}(z) &= \pm\sqrt{\frac{6}{a_2}}\left(\frac{1}{z - z_0} - \frac{1}{z - z_0 - z_1} - \frac{1}{z_1}\right) \\ &\quad + \frac{lb}{2a_2}, \end{aligned} \tag{29}$$

where $z_0 \in \mathbb{C}$, $z_1 \neq 0$. $\lambda = -l^2(4a_2a_5 - b^2)/4a_1a_2$, $\delta = -l^3b^3/24a_2^2$ in the former case, or $\lambda = -((4a_2a_5 - b^2)l^2z_1^2 + 24a^2)/4a_1a_2z_1^2$, $\delta = -(l^3b^3z_1^3 - 72lba_2z_1 - 96\sqrt{6}a_2^{3/2})/24a_2^2z_1^3$ in the latter case.

To obtain simply periodic solutions, let $\eta = e^{\mu z}$, and substitute $w = R(\eta)$ into Eq. (4), then

$$\begin{aligned} &\mu^2\left(\eta R' + \eta^2 R''\right) - \left(a_1\lambda + a_5l^2\right)\eta + \frac{lb}{2}\eta^2 - \frac{a_2}{3}\eta^3 - \delta \\ &= 0. \end{aligned} \tag{30}$$

Substituting

$$R_2(z) = \frac{\beta_{21}}{\eta - 1} + \frac{\beta_{22}}{(\eta - \eta_1)} + \beta_{20}, \tag{31}$$

into (30), we obtain that

$$\begin{aligned} R_{2,1}(z) &= \pm\sqrt{\frac{6}{a_2}}\mu\left(\frac{1}{\eta - 1} + \frac{1}{2}\right) + \frac{lb}{2a_2}, \\ R_{2,2}(z) &= \pm\sqrt{\frac{6}{a_2}}\mu\left(\frac{1}{\eta - 1} - \frac{\eta_1}{\eta - \eta_1} - \frac{\eta_1 + 1}{2(\eta_1 - 1)}\right) \\ &\quad + \frac{lb}{2a_2}, \end{aligned} \tag{32}$$

where $\lambda = -((4a_2a_5 - b^2)l^2 + 6a_2\mu^2)/4a_1a_2$, $\delta = -(l^3b^3 + 12\sqrt{6}\mu^3a_2^{3/2} - 18lba_2\mu^2)/24a_2^2$ in the former case, or $\lambda = 3\mu^2(\eta_1 + 1)^2/2(\eta_1 - 1)^2 - l^2b^2/4a_2$, $\delta = 3(\eta_1 + 1)^2((\sqrt{2a_2/3}\mu + lb/2)\eta_1 + \sqrt{2a_2/3}\mu - lb/2)\mu^2/2a_2(\eta_1 - 1)^3 - l^3b^3/24a_2^3$ in the latter case.

Inserting $\eta = e^{\mu z}$ into (32), we can get simply periodic solutions to (4) with pole at $z = 0$

$$\begin{aligned} w_{s0,1}(z) &= \pm\sqrt{\frac{3}{2a_2}}\mu\coth\frac{\mu}{2}z + \frac{lb}{2a_2}, \\ w_{s0,2}(z) & \\ &= \pm\sqrt{\frac{3}{2a_2}}\mu\left(\coth\frac{\mu}{2}z - \coth\frac{\mu}{2}(z - z_1) - \coth\frac{\mu}{2}z_1\right) \\ &\quad + \frac{lb}{2a_2}, \end{aligned} \tag{33}$$

where $\lambda = -((4a_2a_5 - b^2)l^2 + 6a_2\mu^2)/4a_1a_2$, $\delta = -(l^3b^3 + 12\sqrt{6}\mu^3a_2^{3/2} - 18lba_2\mu^2)/24a_2^2$ in the former case, or $\lambda = (3\mu^2/2)\coth^2(\mu/2)z_1 - l^2b^2/4a_2$, $\delta = \sqrt{3/2a_2}\mu^2\coth^2(\mu/2)z_1(\mu\coth(\mu/2)z_1 + \sqrt{3/2a_2}(lb/2)) - l^3b^3/24a_2^2$ in the latter case.

