Hindawi
Advances in Mathematical Physics
Volume 2018, Article ID 6275432, 6 pages
https://doi.org/10.1155/2018/6275432

Research Article

Explicit Solution of Elastostatic Boundary Value Problems for the Elastic Circle with Voids

Ivane Tsagareli

I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University, 2, University St., 0186 Tbilisi, Georgia

Correspondence should be addressed to Ivane Tsagareli; i.tsagareli@yahoo.com

Received 16 November 2017; Accepted 15 May 2018; Published 10 June 2018

Academic Editor: Boris G. Konopelchenko

We solve the static two-dimensional boundary value problems for an elastic porous circle with voids. Special representations of a general solution of a system of differential equations are constructed via elementary functions which make it possible to reduce the initial system of equations to equations of simple structure and facilitate the solution of the initial problems. Solutions are written explicitly in the form of absolutely and uniformly converging series. The question pertaining to the uniqueness of regular solutions of the considered problems is investigated.

1. Introduction

The physicomathematical foundations of the linear theory of elastic materials with voids or empty pores and applications of this theory to some technological problems were originally proposed in the work Cowin and Nunziato [1]. Such materials include, in particular, rocks and soils, granulated and some other manufactured porous materials.

This theory differs essentially form the classical theory of elasticity in that the volume fraction corresponding to the void volume is considered as an independent variable. Voids have no mechanical or energetic meaning.

In recent years, problems of elasticity for materials with voids were investigated by many authors. Below we mention only a few works where the bibliographical information can also be found.

In [2], the traditional problems of thick-walled spherical and circular cylindrical shells under the action of internal pressure are solved in the framework of the theory of linear elastic materials with voids. Some fundamental solution existence and uniqueness theorems, duality relations, and variational characterization of a solution are considered in terms of the aforementioned theory in [3]. In [4], solutions are obtained for equations of the classical theory of elastodynamics of homogeneous and isotropic elastic materials with voids. Using the results established by Kupradze et al. [5] in classical elasticity, in [6] boundary value problems of stationary oscillations are studied in terms of the linear theory of homogeneous and isotropic materials with voids, while [7] is concerned with the study of the behavior of plane harmonic waves in an elastic material with voids. The basic properties of these waves are established. Also established are the existence and uniqueness of solutions for external problems. Existence theorems in equilibrium theory are proved in [8], and in [9] the fundamental solution is obtained for a differential system of micropolar elasticity with voids in the case of stationary vibrations. The duality properties are also studied.

The linear theory of thermoelastic materials with voids was for the first time considered in the work Ieşan [10]. Solutions of Galerkin type and uniqueness theorems in the theory of thermoelasticity for materials with voids are proved in [11, 12]. Problems of steady vibrations of elastic and thermoelastic materials with voids are investigated in [13]. The linear theory of micropolar thermoelasticity is considered for materials with voids in the papers [14, 15]. Fundamental solutions of systems of differential equations of steady-state oscillations [14] and diffusion [15] are constructed in terms of elementary functions. The nonlinear theory of elastic porous materials with voids was proposed in the work Nunziato and Cowin [16].

Along with the generalization and development of the linear theory of elasticity for materials with voids in various directions, a great deal of attention is attached to the mathematical investigation and construction of explicit solutions of boundary value problems for concrete domains.

In the present paper, the special representations of a general solution of a system of differential equations of the theory of elastic materials with voids are constructed by using harmonic, biharmonic, and metaharmonic functions which make it possible to reduce the initial system of equations to equations of simple structure and facilitate the solution of initial problems. These representations are used in the paper to solve problems for an elastic circle with voids. The solutions are written explicitly in the form of absolutely and uniformly converging series.

The uniqueness of regular solutions of the considered problems is also investigated.

2. Formulation of Boundary Value Problems

Let the isotropic elastic circle D, consisting of empty pores, be bounded by the circumference K of radius R. Denote the area of the macropoint of $\mathbf{x} = (x_1, x_2)$ by $S(\mathbf{x})$, and the area of pores at this point by $S_p(\mathbf{x})$. The value $s(\mathbf{x})$, which is defined by the equality $s(\mathbf{x}) = S_p(\mathbf{x})/S(\mathbf{x})$, is called the relative area of pores. Generally speaking, as a result of deformation of the body the relative area of pores changes, too. We denote this change by $\varphi(\mathbf{x})$.

