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Research Article

Low-Density Asymptotic Behavior of Observables of Hard Sphere Fluids

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The paper deals with a rigorous description of the kinetic evolution of a hard sphere system in the low-density (Boltzmann–Grad) scaling limit within the framework of marginal observables governed by the dual BBGKY (Bogolyubov–Born–Green–Kirkwood–Yvon) hierarchy. For initial states specified by means of a one-particle distribution function, the link between the Boltzmann–Grad asymptotic behavior of a nonperturbative solution of the Cauchy problem of the dual BBGKY hierarchy for marginal observables and a solution of the Boltzmann kinetic equation for hard sphere fluids is established. One of the advantages of such an approach to the derivation of the Boltzmann equation is an opportunity to describe the process of the propagation of initial correlations in scaling limits.

1. Introduction

The main consistent approaches to the derivation of kinetic equations from underlying large particle dynamics were formulated by Bogolyubov [1] and Grad [2, 3]. For a hard sphere system Grad's method was developed by Cercignani [4] and Lanford [5, 6]. The rigorous results on the derivation of the Boltzmann equation with hard sphere collisions by methods of perturbation theory of the BBGKY hierarchy was proved in [7–10]. The most recent advances on the low-density (Boltzmann–Grad) scaling asymptotic behavior [11] of many-particle systems, in particular, systems with short-range interaction potentials, came in [12–24].

As is well known, many-particle systems are described by means of two objects: observables and states. A functional of the mean value of observables defines a duality between observables and states and as a consequence there exist two approaches to the description of the evolution within the framework of the evolution of observables and states, respectively [25]. Traditionally the evolution of many-particle systems is described within the framework of the evolution of states governed by the BBGKY hierarchy for marginal distribution functions. An equivalent approach to the description of the evolution of many-particle systems is given in terms of marginal observables governed by the dual BBGKY hierarchy [26].

The objective of the paper is to develop an approach to the description of the kinetic evolution of a hard sphere system within the framework of the evolution of observables. For this purpose in Section 2 we consider the microscopic description of the evolution of a hard sphere system within the framework of marginal observables governed by the dual BBGKY hierarchy. Then in Section 3 the origin of the dual kinetic evolution is stated; namely, a low-density (Boltzmann–Grad) limit of a nonperturbative solution of the Cauchy problem of the dual BBGKY hierarchy is established. In Sections 4 and 5 for initial states specified by means of a one-particle distribution function the link between the dual Boltzmann hierarchy for the limit marginal observables and the Boltzmann kinetic equation and the process of the propagation of initial chaos is established. In Sections 6 and 7 obtained results extended on hard spheres fluids, namely, for initial states specified by means of a one-particle distribution function and initial correlation functions, characterized condensed states. Finally, in Section 8 we conclude with some perspectives for future research.

2. The Dual BBGKY Hierarchy with Hard Sphere Collisions

As is well known, the evolution of many-particle systems can be described within the framework of a sequence of marginal (s-particle) distribution functions as well as in terms of a sequence of marginal observables. In this section we construct a nonperturbative solution of the Cauchy problem of a hierarchy of evolution equations for marginal observables of a hard sphere system.

We consider identical particles of a unit mass with a diameter $\sigma > 0$, interacting as hard spheres with elastic collisions. Every particle is characterized by its phase coordinates $(q_i, p_i) \equiv x_i \in \mathbb{R}^3 \times \mathbb{R}^3$, $i \geq 1$. For configurations of such a system the following inequalities are satisfied: $|q_i - q_j| \geq \sigma$, $i \neq j \geq 1$; that is, the set $\mathbb{W}_n \equiv \{(q_1, \ldots, q_n) \in \mathbb{R}^{3n} \mid |q_i - q_j| < \sigma$ for at least one pair $(i, j) : i \neq j \in (1, \ldots, n)\}$ is the set of forbidden configurations in the configuration space of $n > 1$ hard spheres. Let $\mathscr{C}_\gamma$ be the space of sequences $b = (b_0, b_1, \ldots, b_n, \ldots)$ of bounded continuous functions on $\mathbb{R}^{3n} \times (\mathbb{R}^{3n} \setminus \mathbb{W}_n)$ which are symmetric with respect to permutations of the arguments $x_1, \ldots, x_n$, equal to zero on the set of forbidden configurations $\mathbb{W}_n$ and equipped with the norm: $\|b\|_{\mathscr{C}_\gamma} = \max_{n\geq 0}(\gamma^n/n!)\|b\|_{\mathscr{C}_n} = \max_{n\geq 0}(\gamma^n/n!)\sup_{x_1,\ldots,x_n}|b_n(x_1, \ldots, x_n)|$, where $0 < \gamma < 1$.

If $t \geq 0$, the evolution of marginal observables $B(t) = (B_0, B_1(t, x_1), \ldots, B_s(t, x_1, \ldots, x_s), \ldots) \in \mathscr{C}_\gamma$ of a system of a nonfixed number of hard spheres is described by the Cauchy problem of the weak formulation of the following hierarchy of evolution equations [26]:

$$\frac{\partial}{\partial t} B_s(t) = \left(\sum_{j=1}^{s} \mathscr{L}(j) + \epsilon^2 \sum_{j_1<j_2=1}^{s} \mathscr{L}_{\mathrm{int}}(j_1, j_2) \right) B_s(t) + \epsilon^2 \sum_{j_1\neq j_2=1}^{s} \mathscr{L}_{\mathrm{int}}(j_1, j_2) \cdot B_{s-1}(t, x_1, \ldots, x_{j_1-1}, x_{j_1+1}, \ldots, x_s), \tag{1}$$

$$B_s(t, x_1, \ldots, x_s)\big|_{t=0} = B_s^{\epsilon,0}(x_1, \ldots, x_s), \quad s \geq 1, \tag{2}$$

where the coefficient $\epsilon > 0$ is a scaling parameter (the ratio of the diameter $\sigma > 0$ to the mean free path of hard spheres) and on the set $\mathscr{C}_{s,0} \subset \mathscr{C}_s$ of the continuously differentiable functions with compact supports the operators $\mathscr{L}(j)$ and $\mathscr{L}_{\mathrm{int}}(j_1, j_2)$ in a dimensionless form are defined by the formulas

$$\begin{aligned} \mathscr{L}(j) b_n &\doteq \left\langle p_j, \frac{\partial}{\partial q_j} \right\rangle b_n, \\ \mathscr{L}_{\mathrm{int}}(j_1, j_2) b_n &\doteq \int_{\mathbb{S}_+^2} d\eta \left\langle \eta, (p_{j_1} - p_{j_2}) \right\rangle \\ &\quad \cdot \left(b_n(x_1, \ldots, q_{j_1}, p_{j_1}^*, \ldots, q_{j_2}, p_{j_2}^*, \ldots, x_n) \right. \\ &\quad \left. - b_n(x_1, \ldots, x_n) \right) \delta(q_{j_1} - q_{j_2} + \epsilon\eta), \end{aligned} \tag{3}$$

respectively. In (3) the symbol $\langle \cdot, \cdot \rangle$ denotes a scalar product, δ is the Dirac measure, $\mathbb{S}_+^2 \doteq \{\eta \in \mathbb{R}^3 \mid |\eta| = 1, \ \langle \eta, (p_{j_1} - p_{j_2}) \rangle > 0\}$, and the momenta $p_{j_1}^*, p_{j_2}^*$ are defined by the equalities

$$\begin{aligned} p_{j_1}^* &\doteq p_{j_1} - \eta \left\langle \eta, (p_{j_1} - p_{j_2}) \right\rangle, \\ p_{j_2}^* &\doteq p_{j_2} + \eta \left\langle \eta, (p_{j_1} - p_{j_2}) \right\rangle. \end{aligned} \tag{4}$$

We refer to recurrence evolution equations (1) as the dual BBGKY hierarchy for hard spheres in a dimensionless form. If $t \leq 0$, a generator of the dual BBGKY hierarchy for hard spheres is defined by the expression of corresponding form [10].