So simply periodic solutions of (4) are

$$\begin{aligned} w_{s,1}(z) &= \pm\sqrt{\frac{3}{2a_2}}\mu\coth\frac{\mu}{2}(z-z_0) + \frac{lb}{2a_2}, \\ w_{s,2}(z) &= \pm\sqrt{\frac{3}{2a_2}}\mu\left(\coth\frac{\mu}{2}(z-z_0) - \coth\frac{\mu}{2}(z-z_0-z_1) - \coth\frac{\mu}{2}z_1\right) + \frac{lb}{2a_2}, \end{aligned} \tag{34}$$

where $z_0 \in \mathbb{C}$, $z_1 \neq 0$. $\lambda = -((4a_2a_5 - b^2)l^2 + 6a_2\mu^2)/4a_1a_2$, $\delta = -(l^3b^3 + 12\sqrt{6}\mu^3a_2^{3/2} - 18lba_2\mu^2)/24a_2^2$ in the former case, or $\lambda = (3\mu^2/2)\coth^2(\mu/2)z_1 - l^2b^2/4a_2$, $\delta = \sqrt{3/2a_2}\mu^2\coth^2(\mu/2)z_1(\mu\coth(\mu/2)z_1 + \sqrt{3/2a_2}(lb/2)) - l^3b^3/24a_2^2$ in the latter case.

From (20), we have the indeterminate relations to elliptic solutions of (4) with pole at $z = 0$

$$w_{d1}(z) = \frac{\beta_{-1}}{2}\frac{\wp'(z) + C_1}{\wp(z) - D_1} + \beta_{30}, \tag{35}$$

where $C_1^2 = 4D_1^3 - g_2D_1 - g_3$. Making use of Lemma 4 to $w_{d1}(z)$, and considering the results obtained above, we infer that $\beta_{30} = lb/2a_2$, $g_3 = 0$, $C_1 = D_1 = 0$. So we obtain

$$w_{d1}(z) = \pm\sqrt{\frac{3}{2a_2}}\frac{\wp'(z)}{\wp(z)} + \frac{lb}{2a_2}, \tag{36}$$

where $g_3 = 0$.

Thus, the elliptic function solutions of (4) are

$$w_d(z) = \pm\sqrt{\frac{3}{2a_2}}\frac{\wp'(z-z_0, g_2, 0)}{\wp(z-z_0, g_2, 0)} + \frac{lb}{2a_2}, \tag{37}$$

where $z_0 \in \mathbb{C}$, $g_3 = 0$, g_2 is arbitrary. Applying the addition formula, we can rewrite it as

$$w_d(z) = \pm\sqrt{\frac{3}{2a_2}}\frac{(-\wp + E)(4\wp E^2 + 4\wp^2E + 2\wp'F - \wp g_2 - Eg_2)}{((12E^2 - g_2)\wp + 4E^3 - 3Eg_2)\wp' + (4\wp^3 + 12E\wp^2 - 3g_2\wp - Eg_2)F} + \frac{lb}{2a_2}, \tag{38}$$

where $g_3 = 0$, $F^2 = 4E^3 - g_2E$, E and g_2 are arbitrary. □

Proof of Theorem 2. By Theorem 1, we can obtain the rational function solutions of (3) which are

$$\begin{aligned} v_{r,1}(z) &= \int w_{r,1}(z)\,dz = \int\left(\pm\sqrt{\frac{6}{a_2}}\frac{1}{z-z_0} + \frac{lb}{2a_2}\right)dz = \pm\sqrt{\frac{6}{a_2}}\ln(z-z_0) + \frac{lb}{2a_2}(z-z_0) + c_1, \\ v_{r,2}(z) &= \int w_{r,2}(z)\,dz \\ &= \int\left(\pm\sqrt{\frac{6}{a_2}}\left(\frac{1}{z-z_0} - \frac{1}{z-z_0-z_1} - \frac{1}{z_1}\right) + \frac{lb}{2a_2}\right)dz = \pm\sqrt{\frac{6}{a_2}}\ln\frac{z-z_0}{z-z_1-z_0} + \left(\frac{lb}{2a_2} \mp\sqrt{\frac{6}{a_2}\frac{1}{z_1}}\right)(z-z_0) + c_2, \end{aligned} \tag{39}$$

where $z_0 \in \mathbb{C}$, $z_1 \neq 0$, c_1 and c_2 are integral constants.