A system of equations of the linear theory of elastic materials with voids has the following form [1, 6]:

$$\mu\Delta\mathbf{u}(\mathbf{x}) + (\lambda+\mu)\operatorname{graddiv}\mathbf{u}(\mathbf{x}) + \beta\operatorname{grad}\varphi(\mathbf{x}) = 0, \tag{1}$$

$$(\alpha\Delta - \xi)\varphi(\mathbf{x}) - \beta\operatorname{div}u(\mathbf{x}) = 0, \quad \mathbf{x}\in D, \tag{2}$$

where $\mathbf{u}(\mathbf{x}) = (u_1, u_2)$ is the displacement vector, and $\varphi(\mathbf{x})$ is a change with respect to the pore area; λ and μ are the Lamé constants; α, β, and ξ are the constants characterizing the body porosity.

Let us now formulate the boundary value problems. Find, in the circle D, a regular vector $\mathbf{U}(\mathbf{x}) = (\mathbf{u}(\mathbf{x}), \varphi(\mathbf{x}))$ that satisfies (1) and (2) and, on the boundary K, one of the following conditions:

$$\begin{aligned} &\mathbf{u}(\mathbf{z}) = \mathbf{f}(\mathbf{z}), \\ &\varphi(\mathbf{z}) = f_3(\mathbf{z}) \\ &\qquad \text{in the Problem I;} \\ &\mathbf{P}_1(\partial_z, \mathbf{n})\mathbf{U}(\mathbf{z}) = \mathbf{f}(\mathbf{z}), \\ &\alpha\partial_n\varphi(\mathbf{z}) = f_3(\mathbf{z}) \\ &\qquad \text{in the Problem II,} \end{aligned} \tag{3}$$

where $(\mathbf{x}) \in C^1(\overline{D}) \cap C^2(D)$, $\mathbf{z} = (z_1, z_2) \in K$, $\mathbf{n}(\mathbf{z}) = (n_1(\mathbf{z}), n_2(\mathbf{z}))$ is the external normal with respect to K at the point z; $\mathbf{f} = (f_1, f_2)$, f_1, f_2, and f_3 are functions given on K;

$$\mathbf{P}(\partial_x, \mathbf{n}) = \begin{pmatrix} \mathbf{T}_{ij}(\partial_{\mathbf{x}}, \mathbf{n}) & \beta n_i \\ 0 & \alpha\partial_n \end{pmatrix}_{3\times 3} \tag{4}$$

is the stress operator in the theory of elasticity for porous bodies with voids [1], $i, j = 1, 2$. Stress in the vector form can be written as

$$\mathbf{P}(\partial_x, \mathbf{n})\mathbf{U}(\mathbf{x}) = \begin{pmatrix} \mathbf{P}_1(\partial_x, n)\mathbf{U}(\mathbf{x}) \\ \alpha\partial_n\varphi(\mathbf{x}) \end{pmatrix}, \tag{5}$$

where

$$\mathbf{P}_1(\partial_x, n)\mathbf{U}(\mathbf{x}) = \mathbf{T}(\partial_{\mathbf{x}}, \mathbf{n})\mathbf{u}(\mathbf{x}) + \beta\mathbf{n}(\mathbf{x})\varphi(\mathbf{x}), \tag{6}$$

and

$$\mathbf{T}(\partial_{\mathbf{x}}, \mathbf{n})\mathbf{u}(\mathbf{x}) = \mu\partial_n\mathbf{u}(\mathbf{x}) + \lambda\mathbf{n}(\mathbf{x})\operatorname{div}\mathbf{u}(\mathbf{x}) + \mu\sum_{i=1}^{2} n_i(\mathbf{x})\operatorname{grad}u_i(\mathbf{x}) \tag{7}$$

is the stress vector in the classical theory of elasticity [17], $\partial_n = (\mathbf{n}\cdot\operatorname{grad})$.

