Research Article

Lump Solutions to the (3+1)-Dimensional Generalized B-Type Kadomtsev-Petviashvili Equation

Xifang Cao

School of Mathematical Sciences, Yangzhou University, China

Correspondence should be addressed to Xifang Cao; xfcao@yzu.edu.cn

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This paper is devoted to the study of lump solutions to the (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation. First we use a direct method to construct a class of exact solutions which contain six arbitrary real constants. Then we use these solutions to generate lump solutions with four real parameters. We also determine the amplitude and velocity of these lumps.

1. Introduction

Lump is a type of localized rational solutions. Lump solutions for the (2+1)-dimensional Kadomtsev-Petviashvili I equation

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0 \quad (1)$$

were first obtained by Manakov et al. [1]. Then Satsuma and Ablowitz [2] constructed more lump solutions by taking a long wave limit of the corresponding soliton solutions. More recently, by use of the bilinear form, Ma [3] generalized the results of [1, 2] and got a larger class of lump solutions. Over the past few decades, lump has been an active area in the study of nonlinear evolution equations. They can be used to describe nonlinear patterns in plasma [4], in optical media [5], in the Bose-Einstein condensate [6], and so forth. Lump solutions have been obtained for some other equations, such as the Ishimori equation [7], the Jimbo-Miwa equation [8], and the Sawada-Kotera equation [9].

Recently much literature is devoted to the study of (3+1)-dimensional B-type Kadomtsev-Petviashvili (BKP) equation

$$u_{yt} + 3u_{xz} - 3u_x u_{xy} - 3u_{xx} u_y - u_{xxx} = 0, \quad (2)$$

which can be used to model fluid dynamics, plasma physics, and weakly dispersive media [10, 11]. Various methods have been applied to (2) to construct its soliton and multiple wave solutions [12–14]. In this paper, we develop another

method to construct its lump solutions. We first use a direct method to obtain a class of exact solutions which contain six arbitrary real constants (see Theorem 1). Then we show that the limits of these solutions can generate lump solutions (see Theorem 4). We also determine the amplitude and velocity of these lumps.

2. Exact Solutions

In this section, we seek exact solutions to the (3+1)-dimensional BKP equation (2) in the following form:

$$u(x, y, z, t) = 2(\ln f)_x, \quad (3)$$

where

$$f(x, y, z, t) = \exp(-\eta) + m \cos \xi + n \exp \eta, \quad (4)$$

$\eta = p(x+qy+rz)$, $\xi = p(ax+by+cz+dt)$, and $m, n, p, q, r, a, b, c, d$ are real constants to be determined. Substituting (3) into the left hand side of (2) and then factoring we get the following three determining equations:

$$6ac - 3r + bd + 3a^2r + p^2q + 4a^3bp^2 - 6a^2p^2q + a^4p^2q + adq - 4abp^2 = 0, \quad (5)$$

$$3c + 6ar + dq - 3a^2c - bp^2 - 4ap^2q + 6a^2bp^2 - a^4bp^2 + 4a^3p^2q - abd = 0, \quad (6)$$

$$m^2 (3r + 18ac + 3bd + 9a^2r - p^2q + 24a^3bp^2 - 6a^2p^2q + 3a^4p^2q + 3adq) + 16n(4p^2q - 3r) = 0. \quad (7)$$

Solving for r, c from (5) and (6) gives

$$r = \frac{p^2q + bd - a^4p^2q - 2a^3bp^2 - 2abp^2 - adq}{3(a^2 + 1)}, \quad (8)$$

and

$$c = \frac{bp^2 + 2a^3p^2q + 2ap^2q - a^4bp^2 - abd - dq}{3(a^2 + 1)}. \quad (9)$$

Substituting (8) and (9) into (7) yields

$$n = -\frac{3ba^5p^2 + 2qa^4p^2 + 4ba^3p^2 + 2qa^2p^2 + bap^2 - dqa + bd}{4(qa^4p^2 + 2ba^3p^2 + 4qa^2p^2 + 2bap^2 + 3qp^2 + dqa - bd)} \cdot m^2. \quad (10)$$

Therefore, we have the following.

Theorem 1. The (3+1)-dimensional BKP equation (2) admits the following exact solutions:

$$u(x, y, z, t) = 2(\ln f)_x, \quad (11)$$

$$f(x, y, z, t) = \exp(-\eta) + m \cos \xi + n \exp \eta,$$

where $\eta = p(x + qy + rz)$, $\xi = p(ax + by + cz + dt)$, m, p, q, a, b, d are arbitrary real constants, and r, c, n are given by (8), (9), and (10), respectively.

3. Lump Solutions

In this section, we use (11) to construct lump solutions to the (3+1)-dimensional BKP equation (2). We first give some examples and then give a general result.