The simply periodic solutions of (3) are

$$\begin{aligned} v_{s,1}(z) &= \int w_{s,1}(z)\,dz = \int\left(\pm\sqrt{\frac{3}{2a_2}}\mu\coth\frac{\mu}{2}(z-z_0) + \frac{lb}{2a_2}\right)dz = \mp\sqrt{\frac{3}{2a_2}}\mu\ln\left(\coth^2\frac{\mu}{2}(z-z_0) - 1\right) + \frac{lb}{2a_2}(z-z_0) + c_3, \\ v_{s,2}(z) &= \int w_{s,2}(z)\,dz \\ &= \int\left(\pm\sqrt{\frac{3}{2a_2}}\mu\left(\coth\frac{\mu}{2}(z-z_0) - \coth\frac{\mu}{2}(z-z_0-z_1) - \coth\frac{\mu}{2}z_1\right)\right)dz \\ &= \mp\sqrt{\frac{3}{2a_2}}\mu\ln\left(\coth^2\frac{\mu}{2}(z-z_0) - 1\right) \pm\sqrt{\frac{3}{2a_2}}\mu \cdot \ln\left(\coth^2\frac{\mu}{2}(z-z_0-z_1) - 1\right) + \left(\frac{lb}{2a_2} \mp\sqrt{\frac{3}{2a_2}}\mu \cdot \coth\frac{\mu}{2}z_1\right)(z-z_0) + c_4, \end{aligned} \tag{40}$$

where $z_0 \in \mathbb{C}$, $z_1 \neq 0$, c_3 and c_4 are integral constants.

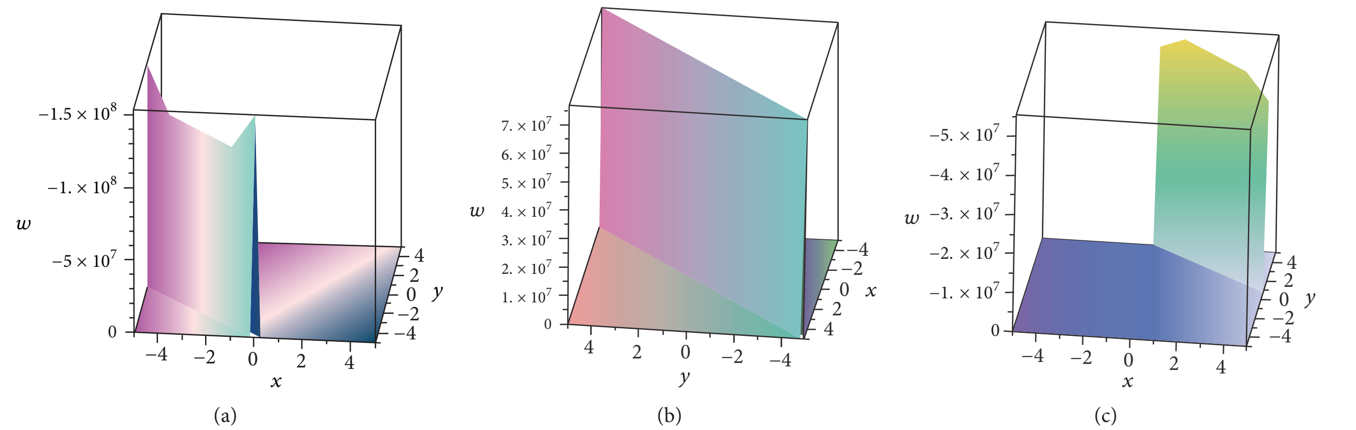

FIGURE 1: The solution of the $(2+1)$-dimensional Jaulent-Miodek equation corresponding to $w_{r,2}(z)$, (a) $t=-5$, (b) $t=0$, and (c) $t=5$.