3. General Representations of a System of Equations

Applying the operator div to $(1)_1$, from (1) and (2) we obtain a system of equations with respect to div $\mathbf{u}$ and φ:

$$\begin{aligned} &\mu_0\Delta\operatorname{div}\mathbf{u} + \beta\Delta\varphi = 0, \\ &\beta\operatorname{div}\mathbf{u} - (\alpha\Delta - \xi)\varphi = 0, \end{aligned} \tag{8}$$

where $\mu_0 = \lambda + 2\mu$. The determinant of system (8) is equal to $-\mu_0\alpha\Delta(\Delta + s_1^2)$, where

$$s_1^2 = -\frac{\mu_0\xi - \beta^2}{\mu_0\alpha}. \tag{9}$$

Assume

$$\begin{aligned} &\lambda > 0, \\ &\mu > 0, \\ &\alpha > 0, \\ &\mu_0\xi > \beta^2. \end{aligned} \tag{10}$$

From (10) it follows that $\xi > 0$. Thus $s_1^2 < 0$, where $s_1 = i\sqrt{(\mu_0\xi - \beta^2)/\mu_0\alpha} = is_0$, $i = \sqrt{-1}$.

Since system (8) is homogeneous, we write

$$\begin{aligned} &\Delta\left(\Delta + {s_1}^2\right)\operatorname{div}\mathbf{u} = 0, \\ &\Delta\left(\Delta + {s_1}^2\right)\varphi = 0. \end{aligned} \tag{11}$$

Taking (11) into account, from (1) we get $\mu\Delta^2(\Delta + {s_1}^2)\mathbf{u} = 0$.

From this equation and also from the second equation (11), we conclude that the representation of a solution of $\mathbf{u}(\mathbf{x})$ contains a harmonic, biharmonic, and a metaharmonic functions, while the representation of $\varphi(\mathbf{x})$ contains a harmonic and a metaharmonic functions.

Solution of system consisting of (1) and (2) are written in the following form:

$$\mathbf{u}(\mathbf{x}) = c_0 \overset{0}{\mathbf{u}}(\mathbf{x}) + c_1 \overset{1}{\mathbf{u}}(\mathbf{x}), \qquad \varphi(\mathbf{x}) = \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}), \tag{12}$$

where φ_1 is a harmonic function, $\Delta\varphi_1 = 0$, and φ_2 is a metaharmonic function with the parameter s_1^2, $(\Delta + {s_1}^2)\varphi_2 = 0$; c_0 and c_1 are the unknown functions for the time being. A general solution $\overset{0}{\mathbf{u}} = (\overset{0}{u}_1, \overset{0}{u}_2)$ of the homogeneous equation, corresponding to the nonhomogeneous equation (1) with respect to $\mathbf{u}(\mathbf{x})$, is represented as follows [18]:

$$\overset{0}{\mathbf{u}}(\mathbf{x}) = \operatorname{grad}\left[\Phi_1(\mathbf{x}) + \Phi_2(\mathbf{x})\right] + rot\Phi_3(\mathbf{x}) + c_2\mathbf{\Gamma}(\mathbf{x}), \tag{13}$$

where functions Φ_2 and Φ_3 are interrelated by

$$\mu_0 \operatorname{grad}\Delta\Phi_2(\mathbf{x}) + \mu rot\Delta\Phi_3(\mathbf{x}) = 0; \tag{14}$$

$\Delta\Phi_1(\mathbf{x}) = 0$, $\Delta\Delta\Phi_2(\mathbf{x}) = 0$, $\Delta\Delta\Phi_3(\mathbf{x}) = 0$, Φ_1, Φ_2, Φ_3 are scalar functions, $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2)$; $\Gamma_1 = x_2$, $\Gamma_2 = -x_1$, $\operatorname{div}\mathbf{\Gamma} = 0$; c_2 is the sought coefficient, $rot = (-\partial/\partial x_2, \partial/\partial x_1)$.