Example 1. Let $m = -2, q = 1, a = 0, b = 1, d = 1$. Then (11) becomes

$$u(x, y, z, t) = 2(\ln f)_x, \quad (12)$$

$$f(x, y, z, t) = \exp(-\eta) - 2 \cos \xi + \frac{1}{1 - 3p^2} \exp \eta,$$

where $\eta = p(x + y + ((1 + p^2)/3)z)$, $\xi = p(y + ((p^2 - 1)/3)z + t)$. As $p \rightarrow 0$, the limit of (12) gives a lump solution to (2):

$$U(x, y, z, t) = \lim_{p \rightarrow 0} u(x, y, z, t) = \frac{4X}{3 + X^2 + Y^2}, \quad (13)$$

where

$$X = x + y + \frac{z}{3}, \quad (14)$$

$$Y = y - \frac{z}{3} + t.$$

The lump (13) attains its maximum value $2/\sqrt{3}$ at $(X, Y) = (\sqrt{3}, 0)$ and minimum value $-2/\sqrt{3}$ at $(X, Y) = (-\sqrt{3}, 0)$. So its amplitude is $2/\sqrt{3}$. The profiles of the lump (13) in the (X, Y) -space are given in Figure 1.

From the point of view of the (x, y, z) -space, at each time t , the lump (13) attains its extreme values at the lines

$$x + y + \frac{z}{3} = \pm\sqrt{3}, \quad (15)$$

$$-3y + z = 3t.$$

Note that the planes $x + y + z/3 = \pm\sqrt{3}$ are fixed (do not depend on t), while the plane $-3y + z = 3t$ moves along its normal vector $\mathbf{N} = (1/\sqrt{10})(0, -3, 1)$ with the speed $3/\sqrt{10}$. Therefore the velocity of the moving lines (15) is

$$\mathbf{V} = (V_x, V_y, V_z) = \frac{63}{5\sqrt{266}}(8, -11, 9). \quad (16)$$

Example 2. Let $m = -2, q = 0, a = 1, b = 1, d = 1$. Then (11) becomes

$$u(x, y, z, t) = 2(\ln f)_x, \quad (17)$$

$$f(x, y, z, t) = \exp(-\eta) - 2 \cos \xi + \frac{1 + 8p^2}{1 - 4p^2} \exp \eta,$$

where $\eta = p(x + ((1 - 4p^2)/6)z)$, $\xi = p(x + y - (1/6)z + t)$. As $p \rightarrow 0$, the limit of (17) gives a lump solution to (2):

$$U(x, y, z, t) = \lim_{p \rightarrow 0} u(x, y, z, t) = \frac{4(X + Y)}{12 + X^2 + Y^2}, \quad (18)$$

where

$$X = x + \frac{z}{6}, \quad (19)$$

$$Y = x + y - \frac{z}{6} + t.$$

The lump (18) attains its maximum value $\sqrt{6}/3$ at $(X, Y) = (\sqrt{6}, \sqrt{6})$ and minimum value $-\sqrt{6}/3$ at $(X, Y) = (-\sqrt{6}, -\sqrt{6})$. So the amplitude of the lump (18) is $\sqrt{6}/3$. The profiles of the lump (18) in the (X, Y) -space are given in Figure 2.

From the point of view of the (x, y, z) -space, at each time t , the lump (18) attains its extreme values at the lines

$$x + \frac{z}{6} = \pm\sqrt{6}, \quad (20)$$

$$x + y - \frac{z}{6} + t = \pm\sqrt{6},$$

which move with the velocity

$$\mathbf{V} = (V_x, V_y, V_z) = \frac{1476}{73\sqrt{1517}}(-2, -37, 12). \quad (21)$$

Example 3. Let $m = -2, q = 1, a = -1, b = 0, d = 1$. Then (11) becomes

$$u(x, y, z, t) = 2(\ln f)_x, \quad (22)$$

$$f(x, y, z, t) = \exp(-\eta) - 2 \cos \xi + \frac{1 + 4p^2}{1 - 8p^2} \exp \eta,$$

FIGURE 1: The 3D plot of lump (13) and the corresponding contour plot.

FIGURE 2: The 3D plot of lump (18) and the corresponding contour plot.

where $\eta = p(x + y + (1/6)z)$, $\xi = p(-x + ((4p^2 - 1)/6)z + t)$. As $p \rightarrow 0$, the limit of (22) gives a lump solution to (2):

$$U(x, y, z, t) = \lim_{p \rightarrow 0} u(x, y, z, t) = \frac{4(X - Y)}{12 + X^2 + Y^2}, \quad (23)$$

where

$$\begin{aligned} X &= x + y + \frac{z}{6}, \\ Y &= -x - \frac{z}{6} + t. \end{aligned} \quad (24)$$

The lump (23) attains its maximum value $\sqrt{6}/3$ at $(X, Y) = (\sqrt{6}, -\sqrt{6})$ and minimum value $-\sqrt{6}/3$ at $(X, Y) = (-\sqrt{6}, \sqrt{6})$. So the amplitude of the lump (23) is $\sqrt{6}/3$. The profiles of the lump (23) in the (X, Y) -space are given in Figure 3.