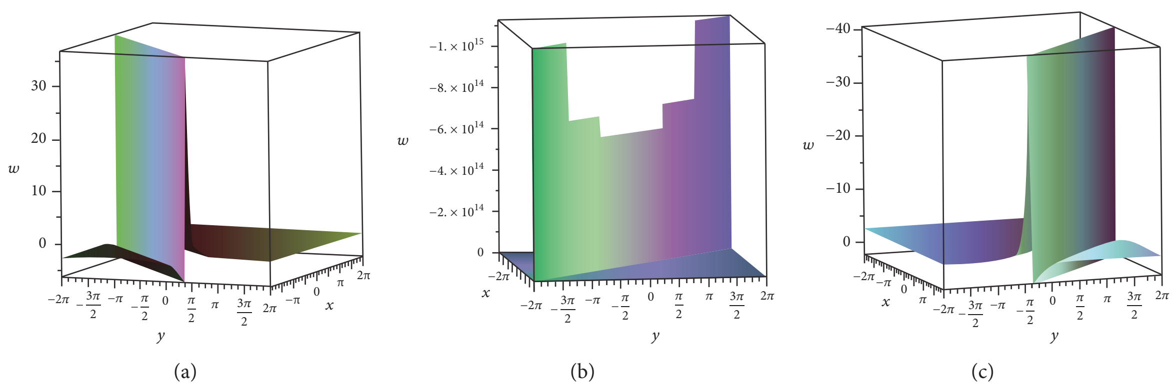

FIGURE 2: The solution of the $(2+1)$-dimensional Jaulent-Miodek equation corresponding to $w_{s,2}(z)$, (a) $t=-5$, (b) $t=0$, and (c) $t=5$.

The elliptic function solutions of (3) are

$$\begin{aligned} v_{d,2}(z) &= \int w_d(z)\,dz \\ &= \int\left(\pm\sqrt{\frac{3}{2a_2}}\frac{\wp'(z-z_0)}{\wp(z-z_0)}+\frac{lb}{2a_2}\right)dz \\ &= \pm\sqrt{\frac{3}{2a_2}}\ln\wp(z-z_0)+\frac{lb}{2a_2}(z-z_0)+c_5 \\ &= \pm\sqrt{\frac{3}{2a_2}}\ln\left(-\wp(z)+\frac{1}{4}\left(\frac{\wp'(z)+G}{\wp(z)-H}\right)^2-H\right) \\ &\quad+\frac{lb}{2a_2}(z-z_0)+c_5, \end{aligned} \tag{41}$$

where c_5 is the integral constant, $G^2=4H^3-g_2H$, $g_3=0$. □

4. Computer Simulations

In this section, we illustrate our main results by some computer simulations. We carry out further analysis to the properties of the new solutions as in the following figures.

(1) By employing the complex method, we are able to obtain the rational solutions $w_{r,1}(z)$ and $w_{r,2}(z)$ of (4). Figure 1 shows shape of solutions $w_{r,2}(z)$ for $a_2=6$, $b=-24$, $l=1$, $\lambda=-1$, $z_0=0$, and $z_1=1$ within the interval $-5\le x, y\le 5$. Note that they have one generation pole which are showed by Figure 1.

(2) By applying the complex method, we achieve the simply periodic solutions $w_{s,1}(z)$ and $w_{s,2}(z)$ of (4). The solutions $w_{s,1}(z)$ and $w_{s,2}(z)$ come from the hyperbolic function. Figure 2 shows the shape of solutions $w_{s,1}(z)$ for $a_2=6$, $b=-24$, $l=1$, $\lambda=-1$, $z_0=0$, and $\mu=1$ within the interval $-2\pi\le x, y\le 2\pi$.

(3) By using the complex method, we are able to get the rational solutions $v_{r,1}(z)$ and $v_{r,2}(z)$ of (3). Figure 3 shows the shape of solutions $v_{r,2}(z)$ for $a_2=6$, $b=6$, $l=1$, $\lambda=-1$, $z_0=0$, $z_1=6$, and $c_2=0$ within the interval $-5\le x, y\le 5$. Note that they have one generation pole which are showed by Figure 3.