$\overset{1}{u} = (\overset{1}{u}_1, \overset{1}{u}_2)$ is one of the particular solutions of (1):

$$\overset{1}{\mathbf{u}}(\mathbf{x}) = -\frac{\beta}{\mu_0} \operatorname{grad}\left(-\frac{1}{{s_1}^2}\varphi_2 + \varphi_0\right), \tag{15}$$

where φ_0 is chosen such that $\Delta\varphi_0 = \varphi_1$. It is obvious that φ_0 is a biharmonic function: $\Delta\Delta\varphi_0 = \Delta\varphi_1 = 0$. For simplicity, the function φ_1 is chosen such that $\varphi_1 = \operatorname{div}\overset{0}{\mathbf{u}} \equiv \Delta\Phi_2$. Then we can take $\varphi_0 = \Phi_2$.

Let us calculate the values of the coefficients c_0 and c_1 in representation (12). We apply the operator div to the first equality in (12) and compare the obtained expression with $\operatorname{div} u$ defined by (2). Using (9), we obtain

$$c_0 = -\frac{\mu_0\xi - \beta^2}{\mu_0\beta}, \qquad c_1 = 1. \tag{16}$$

By an immediate verification we make sure that representations (12) satisfy (1) and (2).

4. Uniqueness Theorems

For a regular solution $U(x)$ of (1) Green's formula [5],

$$\int_D \left[E(\mathbf{u},\mathbf{u}) + \beta\varphi \operatorname{div}\mathbf{u}\right] d\mathbf{x} = \int_K \mathbf{u}\left[\mathbf{T}\left(\partial_y, \mathbf{n}\right)\mathbf{u} + \beta\varphi n\right] d_y K, \tag{17}$$

is valid, where

$$E(\mathbf{u},\mathbf{u}) = (\lambda+\mu)\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)^2 + \mu\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right)^2 + \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)^2, \tag{18}$$

is the nonnegative quadratic form [17] when conditions (10) are used.

On the other hand, we multiply (2) by $\varphi(x)$ and integrate in D. Applying the equality

$$\int_D \varphi\Delta\varphi d\mathbf{x} = \int_K \varphi\frac{\partial\varphi}{\partial n} d_y K - \int_D \left|\operatorname{grad}\varphi\right|^2 d\mathbf{x}, \tag{19}$$

we obtain the following formula:

$$\int_D \left[\alpha\left|\operatorname{grad}\varphi\right|^2 + \xi\varphi^2 + \beta\varphi \operatorname{div}\mathbf{u}\right] d\mathbf{x} = \int_K \alpha\varphi\frac{\partial\varphi}{\partial\mathbf{n}} d_y K. \tag{20}$$

Using formulas (17) and (20), we write

$$\int_D \left[E(\mathbf{u},\mathbf{u}) + \alpha\left|\operatorname{grad}\varphi\right|^2 + \xi\varphi^2 + 2\beta\varphi \operatorname{div}\mathbf{u}\right] d\mathbf{x} = \int_K \left\{\mathbf{u}\left[\mathbf{T}\left(\partial_y, \mathbf{n}\right)\mathbf{u} + \beta\varphi n\right] + \alpha\varphi\frac{\partial\varphi}{\partial\mathbf{n}}\right\} d_y K. \tag{21}$$

Let us assume that each of the above-stated problems has two solutions. To define their difference, the right-hand parts of formulas (17) and (20) are assumed to be equal to zero on K:

$$\int_D \left[E(\mathbf{u},\mathbf{u}) + \beta\varphi \operatorname{div}\mathbf{u}\right] d\mathbf{x} = 0, \qquad \int_D \left[\alpha\left|\operatorname{grad}\varphi\right|^2 + \xi\varphi^2 + \beta\varphi \operatorname{div}\mathbf{u}\right] d\mathbf{x} = 0. \tag{22}$$

From these equalities, taking (10) into account, we obtain $\varphi_1(\mathbf{x}) = k$, $\varphi_2(\mathbf{x}) = 0$,

$$\mathbf{E}(\mathbf{u},\mathbf{u}) + \beta k \operatorname{div} u = 0. \tag{23}$$

where k is arbitrary constant.