To construct a solution of recurrence evolution equations (1) on the space $\mathscr{C}_n \equiv \mathscr{C}(\mathbb{R}^{3n} \times (\mathbb{R}^{3n} \setminus \mathbb{W}_n))$ we introduce the group of operators $S_n(t)$ that describes dynamics of n hard spheres. It is defined by means of the phase trajectories of a hard sphere system almost everywhere on the phase space $\mathbb{R}^{3n} \times (\mathbb{R}^{3n} \setminus \mathbb{W}_n)$, namely, beyond of the set $\mathbb{M}_n^0$ of the zero Lebesgue measure, as follows:

$$\begin{aligned} (S_n(t) b_n)(x_1, \ldots, x_n) &\equiv S_n(t, 1, \ldots, n) b_n(x_1, \ldots, x_n) \\ &\doteq \begin{cases} b_n(X_1(t, x_1, \ldots, x_n), \ldots, X_n(t, x_1, \ldots, x_n)), & \text{if } (x_1, \ldots, x_n) \in (\mathbb{R}^{3n} \times (\mathbb{R}^{3n} \setminus \mathbb{W}_n)), \\ 0, & \text{if } (q_1, \ldots, q_n) \in \mathbb{W}_n, \end{cases} \end{aligned} \tag{5}$$

where $X_i(t) \equiv X_i(t, x_1, \ldots, x_n)$ is a phase trajectory of ith particle constructed in [7, 10], and the set $\mathbb{M}_n^0$ consists of the phase space points which are specified by initial data, generating multiple collisions of hard spheres in the evolutionary process, that is, collisions of more than two particles, more than one two-particle collision at the same instant, and infinite number of collisions on a finite time interval.

On the space $\mathscr{C}_n$ one-parameter mapping (5) is an isometric $*$-weak continuous group of operators; that is, it is a C_0^*-group [27].

The infinitesimal generator $\mathscr{L}_n$ of a group of operators (5) is defined in the sense of a $*$-weak convergence of the space $\mathscr{C}_n$ and it has the structure $\mathscr{L}_n = \sum_{j=1}^{n}\mathscr{L}(j) + \sum_{j_1<j_2=1}^{n}\mathscr{L}_{\mathrm{int}}(j_1,j_2)$, and the operators $\mathscr{L}(j)$ and $\mathscr{L}_{\mathrm{int}}(j_1,j_2)$ are defined by formulas (3).

A nonperturbative solution of the Cauchy problems (1) and (2) is determined by the following expansions:

$$B_s(t,x_1,\ldots,x_s) = \sum_{n=0}^{s}\frac{1}{n!}\sum_{j_1\neq\cdots\neq j_n=1}^{s}\mathfrak{A}_{1+n}(t,\{Y\setminus Z\},Z) \cdot B_{s-n}^{\epsilon,0}\left(x_1,\ldots,x_{j_1-1},x_{j_1+1},\ldots,x_{j_n-1},x_{j_n+1},\ldots,x_s\right), \quad s\geq 1. \tag{6}$$

The generating operators of expansions (6) is the $(1+n)$th-order cumulant of groups of operators (5) defined by the following expansion:

$$\mathfrak{A}_{1+n}(t,\{Y\setminus Z\},Z) \doteq \sum_{P:(\{Y\setminus Z\},Z)=\bigcup_i X_i}(-1)^{|P|-1}(|P|-1)! \cdot \prod_{X_i\subset P}S_{|\theta(X_i)|}(t,\theta(X_i)), \quad n\geq 0, \tag{7}$$

where $Y\equiv(1,\ldots,s)$, $Z\equiv(j_1,\ldots,j_n)\subset Y$, the set, consisting of one element of the set of indices $Y\setminus Z = (1,\ldots,j_1-1,j_1+1,\ldots,j_n-1,j_n+1,\ldots,s)$, we denoted by $\{Y\setminus Z\}$, the declusterization mapping θ is defined by the formula $\theta(\{Y\setminus Z\},Z)=Y$, and the symbol $\sum_P$ means the sum over all possible partitions P of the set $(1,\ldots,n)$ into $|P|$ nonempty mutually disjoint subsets $X_i\subset(1,\ldots,n)$.

The simplest examples of expansions (6) for marginal observables have the following form:

$$\begin{aligned} B_1(t,x_1) &= \mathfrak{A}_1(t,1)B_1^{\epsilon,0}(x_1),\\ B_2(t,x_1,x_2) &= \mathfrak{A}_1(t,\{1,2\})B_2^{\epsilon,0}(x_1,x_2)\\ &\quad + \mathfrak{A}_2(t,1,2)\left(B_1^{\epsilon,0}(x_1)+B_1^{\epsilon,0}(x_2)\right). \end{aligned} \tag{8}$$

On the space $\mathscr{C}_\gamma$ for the Cauchy problems (1) and (2) the following statement is true [28].

Theorem 1. *For finite sequences of infinitely differentiable functions with compact supports $B(0)=(B_0,B_1^{\epsilon,0},\ldots,B_s^{\epsilon,0},\ldots)\in\mathscr{C}_\gamma^0\subset\mathscr{C}_\gamma$ a sequence of functions determined by expansions (6) is a classical solution and for arbitrary initial data $B(0)\in\mathscr{C}_\gamma$ it is a generalized solution.*

Under the condition that $\gamma < e^{-1}$, for a sequence of marginal observables (6), the estimate holds

$$\|B(t)\|_{\mathscr{C}_\gamma} \leq e^2(1-\gamma e)^{-1}\|B(0)\|_{\mathscr{C}_\gamma}. \tag{9}$$

We remark that a one-component sequence of marginal observables corresponds to observables of certain structure; namely, the marginal observable $B^{(1)}(0) = (0,b_1^\epsilon(x_1),0,\ldots)$ corresponds to the additive-type observable, and a one-component sequence of marginal observables $B^{(k)}(0) = (0,\ldots,0,b_k^\epsilon(x_1,\ldots,x_k),0,\ldots)$ corresponds to the k-ary-type observable [26]. If in capacity of initial data (2) we consider the additive-type marginal observables, then the structure of solution expansion (6) is simplified and attains the form

$$B_s^{(1)}(t,x_1,\ldots,x_s) = \mathfrak{A}_s(t,1,\ldots,s)\sum_{j=1}^{s}b_1^\epsilon(x_j), \quad s\geq 1, \tag{10}$$

where the generating operator of this expansion is the sth-order cumulant of groups of operators (7).

In the case of k-ary-type marginal observables solution expansion (6) has the form

$$B_s^{(k)}(t,x_1,\ldots,x_s) = \frac{1}{(s-k)!}\sum_{j_1\neq\cdots\neq j_{s-k}=1}^{s}\mathfrak{A}_{1+s-k}(t, \{(1,\ldots,s)\setminus(j_1,\ldots,j_{s-k})\},j_1,\ldots,j_{s-k})\,b_k^\epsilon(x_1,\ldots, x_{j_1-1},x_{j_1+1},\ldots,x_{j_s-k-1},x_{j_s-k+1},\ldots,x_s), \quad s\geq k, \tag{11}$$

where the generating operator of this expansion is the $(1+s-k)$th-order cumulant of groups of operators (7), and, if $1\leq s<k$, we have $B_s^{(k)}(t)=0$.

We remark also that expansion (6) can be also represented in the form of the perturbation (iteration) series [26] as a result of applying of analogs of the Duhamel equation to cumulants (7) of groups of operators (5).

Let $L_n^1\equiv L^1(\mathbb{R}^{3n}\times(\mathbb{R}^{3n}\setminus\mathbb{W}_n))$ be the space of integrable functions that are symmetric with respect to permutations of the arguments $x_1,\ldots,x_n$, equal to zero on the set of forbidden configurations $\mathbb{W}_n$ and equipped with the norm: $\|f_n\|_{L^1(\mathbb{R}^{3n}\times\mathbb{R}^{3n})} = \int dx_1\cdots dx_n|f_n(x_1,\ldots,x_n)|$. A subspace of continuously differentiable functions with compact supports we denote by $L_{n,0}^1\subset L_n^1$.