(4) By employing the complex method, we obtain the simply periodic solutions $v_{s,1}(z)$ and $v_{s,2}(z)$ of (3). The solutions $v_{s,1}(z)$ and $v_{s,2}(z)$ are in terms of the hyperbolic function solution. Figure 4 shows the shape of solutions $v_{s,1}(z)$ for $a_2=6$, $b=6$, $l=1$, $\lambda=-1$, $z_0=0$, $\mu=1$, and $c_3=0$ within the interval $-2\pi\le x, y\le 2\pi$. It may be observed from Figure 4 that when t increases, there would

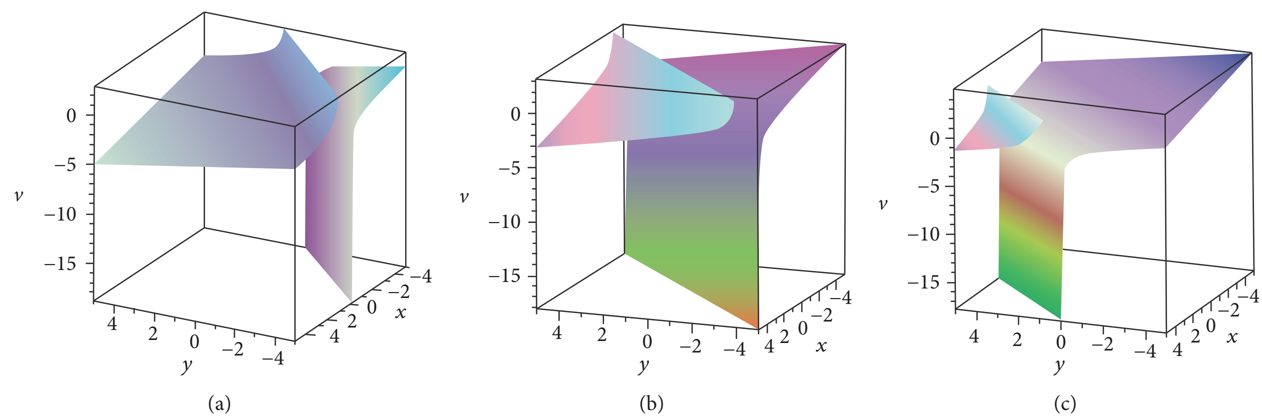

FIGURE 3: The solution of the (2 + 1)-dimensional Jaulent-Miodek equation corresponding to $v_{r,2}(z)$, (a) $t = -5$, (b) $t = 0$, and (c) $t = 5$.

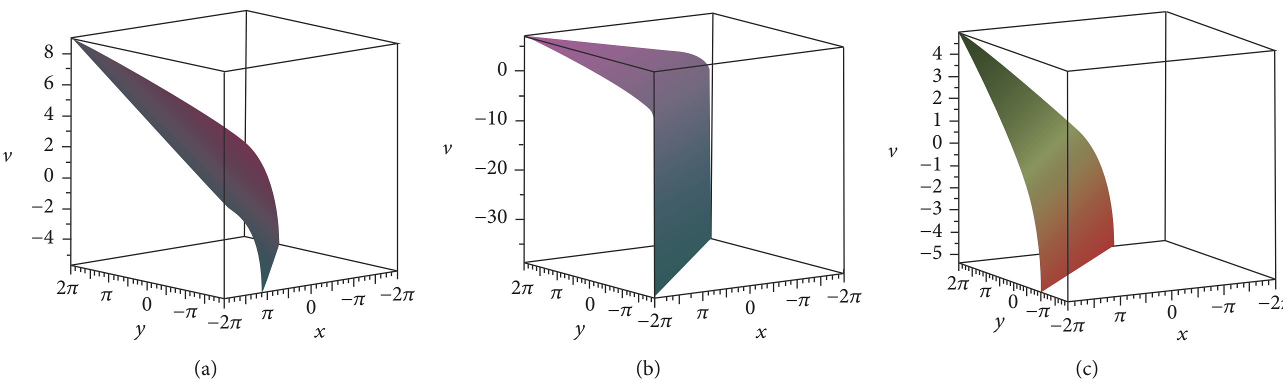

FIGURE 4: The solution of the (2 + 1)-dimensional Jaulent-Miodek equation corresponding to $v_{s,1}(z)$, (a) $t = -5$, (b) $t = 0$, and (c) $t = 5$.

be a delay for the appearance of the peak within the natural topology of traveling wave solution.