Let $\mathbf{U}' = (\mathbf{u}', \varphi')$ and $\mathbf{U}'' = (\mathbf{u}'', \varphi'')$ be two arbitrary solutions of any of problems I and II. Then the differences $\mathbf{U} = (\mathbf{u}, \varphi) = U' - U''$ are the solutions of the corresponding homogeneous problems:

$$\begin{aligned} &\mu\Delta\mathbf{u}(\mathbf{x}) + (\lambda+\mu)\operatorname{graddiv}\mathbf{u}(\mathbf{x}) = 0, \quad \mathbf{x} \in D, \\ &u(\mathrm{z}) = 0, \\ &\varphi(z) = 0, \\ &\qquad \text{in the problem I;} \\ &T\left(\partial_z, \mathbf{n}\right)\mathbf{U}(\mathbf{z}) + \beta\mathbf{n}(\mathbf{z})\varphi(z) = 0, \\ &\alpha\partial_n\varphi(\mathbf{z}) = 0, \\ &\qquad \mathbf{z} \in K, \text{ in the problem II.} \end{aligned} \tag{24}$$

For homogeneous boundary conditions, for Problem I, we obtain from (23): $\mathrm{E}(\mathbf{u},\mathbf{u}) = 0$. The solution of the above equation has the following form [17]:

$$u_1(\mathbf{x}) = -px_2 + q_1, \qquad u_2(\mathbf{x}) = px_1 + q_2, \tag{25}$$

where p, q_1, q_2 are arbitrary constants. In addition, in Problem I, by virtue of homogeneous boundary condition, we get $p = q_1 = q_2 = 0$, and hence, $\mathbf{u}(\mathbf{x}) = 0$, $\mathbf{x} \in D$.

The solution of the homogeneous Problem II, which also satisfies (23), has the following form:

$$\mathbf{u}(\mathbf{x}) = l\mathbf{x} + \mathbf{m}, \tag{26}$$

where $l = \beta k/2(\lambda + \mu)$, $m = (m_1, m_2)$, m_1, m_2, k are arbitrary constants.

Thus, the following assertions are true.

Theorem 1. *Problem I has a unique solution.*

Theorem 2. *Two arbitrary solutions of Problem II may differ from each other only in equality (26) which express the rigid displacement of the body as a whole.*

5. Solution of the Problems

Problem I. Let us rewrite representations (12) in terms of polar coordinates r and ψ as normal and tangent components:

$$\begin{aligned} u_n &= \partial_r (c_0\Phi_1 + c_3\Phi_2 + c_4\varphi_2) - \frac{c_0}{r}\partial_\psi \Phi_3, \\ u_s &= \frac{1}{r}\partial_\psi (c_0\Phi_1 + c_3\Phi_2 + c_4\varphi_2) + c_0\partial_r\Phi_3 - c_2 r, \\ \varphi &= \varphi_1 + \varphi_2, \end{aligned} \tag{27}$$

where $c_3 = -\xi/\beta$, $c_4 = \beta/\mu_0 s_1^2$, $r^2 = x_1^2 + x_2^2$.

Using formula (14) and the equality $\varphi_1 = \Delta\Phi_2$, the harmonic and biharmonic functions and also the metaharmonic function contained in (27) are represented in the circular disc D as series [19, 20]:

$$\begin{aligned} \varphi_1(\mathbf{x}) &= \sum_{m=0}^{\infty} \left(\frac{r}{R}\right)^m (\mathbf{X}_{m1} \cdot \boldsymbol{\nu}_m(\psi)), \\ \Phi_2(x) &= \frac{R^2}{4} \sum_{m=0}^{\infty} \frac{1}{m+1} \left(\frac{r}{R}\right)^{m+2} (\mathbf{X}_{m1} \cdot \nu_m(\psi)), \\ \Phi_3(\mathbf{x}) &= \frac{\mu_0 R^2}{4\mu} \sum_{m=0}^{\infty} \frac{1}{m+1} \left(\frac{r}{R}\right)^{m+2} (\mathbf{X}_{m1} \cdot \mathbf{s}_m(\psi)), \\ \Phi_1(\mathbf{x}) &= \sum_{m=0}^{\infty} \left(\frac{r}{R}\right)^m (\mathbf{X}_{m3} \cdot \boldsymbol{\nu}_m(\psi)), \\ \varphi_2(\mathbf{x}) &= \sum_{m=0}^{\infty} \mathrm{I}_m(s_0 r) (\mathbf{X}_{m2} \cdot \boldsymbol{\nu}_m(\psi)), \end{aligned} \tag{28}$$

where $\mathbf{X}_{mk}$ is the sought two-component vector, $k = 1, 2, 3$, $\mathbf{x} = (r, \psi)$; $\boldsymbol{\nu}_m(\psi) = (\cos m\psi, \sin m\psi)$, $\mathbf{s}_m(\psi) = (-\sin m\psi, \cos m\psi)$, $\mathrm{I}_m(s_0 r)-$ is the Bessel function of the imaginary argument.