The mean value of the marginal observable $B(t)\in\mathscr{C}_\gamma$ at $t\in\mathbb{R}$ is determined by the functional

$$(B(t),F(0)) = \sum_{s=0}^{\infty}\frac{1}{s!}\int_{(\mathbb{R}^3\times\mathbb{R}^3)^s}dx_1\cdots dx_s \cdot B_s(t,x_1,\ldots,x_s)F_s^{\epsilon,0}(x_1,\ldots,x_s), \tag{12}$$

where initial state of finitely many hard spheres is described by means of a sequence of the marginal distribution functions $F(0) = (1,F_1^{\epsilon,0},\ldots,F_n^{\epsilon,0},\ldots)\in L^1 = \bigoplus_{n=0}^{\infty}L_n^1$. Owing to estimate (9), functional (12) exists under the condition that $\gamma<e^{-1}$.

We remark that for mean value functional (12) the following equality holds:

$$(B(t),F(0)) = (B(0),F(t)), \tag{13}$$

where the sequence $F(t) = (1, F_1(t), \dots, F_n(t), \dots) \in L^1 = \bigoplus_{n=0}^{\infty} L_n^1$ is a solution of the BBGKY hierarchy for hard spheres. Generally such a solution is constructed by methods of perturbation theory [5–12, 29–32] (a nonperturbative solution was constructed in [33]). In case of infinitely many hard spheres [29, 30] a local in time solution of the Cauchy problem of the BBGKY hierarchy [7–12] is determined by perturbation series for arbitrary initial data from the space L_ξ^∞ of sequences of bounded functions equipped with the norm:$\|f\|_{L_\xi^\infty} = \sup_{n\geq 0}\xi^{-n}\sup_{x_1,\dots,x_n}|f_n(x_1,\dots,x_n)|\exp(\beta\sum_{i=1}^{n}(p_i^2/2))$. In this case a local in time existence of the mean value functionals $(B(0), F(t))$ and $(B(t), F(0))$ was proved in papers [7], [10] and [34], [35], respectively.

3. The Kinetic Evolution of Hard Sphere Observables

We consider the problem of the rigorous description of the kinetic evolution of hard spheres within the framework of marginal observables by giving of a low-density (Boltzmann–Grad) asymptotic behavior of the Cauchy problem of the dual BBGKY hierarchy (1), (2).

Theorem 2. *If for initial data $B_n^{\epsilon,0} \in \mathscr{C}_n$, $n \geq 1$, there exists the limit $b_n^0 \in \mathscr{C}_n$*

$$w^* - \lim_{\epsilon\to 0}\left(\epsilon^{-2n} B_n^{\epsilon,0} - b_n^0\right) = 0, \tag{14}$$

and then for arbitrary finite time interval there exists the Boltzmann–Grad limit of marginal observables (6) in the sense of a $*$-weak convergence of the space $\mathscr{C}_s$

$$w^* - \lim_{\epsilon\to 0}\left(\epsilon^{-2s} B_s(t) - b_s(t)\right) = 0, \tag{15}$$

which is determined by the expansions

$$\begin{aligned} b_s(t, x_1, \dots, x_s) = {} & \sum_{n=0}^{s-1}\int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \\ & \cdot \prod_{j\in(1,\dots,s)} S_1(t-t_1, j) \sum_{i_1\neq j_1=1}^{s} \mathscr{L}_{\mathrm{int}}^0(i_1, j_1) \\ & \times \prod_{j\in(1,\dots,s)\setminus(j_1)} S_1(t_1-t_2, j) \\ & \cdots \prod_{j\in(1,\dots,s)\setminus(j_1,\dots,j_{n-1})} S_1(t_{n-1}-t_n, j) \\ & \cdot \sum_{\substack{i_n\neq j_n=1,\\ i_n, j_n\neq(j_1,\dots,j_{n-1})}}^{s} \mathscr{L}_{\mathrm{int}}^0(i_n, j_n) \\ & \times \prod_{j\in(1,\dots,s)\setminus(j_1,\dots,j_n)} S_1(t_n, j) \\ & \cdot b_{s-n}^0\left((x_1,\dots,x_s)\setminus\left(x_{j_1},\dots,x_{j_n}\right)\right), \quad s\geq 1, \end{aligned} \tag{16}$$

where the operator $\mathscr{L}_{\mathrm{int}}^0(j_1, j_2)$ is the collision operator of point particles, namely,

$$\begin{aligned} \mathscr{L}_{\mathrm{int}}^0(j_1, j_2)\, b_n \doteq {} & \int_{\mathbb{S}_+^2} d\eta \left\langle \eta, \left(p_{j_1} - p_{j_2}\right)\right\rangle \\ & \cdot \left(b_n\left(x_1,\dots,q_{j_1}, p_{j_1}^*,\dots,q_{j_2}, p_{j_2}^*,\dots,x_n\right)\right. \\ & \left. - b_n(x_1,\dots,x_n)\right)\delta\left(q_{j_1} - q_{j_2}\right). \end{aligned} \tag{17}$$

Before proving this statement we give some comments.

We consider the Boltzmann–Grad limit of a special case of marginal observables, namely, the additive-type marginal observables. If for the initial additive-type marginal observable b_1^ϵ the following condition is satisfied:

$$w^* - \lim_{\epsilon\to 0}\left(\epsilon^{-2} b_1^\epsilon - b_1^0\right) = 0, \tag{18}$$

then, according to statement (15), for additive-type marginal observables (10) we derive

$$w^* - \lim_{\epsilon\to 0}\left(\epsilon^{-2s} B_s^{(1)}(t) - b_s^{(1)}(t)\right) = 0, \tag{19}$$

where the limit marginal observable $b_s^{(1)}(t)$ is determined as a special case of expansion (16):

$$\begin{aligned} b_s^{(1)}(t, x_1, \dots, x_s) = {} & \int_0^t dt_1 \cdots \int_0^{t_{s-2}} dt_{s-1} \\ & \cdot \prod_{j\in(1,\dots,s)} S_1(t-t_1, j) \sum_{i_1\neq j_1=1}^{s} \mathscr{L}_{\mathrm{int}}^0(i_1, j_1) \\ & \times \prod_{j\in(1,\dots,s)\setminus(j_1)} S_1(t_1-t_2, j) \\ & \cdots \prod_{j\in(1,\dots,s)\setminus(j_1,\dots,j_{s-2})} S_1(t_{s-2}-t_{s-1}, j) \\ & \times \sum_{\substack{i_{s-1}\neq j_{s-1}=1,\\ i_{s-1}, j_{s-1}\neq(j_1,\dots,j_{s-2})}}^{s} \mathscr{L}_{\mathrm{int}}^0(i_{s-1}, j_{s-1}) \\ & \cdot \prod_{j\in(1,\dots,s)\setminus(j_1,\dots,j_{s-1})} S_1(t_{s-1}, j) \\ & \cdot b_1^0\left((x_1,\dots,x_s)\setminus\left(x_{j_1},\dots,x_{j_{s-1}}\right)\right), \quad s\geq 1. \end{aligned} \tag{20}$$

We make several examples of expansions (20) of the limit additive-type marginal observable:

$$\begin{aligned} b_1^{(1)}(t, x_1) &= S_1(t, 1)\, b_1^0(x_1), \\ b_2^{(1)}(t, x_1, x_2) &= \int_0^t dt_1 \prod_{i=1}^{2} S_1(t-t_1, i)\, \mathscr{L}_{\mathrm{int}}^0(1,2) \\ & \quad \cdot \sum_{j=1}^{2} S_1(t_1, j)\, b_1^0(x_j). \end{aligned} \tag{21}$$

If for the initial k-ary-type marginal observable b_k^ϵ the following condition is satisfied:

$$w^* - \lim_{\epsilon\to 0}\left(\epsilon^{-2}b_k^\epsilon - b_k^0\right) = 0, \tag{22}$$

then, according to statement (15), for k-ary-type marginal observables (11) we derive

$$w^* - \lim_{\epsilon\to 0}\left(\epsilon^{-2s}B_s^{(k)}(t) - b_s^{(k)}(t)\right) = 0, \tag{23}$$

where the limit marginal observable $b_s^{(k)}(t)$ is determined as a special case of expansion (16):

$$\begin{aligned}
b_s^{(k)}(t, x_1, \ldots, x_s) &= \int_0^t dt_1 \cdots \int_0^{t_{s-k-1}} dt_{s-k} \\
&\cdot \prod_{j\in(1,\ldots,s)} S_1(t-t_1, j) \sum_{i_1\neq j_1=1}^{s} \mathscr{L}_{\mathrm{int}}^0(i_1, j_1) \\
&\times \prod_{j\in(1,\ldots,s)\setminus(j_1)} S_1(t_1-t_2, j) \\
&\cdots \prod_{j\in(1,\ldots,s)\setminus(j_1,\ldots,j_{s-k-1})} S_1(t_{s-k-1}-t_{s-k}, j) \\
&\times \sum_{\substack{i_{s-k}\neq j_{s-k}=1,\\ i_{s-k},j_{s-k}\neq(j_1,\ldots,j_{s-k-1})}}^{s} \mathscr{L}_{\mathrm{int}}^0(i_{s-k}, j_{s-k}) \\
&\cdot \prod_{j\in(1,\ldots,s)\setminus(j_1,\ldots,j_{s-k})} S_1(t_{s-k}, j) \\
&\cdot b_k^0\left((x_1,\ldots,x_s)\setminus\left(x_{j_1},\ldots,x_{j_{s-k}}\right)\right), \quad 1 \le s \le k.
\end{aligned} \tag{24}$$