5. Conclusions

In summary, we have utilized the complex method to construct exact solutions of the nonlinear evolution equation. We first show that meromorphic solutions of the (2 + 1)-dimensional Jaulent-Miodek equation belong to the class W, and then we obtain the exact traveling wave solutions for this equation. To our knowledge, the solutions in this paper have not been reported in former literature. The simply periodic solutions $w_{s,2}(z)$, $v_{s,2}(z)$ and the rational solutions $w_{r,2}(z)$, $v_{r,2}(z)$ are not only new but also not degenerated successively by the elliptic function solutions. We expand the results in [32, 33].

Based on the previous works [32, 33], the complex method allows us to confirm that meromorphic solutions of the differential equation belong to the class W easily. By the indeterminate forms of the solutions, we can find meromorphic solutions $w(z)$ for the differential equation with a pole at $z = 0$; then we are able to obtain all meromorphic solutions $w(z-z_0), z_0 \in \mathbb{C}$ for the differential equation with an arbitrary pole. The results demonstrate that the applied method is direct and efficient method, which allows us to do tedious and complicated algebraic calculation. We can apply the idea of this study to other nonlinear evolution equations.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the NSF of China (11271090); the NSF of Guangdong Province (2016A030310257); and the Foundation for Young Talents in Educational Commission of Guangdong Province (2015KQNCX116).

References

[1] J. Jiang, L. Liu, and Y. Wu, "Positive solutions for p -laplacian fourth-order differential system with integral boundary conditions," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 293734, 2012.

[2] J. Jiang, L. Liu, and Y. Wu, "Multiple positive solutions of singular fractional differential system involving Stieltjes integral conditions," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 2012, no. 43, 18 pages, 2012.

[3] J. Jiang, L. Liu, and Y. Wu, "Positive solutions to singular fractional differential system with coupled boundary conditions," *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 11, pp. 3061–3074, 2013.

[4] J. Jiang and L. Liu, "Existence of solutions for a sequential fractional differential system with coupled boundary conditions," *Boundary Value Problems*, Paper No. 159, 15 pages, 2016.

[5] J. Jiang, L. Liu, and Y. Wu, "Symmetric positive solutions to singular system with multi-point coupled boundary conditions," *Applied Mathematics and Computation*, vol. 220, pp. 536–548, 2013.

[6] J.-Q. Jiang, L.-S. Liu, and Y.-H. Wu, "Positive solutions for second-order differential equations with integral boundary conditions," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 37, no. 3, pp. 779–796, 2014.

[7] X. Hao, "Positive solution for singular fractional differential equations involving derivatives," *Advances in Difference Equations*, article no. 139, 12 pages, 2016.

[8] L. Liu, Z. Wang, and Y. Wu, "Multiple positive solutions of the singular boundary value problems for second-order differential equations on the half-line," *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal*, vol. 71, no. 7-8, pp. 2564–2575, 2009.

[9] L. Liu, X. Hao, and Y. Wu, "Positive solutions for singular second order differential equations with integral boundary conditions," *Mathematical and Computer Modelling*, vol. 57, no. 3-4, pp. 836–847, 2013.

[10] L. Liu, F. Sun, X. Zhang, and Y. Wu, "Bifurcation analysis for a singular differential system with two parameters via to topological degree theory," *Lithuanian Association of Nonlinear Analysts. Nonlinear Analysis: Modelling and Control*, vol. 22, no. 1, pp. 31–50, 2017.