Let us write the boundary conditions of Problem I in the form of normal and tangent components:

$$\begin{aligned} u_n(\mathbf{z}) &= f_n(\mathbf{z}), \\ u_s(\mathbf{z}) &= f_s(\mathbf{z}), \\ \varphi(\mathbf{z}) &= f_3(\mathbf{z}). \end{aligned} \tag{29}$$

Let functions f_n, f_s, and f_3 be expanded into Fourier series:

$$\begin{aligned} f_n(\mathbf{z}) &= \frac{\alpha_0}{2} + \sum_{m=1}^{\infty} (\boldsymbol{\alpha}_m \cdot \boldsymbol{\nu}_m(\psi)), \\ f_s(\mathbf{z}) &= \frac{\beta_0}{2} + \sum_{m=1}^{\infty} (\boldsymbol{\beta}_m \cdot \mathbf{s}_m(\psi)), \\ f_3(\mathbf{z}) &= \frac{\gamma_0}{2} + \sum_{m=1}^{\infty} (\boldsymbol{\gamma}_m \cdot \boldsymbol{\nu}_m(\psi)), \end{aligned} \tag{30}$$

where $\boldsymbol{\alpha}_m$, $\boldsymbol{\beta}_m$, and $\boldsymbol{\gamma}_m$ are the Fourier coefficients of the functions f_n, f_s, and f_3, respectively; $\boldsymbol{\alpha}_m = (\alpha_{m1}, \alpha_{m2})$, $\boldsymbol{\beta}_m = (\beta_{m1}, \beta_{m2})$, $\boldsymbol{\gamma}_m = (\gamma_{m1}, \gamma_{m2})$, $m = 0, 1, \ldots$.

Let us substitute expressions (28) into (27) and pass to the limit as $r \to R$. Taking (30) into account, we substitute the results into (29). For $m = 0$ we obtain the linear algebraic system:

$$\begin{aligned} c_3\mathbf{X}_{01} + 2c_4 s_0 \mathrm{I}_0'(s_0 R)\mathbf{X}_{02} &= \boldsymbol{\alpha}_0, \\ c_0\mu_0 R^2 \mathbf{X}_{01} - 2\mu R \mathbf{X}_{03} &= \boldsymbol{\beta}_0 \mu, \\ \mathbf{X}_{01} + \mathrm{I}_0(s_0 R)\mathbf{X}_{02} &= \frac{\boldsymbol{\gamma}_0}{2}. \end{aligned} \tag{31}$$

where, for convenience, we have introduced the notation $\mathbf{X}_{03} \equiv c_2$.

For $m = 1, 2, \ldots$ for each m we obtain

$$\begin{aligned} &\frac{R}{4(m+1)}\left[c_3(m+2) + \frac{c_0 m\mu_0}{\mu}\right]\mathbf{X}_{m1} \\ &\quad + c_4 s_0 \mathrm{I}_m'(s_0 R)\mathbf{X}_{m2} + \frac{c_0 m}{R}\mathbf{X}_{m3} = \boldsymbol{\alpha}_m, \\ &\frac{R\left[c_3 m\mu + c_0\mu_0 R(m+2)\right]}{4\mu(m+1)}\mathbf{X}_{m1} + \frac{c_4 m}{R}\mathrm{I}_m(s_0 R)\mathbf{X}_{m2} \\ &\quad + \frac{c_0 m}{R}\mathbf{X}_{m3} = \boldsymbol{\beta}_m, \\ &\mathbf{X}_{m1} + \mathrm{I}_m(s_0 R)\mathbf{X}_{m2} = \boldsymbol{\gamma}_m, \quad m = 1, 2, \ldots. \end{aligned} \tag{32}$$

We solve systems (31) and (32). The obtained values of vectors X_{mk} are substituted into (28). Now, using formulas (13), (15), and (12), we obtain the solution of the considered problem.