If $b^0 \in \mathscr{C}_\gamma$, then the sequence $b(t) = (b_0, b_1(t), \ldots, b_s(t), \ldots)$ of limit marginal observables (16) is a generalized global solution of the Cauchy problem of the dual Boltzmann hierarchy with hard sphere collisions

$$\begin{aligned}
\frac{\partial}{\partial t} b_s(t) &= \sum_{j=1}^{s}\mathscr{L}(j)\, b_s(t) \\
&+ \sum_{j_1\neq j_2=1}^{s} \mathscr{L}_{\mathrm{int}}^0(j_1, j_2)\, b_{s-1}\left(t, (x_1,\ldots,x_s)\setminus\left(x_{j_1}\right)\right),
\end{aligned} \tag{25}$$

$$b_s(t, x_1, \ldots, x_s)\big|_{t=0} = b_s^0(x_1, \ldots, x_s), \quad s \ge 1, \tag{26}$$

where the operator $\mathscr{L}_{\mathrm{int}}^0$ is defined by (17). This fact is proved similar to the case of an iteration series of the dual BBGKY hierarchy [26].

It should be noted that equations set (25) has the structure of recurrence evolution equations. Indeed, we make a few examples of the dual Boltzmann hierarchy with hard sphere collisions (25):

$$\begin{aligned}
\frac{\partial}{\partial t} b_1(t, x_1) &= \left\langle p_1, \frac{\partial}{\partial q_1}\right\rangle b_1(t, x_1), \\
\frac{\partial}{\partial t} b_2(t, x_1, x_2) &= \sum_{j=1}^{2}\left\langle p_j, \frac{\partial}{\partial q_j}\right\rangle b_2(t, x_1, x_2) \\
&+ \int_{\mathbb{S}_+^2} d\eta\, \langle \eta, (p_1 - p_2)\rangle\, \delta(q_1 - q_2) \\
&\cdot \left(b_1(q_1, p_1^*) - b_1(x_1) + b_1(q_2, p_2^*) - b_1(x_2)\right).
\end{aligned} \tag{27}$$

Thus, in the Boltzmann–Grad scaling limit the kinetic evolution of hard spheres is described in terms of limit marginal observables (16) governed by the dual Boltzmann hierarchy with hard sphere collisions (25). Similar approach to the description of the mean field asymptotic behavior of quantum many-particle systems was developed in paper [36].

We outline the sketch of the proof of the limit theorem. For the group of operators (5) the analog of the Duhamel equation is valid [27]

$$\begin{aligned}
&\left(S_s(t, 1, \ldots, s) - S_{s-1}(t, (1, \ldots, s)\setminus j_1)\, S_1(t, j_1)\right) b_s \\
&= \epsilon^2 \int_0^t d\tau S_s(t-\tau, 1, \ldots, s) \sum_{\substack{i_1=1,\\ i_1\neq j_1}}^{s} \mathscr{L}_{\mathrm{int}}(i_1, j_1) \\
&\cdot S_{s-1}(\tau, (1, \ldots, s)\setminus j_1)\, S_1(\tau, j_1)\, b_s,
\end{aligned} \tag{28}$$

where the operator $\mathscr{L}_{\mathrm{int}}(i, j)$ is defined by formula (3). Then for the $(1 + n)$th-order cumulant of groups of operators (5) the analog of the Duhamel equation holds

$$\begin{aligned}
&\mathfrak{A}_{1+n}\left(t, \{(1, \ldots, s)\setminus(j_1, \ldots, j_n)\}, j_1, \ldots, j_n\right) \\
&\cdot b_{s-n}\left((x_1, \ldots, x_s)\setminus\left(x_{j_1}, \ldots, x_{j_n}\right)\right) \\
&= \epsilon^{2n} n! \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n S_s(t - t_1) \\
&\cdot \sum_{\substack{i_1=1,\\ i_1\neq j_1}}^{s} \mathscr{L}_{\mathrm{int}}(i_1, j_1)\, S_{s-1}(t_1 - t_2) \cdots S_{s-n+1}(t_{n-1} - t_n) \\
&\cdot \sum_{\substack{i_n=1,\\ i_n\neq(j_1,\ldots,j_n)}}^{s} \mathscr{L}_{\mathrm{int}}(i_n, j_n)\, S_{s-n}(t_n) \\
&\cdot b_{s-n}\left((x_1, \ldots, x_s)\setminus\left(x_{j_1}, \ldots, x_{j_n}\right)\right),
\end{aligned} \tag{29}$$

where notations accepted above are used, $S_{s-n}(t_n) \equiv S_{s-n}(t_n, (1, \ldots, s)\setminus(j_1, \ldots, j_n))$, and we take into consideration the identity

$$\begin{aligned}
&S_n(t, j_1, \ldots, j_n)\, B_{s-n}^{\epsilon,0}\left((x_1, \ldots, x_s)\setminus\left(x_{j_1}, \ldots, x_{j_n}\right)\right) \\
&= B_{s-n}^{\epsilon,0}\left((x_1, \ldots, x_s)\setminus\left(x_{j_1}, \ldots, x_{j_n}\right)\right).
\end{aligned} \tag{30}$$

For arbitrary finite time interval $*$-weak continuous group of operators (5) has the following Boltzmann–Grad scaling limit in the sense of a $*$-weak convergence of the space $\mathscr{C}_s$

$$w^* - \lim_{\epsilon\to 0}\left(S_s(t)\, b_s - \prod_{j=1}^{s} S_1(t,j)\, b_s \right) = 0. \tag{31}$$

Taking into account assumption (14) and an analog of the Duhamel equation (29), then in view of formula (31), for cumulants of asymptotically perturbed groups of operators we have

$$\begin{aligned} w^* - \lim_{\epsilon\to 0}\Bigg(& \epsilon^{-2n}\mathfrak{A}_{1+n}(t, \{(1,\ldots,s)\setminus(j_1,\ldots,j_n)\}, j_1, \\ & \ldots, j_n) - n! \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \prod_{j\in(1,\ldots,s)} S_1(t-t_1, j) \\ & \cdot \sum_{i_1\neq j_1=1}^{s} \mathscr{L}^0_{\mathrm{int}}(i_1, j_1) \prod_{j\in(1,\ldots,s)\setminus(j_1)} S_1(t_1-t_2, j) \\ & \cdots \prod_{j\in(1,\ldots,s)\setminus(j_1,\ldots,j_{n-1})} S_1(t_{n-1}-t_n, j) \\ & \cdot \sum_{\substack{i_n\neq j_n=1,\\ i_n,j_n\neq(j_1,\ldots,j_{n-1})}}^{s} \mathscr{L}^0_{\mathrm{int}}(i_n, j_n) \\ & \times \prod_{j\in(1,\ldots,s)\setminus(j_1,\ldots,j_n)} S_1(t_n, j) \Bigg) b_{s-n}\big((x_1,\ldots, \\ & x_s)\setminus(x_{j_1},\ldots,x_{j_n})\big) = 0. \end{aligned} \tag{32}$$

As a result of equality (32) we establish the validity of statement (15) for nonperturbative solution (6) of the Cauchy problem of the dual BBGKY hierarchies (1) and (2).