From the point of view of the (x, y, z) -space, at each time t , the lump (23) attains its extreme values at the lines

$$\begin{aligned} x + y + \frac{z}{6} &= \pm\sqrt{6}, \\ -x - \frac{z}{6} + t &= \mp\sqrt{6}, \end{aligned} \quad (25)$$

which move with the velocity

$$\mathbf{V} = (V_x, V_y, V_z) = \frac{1332}{37\sqrt{2701}} (36, -37, 6). \quad (26)$$

In general, for arbitrary real constants q, a, b, d , let $m = -2$. Then as $p \rightarrow 0$, the limit of (11) yields the following result.

Theorem 4. *The (3+1)-dimensional BKP equation (2) admits the following rational solutions:*

$$U(x, y, t) = 4 \frac{X + aY}{A + X^2 + Y^2}, \quad (27)$$

FIGURE 3: The 3D plot of lump (23) and the corresponding contour plot.

where

$$A = \frac{3(a^2 + 1)^2(q + ab)}{d(b - aq)}, \quad (28)$$

$$X = x + qy - \frac{d(aq - b)}{3(a^2 + 1)}z, \quad (29)$$

$$Y = ax + by - \frac{d(q + ab)}{3(a^2 + 1)}z + dt, \quad (30)$$

and q, a, b, d are arbitrary real constants.

Remark 5. If the parameters q, a, b, d satisfy the following condition,

$$\frac{q + ab}{d(b - aq)} > 0, \quad (31)$$

then (27) are lump solutions.

For given parameters q, a, b, d satisfying condition (31), the lump (27) attains its extreme values $\pm 2\sqrt{d(b - aq)/3(a^2 + 1)(q + ab)}$ at the lines

$$\begin{aligned} x + qy - \frac{d(aq - b)}{3(a^2 + 1)}z &= \pm \sqrt{\frac{3(a^2 + 1)(q + ab)}{d(b - aq)}}, \\ ax + by - \frac{d(q + ab)}{3(a^2 + 1)}z + dt & \\ &= \pm a \sqrt{\frac{3(a^2 + 1)(q + ab)}{d(b - aq)}} \end{aligned} \quad (32)$$

in the (x, y, z) -space. Therefore the amplitude of the lump (27) is

$$U_{\text{amplitude}} = 2\sqrt{\frac{d(b - aq)}{3(a^2 + 1)(q + ab)}}. \quad (33)$$

The velocity $\mathbf{V} = (V_x, V_y, V_z)$ of the lump (27) in the (x, y, z) -space can also be determined. Here we omit the details.

4. Discussion

The (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation is the Jimbo-Miwa equation. If we replace t with $-t/2$, then (2) becomes

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0. \quad (34)$$

Based on the bilinear form of (34), Yang and Ma [8] construct ten classes of its lump solutions in the form

$$u = 2(\ln f)_x, \quad f = g^2 + h^2 + a_{11}, \quad (35)$$

where $g = a_1x + a_2y + a_3z + a_4t + a_5$, $h = a_6x + a_7y + a_8z + a_9t + a_{10}$, and a_i ($i = 1, \dots, 11$) are some constants.

In this section, we point out that up to the invariance after a translation and a Galilean transformation, all the solutions obtained in [8] can be simplified and be rewritten in a unified form of (27). This is because, firstly, the parameters a_5 and a_{10} can be set zero by the translation $(x, y, z, t) \rightarrow (x', y', z', t')$ defined by

$$\begin{aligned} x' &= x + x_0, \\ y' &= y + y_0, \\ z' &= z + z_0, \\ t' &= t + t_0, \end{aligned} \quad (36)$$

where x_0, y_0, z_0, t_0 are some constants; secondly, since the parameters a_1 and a_6 can not be zero simultaneously, without loss of generality, we may assume $a_1 \neq 0$, and then from the form of (35), we can set $a_1 = 1$; thirdly, since (34) is invariant under the Galilean transformation $(x, y, z, t) \rightarrow (x', y', z', t')$ given by

$$\begin{aligned}
 x' &= x + 3\lambda t, \\
 y' &= y + 2\lambda z, \\
 z' &= z, \\
 t' &= t,
 \end{aligned} \tag{37}$$

where λ is an arbitrary real number, we can set $a_4 = 0$. Then under the condition

$$\begin{aligned}
 a_1 &= 1, \\
 a_4 &= 0, \\
 a_5 &= 0, \\
 a_{10} &= 0,
 \end{aligned} \tag{38}$$

g and h are of the forms of (29) and (30), and a direct calculation shows that each class of solutions in [8] can be rewritten in the unified form of (27).

From the mathematical point of view, the lump solutions (27) are limits of the exact solutions (11) as the parameter p approaches zero. It is worth studying the physical meaning of this limit.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

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