Problem II. Passing in (5) to normal and tangent components, we obtain

$$\begin{aligned}
&\{\mathbf{P}_1(\partial_x,\mathbf{n}(\mathbf{x}))\mathbf{U}(\mathbf{x})\}_n \\
&\quad=\mu_0\partial_r u_n(\mathbf{x})+\frac{\lambda}{r}\partial_\psi u_s(\mathbf{x})+\beta\varphi(\mathbf{x}), \\
&\{\mathbf{P}_1(\partial_x,\mathbf{n}(\mathbf{x}))\mathbf{U}(\mathbf{x})\}_s=\mu\left[\partial_r u_s(\mathbf{x})+\frac{1}{r}\partial_\psi u_n(\mathbf{x})\right], \\
&\partial_n\varphi(\mathbf{x})=\partial_r\varphi(\mathbf{x}),
\end{aligned} \tag{33}$$

where u_n, u_s, and φ are defined in (27). Rewrite boundary conditions $(3)_2$ in the form

$$\begin{aligned}
\{\mathbf{P}_1(\partial_z,\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}_n&=f_n(\mathbf{z}), \\
\{\mathbf{P}_1(\partial_z,\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}_s&=f_s(\mathbf{z}), \\
\alpha\partial_r\varphi(\mathbf{z})&=f_3(\mathbf{z}).
\end{aligned} \tag{34}$$

Substitute (27) and (28) into (34). Using (30) and passing to the limit as $r\to R$, from (30) we obtain the algebraic system. For $m=0$ we have

$$\begin{aligned}
&\left[\frac{\mu_0 c_3}{2}-\frac{\lambda\mu_0 R c_0}{2\mu}+\beta\right]\mathbf{X}_{01} \\
&\quad+\left[\mu_0 c_4 s_0^2\mathrm{I}''(s_0R)+\beta\mathrm{I}_0(s_0R)\right]\mathbf{X}_{02}=\frac{\boldsymbol{\alpha}_0}{2}, \\
&\frac{\mu_0 R c_0}{2}\mathbf{X}_{01}-\mu\mathbf{X}_{03}=\frac{\boldsymbol{\beta}_0}{2}, \\
&s_0\mathrm{I}_0'(s_0R)\mathbf{X}_{02}=\frac{\boldsymbol{\gamma}_0}{2\alpha}.
\end{aligned} \tag{35}$$

For $m=1,2,\ldots$ we get

$$\begin{aligned}
&\left[\frac{\mu_0(m+2)c_3}{4}+\frac{\mu_0 m c_0}{4\mu R}-\frac{\lambda m^2 c_0}{R^2}-\frac{\lambda\mu_0 R(m+2)c_0}{4\mu(m+1)}\right. \\
&\quad\left.+\beta\right]\mathbf{X}_{m1}+\left[\mu_0 s_0^2\mathrm{I}_m''(s_0R)c_4-\frac{\lambda m^2 c_4}{R^2}\right. \\
&\quad\left.+\beta\mathrm{I}_0(s_0R)\right]\mathbf{X}_{m2}+\frac{\mu_0 m(m-1)c_0}{R^2}\mathbf{X}_{m3}=\boldsymbol{\alpha}_m, \\
&\left[\mu_0 s_0^2\mathrm{I}_m''(s_0R)c_4-\frac{\lambda m^2 c_4}{R^2}+\beta\mathrm{I}_0(s_0R)\right]\mathbf{X}_{m1} \\
&\quad+\left[\frac{2ms_0c_4}{R}\mathrm{I}_m'(s_0R)-\frac{mc_4}{R^2}\mathrm{I}_m(s_0R)\right]\mathbf{X}_{m2} \\
&\quad+\frac{mc_0}{R^2}(2m-1)\mathbf{X}_{m3}=\frac{\boldsymbol{\beta}_m}{\mu}, \\
&\frac{m}{R}\mathbf{X}_{m1}+s_0\mathrm{I}_m'(s_0R)\mathbf{X}_{m2}=\frac{\boldsymbol{\gamma}_m}{\alpha}.
\end{aligned} \tag{36}$$

Solving these systems and substituting the obtained values of the vectors $\mathbf{X}_{mk}$ into (28), by formulas (13), (15), and (12) we obtain the solution of Problem II.