4. The Derivation of the Boltzmann Kinetic Equation

We shall establish the link between the constructed asymptotic behavior of marginal observables of a hard sphere system (Theorem 2) and the description of kinetic evolution of states by means of a one-particle marginal distribution function governed by the Boltzmann kinetic equation.

In case of the absence of correlations between particles at initial time, that is, for initial states satisfying a chaos condition [10], the sequence of initial marginal distribution functions for a system of hard spheres has the form

$$F^{(c)} \equiv \left(1, F_1^{\epsilon,0}(x_1), \ldots, \prod_{i=1}^{s} F_1^{\epsilon,0}(x_i)\, \mathscr{X}_{\mathbb{R}^{3s}\setminus\mathbb{W}_s}, \ldots \right), \tag{33}$$

where $\mathscr{X}_{\mathbb{R}^{3s}\setminus\mathbb{W}_s}$ is the Heaviside step function of the allowed configurations. This assumption about initial state is intrinsic for the kinetic theory, because in this case all possible states of gases are described by means of a one-particle distribution function.

Let $F_1^{0,\epsilon} \in L^\infty_\xi(\mathbb{R}^3\times\mathbb{R}^3)$; that is, the inequality holds: $|F_1^{0,\epsilon}(x_i)| \leq \xi \exp(-\beta(p_i^2/2))$, where $\xi > 0$, $\beta \geq 0$ are parameters. We assume that the Boltzmann–Grad limit of the initial one-particle (marginal) distribution function $F_1^{0,\epsilon} \in L^\infty_\xi(\mathbb{R}^3\times\mathbb{R}^3)$ exists in the sense of a weak convergence of the space $L^\infty_\xi(\mathbb{R}^3\times\mathbb{R}^3)$, namely,

$$w - \lim_{\epsilon\to 0}\left(\epsilon^2 F_1^{0,\epsilon} - f_1^0 \right) = 0, \tag{34}$$

and then the Boltzmann–Grad limit of the initial state (33) satisfies a chaos property too, that is, $f^{(c)} \equiv (1, f_1^0(x_1), \ldots, \prod_{i=1}^{s} f_1^0(x_i), \ldots)$.

We note that assumption (34) with respect to the Boltzmann–Grad limit of initial states holds true for the equilibrium states [37].

If $b(t) \in \mathscr{C}_\gamma$ and $|f_1^0(x_i)| \leq \xi\exp(-\beta(p_i^2/2))$, then the Boltzmann–Grad limit of mean value functional (12) exists under the condition that [7]: $t < t_0 \equiv (\mathrm{const}(\xi,\beta)\|f_1^0\|_{L^\infty_\xi(\mathbb{R}^3\times\mathbb{R}^3)})^{-1}$, and it is determined by the following series expansion:

$$\begin{aligned} \left(b(t), f^{(c)} \right) = \sum_{s=0}^{\infty} \frac{1}{s!} \int_{(\mathbb{R}^3\times\mathbb{R}^3)^s} dx_1 \cdots dx_s \\ \cdot b_s(t, x_1, \ldots, x_s) \prod_{i=1}^{s} f_1^0(x_i). \end{aligned} \tag{35}$$

In consequence of the following equality for the limit additive-type marginal observables (20) (below it is proved in more general case)

$$\begin{aligned} \left(b^{(1)}(t), f^{(c)} \right) &= \sum_{s=0}^{\infty} \frac{1}{s!} \int_{(\mathbb{R}^3\times\mathbb{R}^3)^s} dx_1 \cdots dx_s \\ &\cdot b_s^{(1)}(t, x_1, \ldots, x_s) \prod_{i=1}^{s} f_1^0(x_i) = \int_{\mathbb{R}^3\times\mathbb{R}^3} dx_1 b_1^0(x_1) \\ &\cdot f_1(t, x_1), \end{aligned} \tag{36}$$

where function $b_s^{(1)}(t)$ is given by expansion (20) and the distribution function $f_1(t, x_1)$ is given by the series

$$\begin{aligned} f_1(t, x_1) = \sum_{n=0}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \\ \cdot \int_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_2 \cdots dx_{n+1} S_1^*(t-t_1, 1) \\ \cdot \mathscr{L}^{0,*}_{\mathrm{int}}(1,2) \prod_{j_1=1}^{2} S_1^*(t_1-t_2, j_1) \cdots \prod_{i_n=1}^{n} S_1^* \\ \cdot (t_{n-1}-t_n, i_n) \sum_{k_n=1}^{n} \mathscr{L}^{0,*}_{\mathrm{int}}(k_n, n+1) \\ \cdot \prod_{j_n=1}^{n+1} S_1^*(t_n, j_n) \prod_{i=1}^{n+1} f_1^0(x_i), \end{aligned} \tag{37}$$

where the following operator was introduced:

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{R}^3} dx_{n+1}\mathscr{L}^{0,*}_{\mathrm{int}}(i,n+1)\,f_{n+1}(x_1,\dots,x_{n+1})\\&\quad\equiv\int_{\mathbb{R}^3\times\mathbb{S}^2_+} dp_{n+1}d\eta\,\langle\eta,(p_i-p_{n+1})\rangle\\&\quad\cdot\left(f_{n+1}(x_1,\dots,q_i,p_i^*,\dots,x_s,q_i,p_{n+1}^*)\right.\\&\quad\left.-f_{n+1}(x_1,\dots,x_s,q_i,p_{n+1})\right).\end{aligned}\tag{38}$$

and the group of operators $S_1^*(t)$ is a group of adjoint operators to operators (5) in the sense of mean value functional (12).

The distribution function $f_1(t)$ is a solution of the Cauchy problem of the Boltzmann kinetic equation

$$\begin{aligned}\frac{\partial}{\partial t}f_1(t,x_1)&=-\left\langle p_1,\frac{\partial}{\partial q_1}\right\rangle f_1(t,x_1)\\&+\int_{\mathbb{R}^3\times\mathbb{S}^2_+} dp_2d\eta\,\langle\eta,(p_1-p_2)\rangle\\&\cdot\left(f_1(t,q_1,p_1^*)\,f_1(t,q_1,p_2^*)\right.\\&\left.-f_1(t,x_1)\,f_1(t,q_1,p_2)\right),\end{aligned}\tag{39}$$

$$f_1(t,x_1)\big|_{t=0}=f_1^0(x_1).\tag{40}$$

Thus, we establish that hierarchy (25) for additive-type marginal observables and initial state (34) describes the evolution of hard sphere systems just as the Boltzmann kinetic equation (39).

We differentiate over the time variable expression (37) in the sense of the pointwise convergence of the space $L^\infty_\xi(\mathbb{R}^3\times\mathbb{R}^3)$

$$\begin{aligned}\frac{\partial}{\partial t}f_1(t,x_1)&=\mathscr{L}^*(1)\,f_1(t,x_1)+\int_{\mathbb{R}^3\times\mathbb{R}^3}dx_2\\&\cdot\mathscr{L}^{0,*}_{\mathrm{int}}(1,2)\sum_{n=0}^{\infty}\int_0^t dt_1\cdots\int_0^{t_{n-1}}dt_n\\&\cdot\int_{(\mathbb{R}^3\times\mathbb{R}^3)^{n-1}}dx_3\cdots dx_{n+2}\\&\cdot\prod_{i_1=1}^{2}S_1^*(t-t_1,i_1)\sum_{k_1=1}^{2}\mathscr{L}^{0,*}_{\mathrm{int}}(k_1,3)\\&\times\prod_{j_1=1}^{3}S_1^*(t_1-t_2,j_1)\cdots\prod_{i_n=1}^{n+1}S_1^*\\&\cdot(t_{n-1}-t_n,i_n)\sum_{k_n=1}^{n+1}\mathscr{L}^{0,*}_{\mathrm{int}}(k_n,n+2)\\&\cdot\prod_{j_n=1}^{n+2}S_1^*(t_n,j_n)\prod_{i=1}^{n+2}f_1^0(x_i),\end{aligned}\tag{41}$$

where the operator $\mathscr{L}^{0,*}_{\mathrm{int}}(k_i,i+2)$ is defined by formula (38).