[11] F. Li and Q. Gao, "Blow-up of solution for a nonlinear Petrovsky type equation with memory," *Applied Mathematics and Computation*, vol. 274, pp. 383–392, 2016.

[12] F. Li, "Limit behavior of the solution to nonlinear viscoelastic Marguerre–von Kármán shallow shell system," *Journal of Differential Equations*, vol. 249, no. 6, pp. 1241–1257, 2010.

[13] F. Li and Y. Bai, "Uniform rates of decay for nonlinear viscoelastic Marguerre-von Kármán shallow shell system," *Journal of Mathematical Analysis and Applications*, vol. 351, no. 2, pp. 522–535, 2009.

[14] Y. Guo, "Nontrivial solutions for boundary-value problems of nonlinear fractional differential equations," *Bulletin of the Korean Mathematical Society*, vol. 47, no. 1, pp. 81–87, 2010.

[15] Y. Guo, "Solvability of boundary-value problems for nonlinear fractional differential equations," *Ukrainian Mathematical Journal*, vol. 62, no. 9, pp. 1409–1419, 2011.

[16] X. Lin and Z. Zhao, "Iterative technique for a third-order differential equation with three-point nonlinear boundary value conditions," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 2016, 2016.

[17] X. Zheng, Y. Shang, and X. Peng, "Orbital stability of solitary waves of the coupled Klein-Gordon-Zakharov equations," *Mathematical Methods in the Applied Sciences*, vol. 40, no. 7, pp. 2623–2633, 2017.

[18] X. Zheng, Y. Shang, and X. Peng, "Orbital stability of periodic traveling wave solutions to the generalized Zakharov equations," *Acta Mathematica Scientia B*, vol. 37, no. 4, pp. 998–1018, 2017.

[19] X. Zheng, Y. Shang, and H. Di, "The time-periodic solutions to the modified Zakharov equations with a quantum correction," *Mediterranean Journal of Mathematics*, vol. 14, no. 4, Art. 152, 17 pages, 2017.

[20] J. Liu and Z. Zhao, "An application of variational methods to second-order impulsive differential equation with derivative dependence," *Electronic Journal of Differential Equations*, vol. 62, pp. 1–13, 2014.

[21] Y. Huang and F. Meng, "Oscillation criteria for forced second-order nonlinear differential equations with damping," *Journal of Computational and Applied Mathematics*, vol. 224, no. 1, pp. 339–345, 2009.

[22] Y. Wang, L. Liu, and Y. Wu, "Positive solutions for a nonlocal fractional differential equation," *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal*, vol. 74, no. 11, pp. 3599–3605, 2011.

[23] Y. Wang, L. Liu, and Y. Wu, "Existence and uniqueness of a positive solution to singular fractional differential equations," *Boundary Value Problems*, vol. 2012, article 81, 12 pages, 2012.

[24] F. Meng and Y. Huang, "Interval oscillation criteria for a forced second-order nonlinear differential equations with damping," *Applied Mathematics and Computation*, vol. 218, no. 5, pp. 1857–1861, 2011.

[25] A. Qian, "Infinitely many sign-changing solutions for a Schrödinger equation," *Advances in Difference Equations*, vol. 2011, article no. 39, 6 pages, 2011.

[26] A. Qian, "Sing-changing solutions for nonlinear problems with strong resonance," *Electronic Journal of Differential Equations*, vol. 17, 8 pages, 2012.

[27] X. Zhang, L. Liu, Y. Wu, and Y. Lu, "The iterative solutions of nonlinear fractional differential equations," *Applied Mathematics and Computation*, vol. 219, no. 9, pp. 4680–4691, 2013.

[28] K. Zhang, "On a sign-changing solution for some fractional differential equations," *Boundary Value Problems*, Paper No. 59, 8 pages, 2017.

[29] X. Geng, C. Cao, and H. H. Dai, "Quasi-periodic solutions for some (2 + 1)-dimensional integrable models generated by the Jaulent-Miodek hierarchy," *Journal of Physics A: Mathematical and General*, vol. 34, no. 5, pp. 989–1004, 2001.