For the obtained series to converge absolutely and uniformly it suffices to require the following.

In Problem I, $f_j\in C^3(K)$, where $j=1,2,3$.

In Problem II, $f_j\in C^2(K)$, where $j=1,2,3$.

Conflicts of Interest

The author declares that there are no conflicts of interest.

References

[1] S. C. Cowin and J. W. Nunziato, "Linear elastic materials with voids," *Journal of Elasticity*, vol. 13, no. 2, pp. 125–147, 1983.

[2] S. C. Cowin and P. Puri, "The classical pressure vessel problems for linear elastic materials with voids," *Journal of Elasticity*, vol. 13, no. 2, pp. 157–163, 1983.

[3] D. Iesan, "Some theorems in the theory of elastic materials with voids," *Journal of Elasticity: The Physical and Mathematical Science of Solids*, vol. 15, no. 2, pp. 215–224, 1985.

[4] D. S. Chandrasekharaiah, "Complete solutions in the theory of elastic materials with voids," *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 40, no. 3, pp. 401–414, 1987.

[5] V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, *Three- Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North-Holland Publishing Company, Amsterdam, The Netherlands, 1979.

[6] A. Pompei and A. Scalia, "On the steady vibrations of elastic materials with voids," *Journal of Elasticity: The Physical and Mathematical Science of Solids*, vol. 36, no. 1, pp. 1–26, 1994.

[7] P. Puri and S. C. Cowin, "Plane waves in linear elastic materials with voids," *Journal of Elasticity*, vol. 15, no. 2, pp. 167–183, 1985.

[8] G. Rusu, "Existence theorems in elastostatics of materials with voids," *Scientific Annals of the Alexandru Ioan Cuza University of Iaşi Mathematics*, vol. 30, pp. 193–204, 1985.

[9] E. Scarpetta, "On the fundamental solutions in micropolar elasticity with voids," *Acta Mechanica*, vol. 82, no. 3-4, pp. 151–158, 1990.

[10] D. Ieşan, "A theory of thermoelastic materials with voids," *Acta Mechanica*, vol. 60, no. 1-2, pp. 67–89, 1986.

[11] M. Ciarletta, "A solution of Galerkin type in the theory of thermoelastic materials with voids," *Journal of Thermal Stresses. An International Quarterly*, vol. 14, no. 4, pp. 409–417, 1991.

[12] M. Ciarletta and A. Scalia, "On uniqueness and reciprocity in linear thermoelasticity of materials with voids," *Journal of Elasticity: The Physical and Mathematical Science of Solids*, vol. 32, no. 1, pp. 1–17, 1993.

[13] A. Pompei and A. Scalia, "On the steady vibrations of the thermoelastic porous materials," *International Journal of Solids and Structures*, vol. 31, no. 20, pp. 2819–2834, 1994.

[14] M. Ciarletta, A. Scalia, and M. Svanadze, "Fundamental solution in the theory of micropolar thermoelasticity for materials with voids," *Journal of Thermal Stresses*, vol. 30, no. 3, pp. 213–229, 2007.

[15] R. Kumar and T. Kansal, "Fundamental solution in the theory of micropolar thermoelastic diffusion with voids," *Computational and Applied Mathematics*, vol. 31, no. 1, pp. 169–189, 2012.

[16] J. W. Nunziato and S. C. Cowin, "A nonlinear theory of elastic materials with voids," *Archive for Rational Mechanics and Analysis*, vol. 72, no. 2, pp. 175–201, 1979.

[17] N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff International Publishing, 1953.

[18] I. Tsagareli and L. Bitsadze, "Explicit solution of one boundary value problem in the full coupled theory of elasticity for solids with double porosity," *Acta Mechanica*, vol. 226, no. 5, pp. 1409–1418, 2015.

[19] S. G. Mikhlin, *Mathematical Physics: An Advanced Course*, North-Holland Publishing Company, Amsterdam, The Netherlands, 1970.

[20] I. Vekua, *New Methods for Solving Elliptic Equations*, North-Holland Publishing Company, 1967.