Using the product formula for the one-particle marginal distribution function $f_1(t,x_i)$ defined by series expansion (37) in case of initial data (34)

$$\begin{aligned}\prod_{i=1}^{k}f_1(t,x_i)&=\sum_{n=0}^{\infty}\int_0^t dt_1\cdots\int_0^{t_{n-1}}dt_n\\&\cdot\int_{(\mathbb{R}^3\times\mathbb{R}^3)^n}dx_{k+1}\cdots dx_{k+n}\prod_{i_1=1}^{k}S_1^*(t-t_1,i_1)\\&\times\sum_{k_1=1}^{k}\mathscr{L}^{0,*}_{\mathrm{int}}(k_1,k+1)\\&\cdot\prod_{j_1=1}^{k+1}S_1^*(t_1-t_2,j_1)\cdots\prod_{i_n=1}^{k+n-1}S_1^*(t_{n-1}-t_n,i_n)\\&\times\sum_{k_n=1}^{k+n-1}\mathscr{L}^{0,*}_{\mathrm{int}}(k_n,k+n)\prod_{j_n=1}^{k+n}S_1^*(t_n,j_n)\prod_{i=1}^{k+n}f_1^0(x_i),\end{aligned}\tag{42}$$

where the group property of one-parameter mapping (5) is applied, we express the second summand in the right-hand side of equality (41) in terms of $\prod_{i=1}^2 f_1(t,i)$, and, consequently, we get (39).

We remark that in a one-dimensional space the collision integral of the Boltzmann equation with elastic hard sphere collisions identically equals zero. In a one-dimensional space the Boltzmann–Grad limit is not trivial in case of hard sphere dynamics with inelastic collisions [38]. In paper [38] for one-dimensional granular gas the process of the creation of correlations in the Boltzmann–Grad limit was also described.

5. On Propagation of Initial Chaos in a Low-Density Limit

If the initial states of hard spheres are specified by a sequence of marginal distribution functions (33), then the property of the propagation of initial chaos holds in the Boltzmann–Grad limit. It is a result of the validity of the following equality for the limit k-ary marginal observables (24); that is, $b^{(k)}(0)=(0,\dots,b_k^0(x_1,\dots,x_k),0,\dots)$,

$$\begin{aligned}\left(b^{(k)}(t),f^{(c)}\right)&=\sum_{s=0}^{\infty}\frac{1}{s!}\\&\cdot\int_{(\mathbb{R}^3\times\mathbb{R}^3)^s}dx_1\cdots dx_s b_s^{(k)}(t,x_1,\dots,x_s)\prod_{i=1}^{s}f_1^0(x_i)\\&=\frac{1}{k!}\int_{(\mathbb{R}^3\times\mathbb{R}^3)^k}dx_1\cdots dx_k b_k^0(x_1,\dots,x_k)\\&\cdot\prod_{i=1}^{k}f_1(t,x_i),\quad k\geq 2,\end{aligned}\tag{43}$$

where for finite time interval the limit one-particle marginal distribution function $f_1(t)$ is defined by series expansion (37) and therefore it is governed by the Cauchy problem of the Boltzmann kinetic equations (39) and (40).

In fact, taking into account the validity of the following equality for expansion (16) of the function $b_s^{(k)}(t)$

$$\begin{aligned}
&\sum_{s=0}^{\infty}\frac{1}{s!}\int_{(\mathbb{R}^3\times\mathbb{R}^3)^s}dx_1\cdots dx_s b_s^{(k)}(t,x_1,\dots,x_k)\prod_{i=1}^{s}f_1^0(x_i)\\
&\quad=\frac{1}{k!}\int_{(\mathbb{R}^3\times\mathbb{R}^k)^s}dx_1\cdots dx_k b_k^0(x_1,\dots,x_k)\\
&\quad\cdot\sum_{n=0}^{\infty}\int_0^t dt_1\cdots\int_0^{t_{n-1}}dt_n\int_{(\mathbb{R}^3\times\mathbb{R}^3)^n}dx_{k+1}\cdots dx_{k+n}\\
&\quad\cdot\prod_{i_1=1}^{k}S_1^*(t-t_1,i_1)\sum_{k_1=1}^{k}\mathscr{L}_{\mathrm{int}}^{0,*}(k_1,k+1)\\
&\quad\cdot\prod_{j_1=1}^{k+1}S_1^*(t_1-t_2,j_1)\cdots\prod_{i_n=1}^{k+n-1}S_1^*(t_{n-1}-t_n,i_n)\\
&\quad\cdot\sum_{k_n=1}^{k+n-1}\mathscr{L}_{\mathrm{int}}^{0,*}(k_n,k+n)\prod_{j_n=1}^{k+n}S_1^*(t_n,j_n)\prod_{i=1}^{k+n}f_1^0(x_i),
\end{aligned}\tag{44}$$

and product formula (42), for the limit one-particle marginal distribution function defined by series expansion (37), we finally verify the validity of equality (43).

Thus, in the Boltzmann–Grad scaling limit an equivalent approach to the description of the kinetic evolution of hard spheres within the framework of the Cauchy problem of the Boltzmann kinetic equations (39) and (40) is given by the Cauchy problem of the dual Boltzmann hierarchy with hard sphere collisions (25) and (26) for the additive-type marginal observables. In case of the nonadditive-type marginal observables a solution of the dual Boltzmann hierarchy with hard sphere collisions (25) is equivalent to the property of a propagation of initial chaos in the sense of equality (43).

6. The Boltzmann Equation for Hard Spheres Fluids

We consider initial states of a hard sphere system specified by the one-particle marginal distribution function $F_1^{0,\epsilon}\in L_\xi^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ in the presence of correlations, that is, initial states defined by the following sequence of marginal distribution functions:

$$\begin{aligned}
&F^{(cc)}\\
&\quad=\left(1,F_1^{0,\epsilon}(x_1),g_2^\epsilon\prod_{i=1}^{2}F_1^{0,\epsilon}(x_i),\dots,g_n^\epsilon\prod_{i=1}^{n}F_1^{0,\epsilon}(x_i),\dots\right),
\end{aligned}\tag{45}$$

where the functions $g_n^\epsilon\equiv g_n^\epsilon(x_1,\dots,x_n)\in C_n(\mathbb{R}^{3n}\times(\mathbb{R}^{3n}\setminus\mathbb{W}_n))$, $n\ge 2$, are specified initial correlations. Since many-particle systems in condensed states are characterized by correlations sequence (45) describes the initial state of the kinetic evolution of hard sphere fluids.

We assume that the Boltzmann–Grad limit of initial one-particle marginal distribution function $F_1^{0,\epsilon}\in L_\xi^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ exists in the sense as above; that is, in the sense of a weak convergence the equality holds: $w-\lim_{\epsilon\to0}(\epsilon^2F_1^{0,\epsilon}-f_1^0)=0$, and in case of correlation functions let $w-\lim_{\epsilon\to0}(g_n^\epsilon-g_n)=0$, $n\ge2$; then in the Boltzmann–Grad limit initial state (45) is defined by the following sequence of the limit marginal distribution functions:

$$\begin{aligned}
&f^{(cc)}\\
&\quad=\left(1,f_1^0(x_1),g_2\prod_{i=1}^{2}f_1^0(x_i),\dots,g_n\prod_{i=1}^{n}f_1^0(x_i),\dots\right).
\end{aligned}\tag{46}$$

We consider relationships of the constructed Boltzmann–Grad asymptotic behavior of marginal observables with the nonlinear Boltzmann-type kinetic equation in case of initial states (46).