[30] M. Jaulent and I. Miodek, "Nonlinear evolution equations associated with 'energy-dependent Schrödinger potentials'," *Letters in Mathematical Physics. A Journal for the Rapid Dissemination of Short Contributions in the Field of Mathematical Physics*, vol. 1, no. 3, pp. 243–250, 1976.

[31] X. Geng, "Algebraic-geometrical solutions of some multidimensional nonlinear evolution equations," *Journal of Physics A: Mathematical and General*, vol. 36, no. 9, pp. 2289–2303, 2003.

[32] H. Liu and F. Yan, "The bifurcation and exact travelling wave solutions for (2+1)-dimensional nonlinear models generated by the Jaulent-Miodek hierarchy," *International Journal of Nonlinear Science*, vol. 11, no. 2, pp. 200–205, 2011.

[33] A.-M. Wazwaz, "Multiple kink solutions and multiple singular kink solutions for (2 + 1)-dimensional nonlinear models generated by the Jaulent-Miodek hierarchy," *Physics Letters A*, vol. 373, no. 21, pp. 1844–1846, 2009.

[34] J. Wu, "N-soliton solution, generalized double Wronskian determinant solution and rational solution for a (2+1)-dimensional nonlinear evolution equation," *Physics Letters A*, vol. 373, no. 1, pp. 83–88, 2008.

[35] Y.-y. Zhang, X.-q. Liu, and G.-w. Wang, "Symmetry reductions and exact solutions of the (2 + 1)-dimensional Jaulent-Miodek

equation," *Applied Mathematics and Computation*, vol. 219, no. 3, pp. 911–916, 2012.

[36] A. Eremenko, "Meromorphic traveling wave solutions of the Kuramoto-Sivashinsky equation," *Journal of Mathematical Physics, Analysis, Geometry*, vol. 2, no. 3, pp. 278–286, 2006.

[37] N. A. Kudryashov, "Meromorphic solutions of nonlinear ordinary differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 10, pp. 2778–2790, 2010.

[38] N. A. Kudryashov, D. I. Sinelshchikov, and M. V. Demina, "Exact solutions of the generalized Bretherton equation," *Physics Letters A*, vol. 375, no. 7, pp. 1074–1079, 2011.

[39] W. Yuan, Y. Li, and J. Lin, "Meromorphic solutions of an auxiliary ordinary differential equation using complex method," *Mathematical Methods in the Applied Sciences*, vol. 36, no. 13, pp. 1776–1782, 2013.

[40] W. J. Yuan, Y. D. Shang, Y. Huang, and H. Wang, "The representation of meromorphic solutions to certain ordinary differential equations and its applications," *Science China Mathematics*, vol. 43, no. 6, pp. 563–575, 2013.

[41] W. Yuan, Y. Wu, Q. Chen, and Y. Huang, "All meromorphic solutions for two forms of odd order algebraic differential equations and its applications," *Applied Mathematics and Computation*, vol. 240, pp. 240–251, 2014.

[42] Z. Huang, L. Zhang, Q. Chen, and W. Yuan, "The representation of meromorphic solutions for a class of odd order algebraic differential equations and its applications," *Mathematical Methods in the Applied Sciences*, vol. 37, no. 10, pp. 1553–1560, 2014.

[43] W. Yuan, F. Meng, Y. Huang, and Y. Wu, "All traveling wave exact solutions of the variant Boussinesq equations," *Applied Mathematics and Computation*, vol. 268, pp. 865–872, 2015.

[44] W. Yuan, Q. Chen, J. Qi, and Y. Li, "The general traveling wave solutions of the fisher equation with degree three," *Advances in Mathematical Physics*, Article ID 657918, 2013.

[45] A. E. Eremenko, L. Liao, and T. W. Ng, "Meromorphic solutions of higher order Briot-BOUquet differential equations," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 146, no. 1, pp. 197–206, 2009.

[46] S. Lang, *Elliptic Functions*, Springer, New York, NY, USA, 2nd edition, 1987.

[47] R. Conte and M. Musette, "Elliptic general analytic solutions," *Studies in Applied Mathematics*, vol. 123, no. 1, pp. 63–81, 2009.