For the limit additive-type marginal observables (20) and initial states (46) the following equality is true:

$$\begin{aligned}
&\left(b^{(1)}(t),f^{(cc)}\right)=\sum_{s=0}^{\infty}\frac{1}{s!}\\
&\quad\cdot\int_{(\mathbb{R}^3\times\mathbb{R}^3)^s}dx_1\cdots dx_s b_s^{(1)}(t,x_1,\dots,x_s)\\
&\quad\cdot g_s(x_1,\dots,x_s)\prod_{i=1}^{s}f_1^0(x_i)=\int_{\mathbb{R}^3\times\mathbb{R}^3}dx_1 b_1^0(x_1)\\
&\quad\cdot f_1(t,x_1),
\end{aligned}\tag{47}$$

where the functions $b_s^{(1)}(t)$ are represented by expansions (20) and the limit marginal distribution function $f_1(t)$ is represented by the following series expansion:

$$\begin{aligned}
&f_1(t,x_1)=\sum_{n=0}^{\infty}\int_0^t dt_1\cdots\int_0^{t_{n-1}}dt_n\\
&\quad\cdot\int_{(\mathbb{R}^3\times\mathbb{R}^3)^n}dx_2\cdots dx_{n+1}S_1^*(t-t_1,1)\mathscr{L}_{\mathrm{int}}^{0,*}(1,2)\\
&\quad\cdot S_1^*(t_1-t_2,j_1)\cdots\prod_{i_n=1}^{n}S_1^*(t_n-t_n,i_n)\\
&\quad\cdot\sum_{k_n=1}^{n}\mathscr{L}_{\mathrm{int}}^{0,*}(k_n,n+1)\prod_{j_n=1}^{n+1}S_1^*(t_n,j_n)g_{1+n}\\
&\quad\cdot(x_1,\dots,x_{n+1})\prod_{i=1}^{n+1}f_1^0(x_i).
\end{aligned}\tag{48}$$

Series (48) is uniformly convergent for finite time interval under the condition as above (37).

The function $f_1(t)$ represented by series (48) is a weak solution of the following Cauchy problem of the Boltzmann kinetic equation with initial correlations [39, 40]

$$\begin{aligned}\frac{\partial}{\partial t} f_1(t,x_1) = & -\left\langle p_1, \frac{\partial}{\partial q_1}\right\rangle f_1(t,x_1) \\ & + \int_{\mathbb{R}^3\times\mathbb{S}^2_+} dp_2 d\eta \, \langle \eta, (p_1 - p_2)\rangle \\ & \cdot \left(g_2\left(q_1 - p_1^* t, p_1^*, q_2 - p_2^* t, p_2^*\right) f_1\left(t, q_1, p_1^*\right)\right. \\ & \cdot f_1\left(t, q_1, p_2^*\right) - g_2\left(q_1 - p_1 t, p_1, q_2 - p_2 t, p_2\right) \\ & \left.\cdot f_1(t,x_1) f_1(t,q_1,p_2)\right),\end{aligned} \tag{49}$$

$$f_1(t,x_1)\big|_{t=0} = f_1^0(x_1). \tag{50}$$

This fact is proved similarly to the case of a perturbative solution of the BBGKY hierarchy for hard spheres represented by the iteration series [10, 29].

Thus, in case of initial states specified by one-particle marginal distribution function (46) we establish that the dual Boltzmann hierarchy with hard sphere collisions (25) for additive-type marginal observables describes the evolution of a hard sphere system just as the Boltzmann kinetic equation with initial correlations (49).

7. Propagation of Initial Correlations in a Low-Density Limit

The property of the propagation of initial correlations in a low-density limit is a consequence of the validity of the following equality for a mean value functional of the limit k-ary marginal observables:

$$\begin{aligned}\left(b^{(k)}(t), f^{(cc)}\right) = & \sum_{s=0}^{\infty} \frac{1}{s!} \\ & \cdot \int_{(\mathbb{R}^3\times\mathbb{R}^3)^s} dx_1 \cdots dx_s b_s^{(k)}(t,x_1,\ldots,x_s)\, g_s(x_1,\ldots,x_s) \\ & \cdot \prod_{j=1}^{s} f_1^0(x_j) = \frac{1}{k!} \\ & \cdot \int_{(\mathbb{R}^3\times\mathbb{R}^3)^k} dx_1 \cdots dx_k b_k^0(x_1,\ldots,x_k) \prod_{i_1=1}^{k} S_1^*(t,i_1) \\ & \cdot g_k(x_1,\ldots,x_k) \prod_{i_2=1}^{k} (S_1^*)^{-1}(t,i_2) \prod_{j=1}^{k} f_1(t,x_j),\end{aligned} \tag{51}$$

where the one-particle marginal distribution function $f_1(t,x_j)$ is solution (48) of the Cauchy problem of the Boltzmann kinetic equation with initial correlations (49) and (50), and the inverse group to the group of operators $S_1^*(t)$ we denote by $(S_1^*)^{-1}(t) = S_1^*(-t) = S_1(t)$.

This fact is proved similarly to the proof of a property of a propagation of initial chaos (43).

We note that, according to equality (51), in the Boltzmann–Grad limit the marginal correlation functions defined as cluster expansions of marginal distribution functions, namely,

$$f_s(t,x_1,\ldots,x_s) = \sum_{P:(x_1,\ldots,x_s)=\bigcup_i X_i} \prod_{X_i\subset P} g_{|X_i|}(t,X_i), \quad s \geq 1, \tag{52}$$

have the following explicit form:

$$\begin{aligned} g_1(t,x_1) &= f_1(t,x_1), \\ g_s(t,x_1,\ldots,x_s) &= \tilde{g}_s(q_1 - p_1 t, p_1, \ldots, q_s - p_s t, p_s) \prod_{j=1}^{s} f_1(t,x_j), \quad s \geq 2,\end{aligned} \tag{53}$$

where for initial correlation functions (46) it is used the following notations:

$$\tilde{g}_s(x_1,\ldots,x_s) = \sum_{P:(x_1,\ldots,x_s)=\bigcup_i X_i} \prod_{X_i\subset P} g_{|X_i|}(X_i), \tag{54}$$

where the symbol $\sum_P$ means the sum over possible partitions P of the set of arguments $(x_1,\ldots,x_s)$ on $|P|$ nonempty subsets X_i, and the one-particle marginal distribution function $f_1(t)$ is a solution of the Cauchy problem of the Boltzmann kinetic equation with initial correlations (49) and (50).

Thus, in case of the limit k-ary marginal observables a solution of the dual Boltzmann hierarchy with hard sphere collisions (25) is equivalent to a property of the propagation of initial correlations for the k-particle marginal distribution function in the sense of equality (51) or in other words the Boltzmann–Grad scaling dynamics does not create new correlations except initial correlations.

8. Conclusion

In the paper a new approach to the problem of the rigorous description of the kinetic evolution of a system of hard spheres with elastic collisions was developed. For this purpose we established the low-density (Boltzmann–Grad) asymptotic behavior of a solution of the Cauchy problem of the dual BBGKY hierarchy for marginal observables of hard spheres (1) and (2). The constructed scaling limit is governed by the set of recurrence evolution equations, namely, by the dual Boltzmann hierarchy with hard sphere collisions (25).

Furthermore, it was established that for initial states specified by a one-particle distribution function the evolution of additive-type marginal observables is equivalent to a solution of the Boltzmann kinetic equation (39) and the evolution of nonadditive-type marginal observables is equivalent to the property of the propagation of initial chaos for states (43). In other words the Boltzmann–Grad dynamics does not create correlations.

One of the advantages of such an approach to the derivation of the Boltzmann equation is an opportunity to construct the kinetic equation, involving correlations at initial time, in particular, that can characterize the condensed states. Moreover, it gives opportunity to describe the propagation of initial correlations in the Boltzmann–Grad scaling limit (53).

Some applications of the developed method to the derivation of kinetic equations in scaling limits of large particle systems of different kinds, in particular, hard spheres with inelastic collisions [38], are considered in papers [38, 41, 42].

We note that one more approach to the description of the kinetic evolution of hard spheres is based on the non-Markovian generalization of the Enskog kinetic equation [43].

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] N. N. Bogolyubov, *Problems of Dynamic Theory in Statistical Physics*, Gostekhizdat, 1946.

[2] H. Grad, "On the kinetic theory of rarefied gases," *Communications on Pure and Applied Mathematics*, vol. 2, pp. 331–407, 1949.

[3] H. Grad, "Principles of the kinetic theory of gases," in *Handbuch der Physik*, vol. 12, pp. 205–294, Springer, Berlin, Germany, 1958.

[4] C. Cercignani, "On the Boltzmann equation for rigid spheres," *Transport Theory and Statistical Physics*, vol. 2, no. 3, pp. 211–225, 1972.

[5] O. E. Lanford, "Time evolution of large classical systems," in *Dynamical Systems, Theory and Applications*, vol. 38 of *Lecture Notes in Physics*, pp. 1–111, Springer, Berlin, Germany, 1975.

[6] O. E. Lanford, "The hard sphere gas in the BoltzmannGrad limit," *Physica A: Statistical Mechanics and its Applications*, vol. 106, no. 1-2, pp. 70–76, 1981.

[7] D. Y. Petrina and V. I. Gerasimenko, "Mathematical problems of the statistical mechanics of a hard-sphere system," *Russian Mathematical Surveys (Uspekhi Matematicheskikh Nauk)*, vol. 45, no. 3, pp. 135–182, 1990.

[8] H. Spohn, *Large Scale Dynamics of Interacting Particles*, Springer-Verlag, Berlin, Germany, 1991.

[9] C. Cercignani, R. Illner, and M. Pulvirenti, *The Mathematical Theory of Dilute Gases*, Springer-Verlag, Berlin, Germany, 1994.

[10] C. Cercignani, V. I. Gerasimenko, and D. Y. Petrina, *Many-particle Dynamics and Kinetic Equations*, Springer, Berlin, Germany, 2012.

[11] H. Spohn, "Kinetic equations from Hamiltonian dynamics: Markovian limits," *Reviews of Modern Physics*, vol. 52, no. 3, pp. 569–615, 1980.

[12] I. Gallagher, L. Saint-Raymond, and B. Texier, *From Newton to Boltzmann: Hard Spheres and Short-range Potentials*, Zurich Lectures in Advanced Mathematics, European Mathematical Society, Helsinki, Finland, 2014.

[13] T. Bodineau, I. Gallagher, and L. Saint-Raymond, "From hard sphere dynamics to the Stokes-Fourier equations: an L^2 analysis of the Boltzmann-Grad limit," *Annals of PDE. Journal Dedicated to the Analysis of Problems from Physical Sciences*, vol. 3, no. 2, 2017.

[14] M. Pulvirenti, C. Saffirio, and S. Simonella, "On the validity of the Boltzmann equation for short range potentials," *Reviews in Mathematical Physics*, vol. 26, no. 2, Article ID 1450001, 2014.

[15] S. Simonella, "Evolution of correlation functions in the hard sphere dynamics," *Journal of Statistical Physics*, vol. 155, no. 6, pp. 1191–1221, 2014.

[16] M. Pulvirenti and S. Simonella, "The Boltzmann-Grad limit of a hard sphere system: analysis of the correlation error," *Inventiones Mathematicae*, vol. 207, no. 3, pp. 1135–1237, 2017.

[17] M. Pulvirenti and S. Simonella, "On the evolution of the empirical measure for the hard-sphere dynamics," *Bulletin of the Institute of Mathematics, Academia Sinica*, vol. 10, no. 2, pp. 171–204, 2015.

[18] K. Aoki, M. Pulvirenti, S. Simonella, and T. Tsuji, "Backward clusters, hierarchy and wild sums for a hard sphere system in a low-density regime," *Mathematical Models and Methods in Applied Sciences*, vol. 25, no. 5, pp. 995–1010, 2015.

[19] R. I. Patterson, S. Simonella, and W. Wagner, "Kinetic theory of cluster dynamics," *Physica D: Nonlinear Phenomena*, vol. 335, pp. 26–32, 2016.

[20] Th. Paul, M. Pulvirenti, and S. Simonella, "On the size of chaos in the mean field dynamics," https://arxiv.org/abs/1708.07701.

[21] M. Tessarotto, "On the BoltzmannGrad limit for the classical hard-spheres systems," in *Proceedings of the Proc. 26 Int. Symposium on Rarefied Gas Dynamics*, pp. 194–199, AIP, NY, USA, 2008.

[22] R. Denlinger, "The propagation of chaos for a rarefied gas of hard spheres in the whole space," https://arxiv.org/abs/1605.00589.

[23] R. Denlinger, Observables and strong one-sided chaos in the Boltzmann-Grad limit, 2017, https://arxiv.org/abs/1706.09961.

[24] C. Saffirio, "Derivation of the Boltzmann equation: hard spheres, short-range potentials and beyond," in *From Particle Systems to Partial Differential Equations III*, vol. 162 of *Springer Proceedings in Mathematics & Statistics*, pp. 301–321, 2016.

[25] V. I. Gerasimenko, "Hierarchies of quantum evolution equations and dynamics of many-particle correlations," in *in Statistical Mechanics and Random Walks: Principles, Processes and Applications*, pp. 233–288, Nova Science Publishers, Hauppauge, NY, USA, 2012.

[26] G. Borgioli and V. I. Gerasimenko, "The dual BBGKY hierarchy for the evolution of observables," *Rivista di Matematica della Università di Parma*, vol. 4, pp. 251–267, 2001.

[27] J. Banasiak and L. Arlotti, *Perturbations of Positive Semigroups with Applications*, Springer-Verlag, Berlin, Germany, 2006.

[28] V. I. Gerasimenko, "On the approaches to the derivation of the Boltzmann equation with hard sphere collisions," *Proc. Inst. Math. NASU*, vol. 10, no. 2, pp. 71–95, 2013.

[29] V. I. Gerasimenko and D. Ya. Petrina, "Thermodynamical limit of nonequilibrium states of three-dimensional hard-sphere system," *Theoretical and Mathematical Physics*, vol. 64, no. 1, pp. 130–149, 1985.

[30] V. I. Gerasimenko, "On a solution of the BBGKY hierarchy for a one-dimensional hard sphere system," *Theoretical and Mathematical Physics*, vol. 91, no. 1, pp. 120–132, 1992.

[31] D. Y. Petrina, *Stochastic Dynamics and Boltzmann Hierarchy*, NASU Institute of Mathematics, Kiev, Ukraine, 2008.

[32] H. Spohn, "On the integrated form of the BBGKY hierarchy for hard spheres., arXiv," Article ID 0605068, https://arxiv.org/abs/math-ph/0605068.

[33] V. I. Gerasimenko, T. V. Ryabukha, and M. O. Stashenko, "On the structure of expansions for the BBGKY hierarchy solutions," *Journal of Physics A: Mathematical and General*, vol. 37, no. 42, pp. 9861–9872, 2004.

[34] V. I. Gerasimenko, Y. G. Kondratiev, T. Kuna, and D. Y. Petrina, "Evolution of observables andquasiobservables in classical statistical mechanics," *Journal of Physics: Condensed Matter*, vol. 3, no. 2, pp. 239–264, 2000.

[35] T. V. Ryabukha, "On mean-value ofobservalesfunctionals of one-dimensional infinite particle systems," *Theoretical and Mathematical Physics*, vol. 162, no. 3, pp. 422–438, 2010.

[36] V. Gerasimenko, "Heisenberg picture of quantum kinetic evolution in mean-field limit," *Kinetic and Related Models*, vol. 4, no. 1, pp. 385–399, 2011.

[37] V. I. Gerasimenko and D. Y. Petrina, "The BoltzmannGrad limit of equilibrium states," *Reports of the National Academy of Sciences of Ukraine SSR*, vol. 12, pp. 17–19, 1988.

[38] M. S. Borovchenkova and V. I. Gerasimenko, "On the non-Markovian Enskog equation for granular gases," *Journal of Physics A: Mathematical and Theoretical*, vol. 47, no. 3, Article ID 035001, 2014.

[39] I. V. Gapyak, "The kinetic equations of a hard sphere system with initial correlations," *Proc. Inst. Math. NASU*, vol. 11, no. 1, pp. 166–177, 2014.

[40] V. Gerasimenko and A. Kornienko, "The Boltzmann kinetic equation with correlations for hard sphere fluids," *Reports of the National Academy of Sciences of Ukraine*, no. 3, pp. 17–23, 2015.

[41] V. I. Gerasimenko, "New approach to derivation of quantum kinetic equations with initial correlations," *Carpathian Journal of Mathematics*, vol. 7, no. 1, pp. 38–48, 2015.

[42] V. I. Gerasimenko and Yu. Yu. Fedchun, "On semigroups of large particle systems and their scaling asymptotic behavior," in *Semigroups of Operators – Theory and Applications*, vol. 113 of *Springer Proceedings in Mathematics & Statistics*, pp. 165–182, Springer, Berlin, Germany, 2015.

[43] V. I. Gerasimenko and I. V. Gapyak, "Hard sphere dynamics and the Enskog equation," *Kinetic and Related Models*, vol. 5, no. 3, pp. 459–484, 2012.

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