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Research Article

Minimization of Stress State of a Hub of Friction Pair

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The function of displacements of external contour points of a friction pair hub that could provide minimization of stress state of a hub was determined on the basis of minimax criterion. The problem is to decrease stress state at that place where it is important. The rough friction surface model is used. To solve a problem of optimal design of friction unit the closed system of algebraic equations is constructed. Increase of serviceability of friction pair parts may be controlled by design-engineering methods, in particular by geometry of triboconjugation elements. Minimization of maximum circumferential stress on contact surface of friction unit is of great importance in the design stage for increasing the serviceability of friction pair. The obtained function of displacements of the hub's external contour points provides the serviceability of friction pair elements. The calculation of friction pair for oil-well sucker-rod pumps is considered as an example.

1. Introduction

Operation efficiency of a friction pair of machines depends in considerable degree on stress state of the friction pairs. For example, typical operational failure of hubs of oil-well sucker-rod pump is appearance of plastic deformations on the internal contour. When operating, the plunger rubs against the hub's surface. The surface layer of the hub's metal is heating. In the process of operation of the friction pair "a hub-a plunger" under repeatedly reciprocating motion of the plunger there arises force interaction between contacting surfaces of the hub and plunger; there occur friction forces that cause wear of mating materials. In its parts there arises stress-strain state caused by the action of force and thermal loads.

According to classic theories of strength, in the simplest case maximum normal stress is responsible for failure of friction pair parts. Consequently, the value of maximum normal peripheral stress achieved in the material may be considered responsible for strength failure of friction pair materials. Increase of serviceability of friction pair parts may be controlled by design-engineering methods, in particular by geometry of triboconjugation elements. At present there are no solutions of tribomechanics problems on construction geometry of the surface of friction pair parts such that the stress field created by it prevented failure or occurrence of irreversible deformations of the materials of contact pair elements. Minimization of maximum circumferential stress on contact surface of friction unit is of great importance in the design stage for increasing serviceability of friction pair. Obviously, the lower the stress state of the hub, the higher its operation life. To improve the efficiency of details of friction pair the optimal design is important [1–11].

The goal of the study is to develop a mathematical model for a hub-plunger pair that allows determination of optimal function of displacements of the points of the external contour of the hub under given operation modes of the plunger.

2. Problem Statement

It is known that real treated surfaces are not absolutely smooth but always have irregularities of technological character. Such micro- or macroscopic irregularities form the rough surface. Despite the very small sizes of the irregularities, they affect the different service properties of

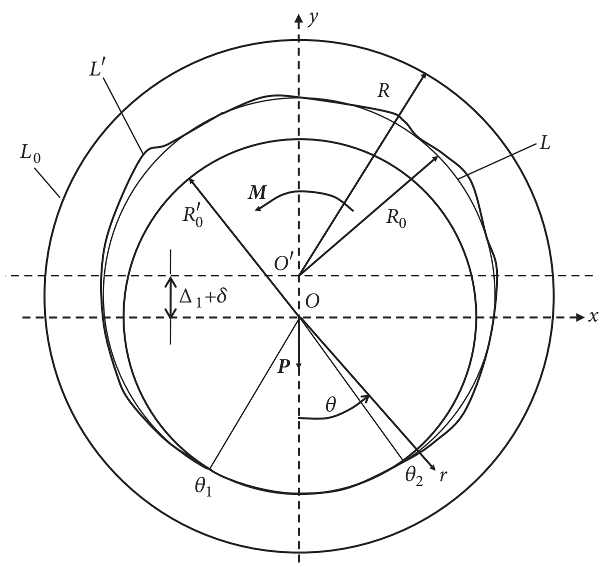

FIGURE 1: Design diagram of contact problem for hub-plunger friction pair.

triboconjugation [12–15]. In the operation process of friction pair, on the internal surface of the hub, on the area of contact with the plunger, there acts surface thermal source caused by external friction of the plunger against the hub's wall in the course of repeatedly reciprocating motion of the plunger. As a result of such interaction, there happen friction forces reducing to wear of mating materials and increase of hub's and plunger's temperature. We adopt the parameters of the function of displacements of the hub's external surface points as controlling variables.

As a mathematical model of the problem on reduction of hub's stress level we assume differential equations of thermoelasticity.

To determine the stress distribution and contact pressure in the elements of friction pair, a wear-contact problem on pressing the plunger into the hub's surface must be considered.

Let in some unknown area a plunger be pressed into the hub internal surface. The shear modulus and Poisson's ratio of the plunger and hub are various.

It is considered that on the external surface the hub has some displacements. The function of these displacements is unknown beforehand and is to be defined. It is assumed that the plane strain conditions are fulfilled. We simulate the hub and plunger by an isotropic elastic homogeneous body. It is assumed that the operation modes of the friction unit at which residual deformations arise are inadmissible. The loading conditions are considered as quasistatic.

Assign the hub of a friction pair to polar system of coordinates $r\theta$ having chosen the origin at the center of concentrical circles L, L_0 of radii R_0 and R, respectively (Figure 1). It is assumed that the internal contour L' of the hub is close to circular one. Let us consider some realization of the hub and plunger's rough internal surface.

The boundary of the internal contour L' we represent is in the form $r = \rho(\theta)$, $\rho = R_0 + \varepsilon H(\theta)$, where $\varepsilon = R^0_{\max}/R_0$ is a small parameter; $R^0_{\max}$ is the greatest irregularity (hollow) of unevenness of the internal contour profile from the circle $r = R_0$.

We assume without loss of generality that the function $H(\theta)$ may be represented in the form of Fourier series

$$H(\theta) = \sum_{k=0}^{n} \left(a_k^0 \cos k\theta + b_k^0 \sin k\theta\right). \tag{1}$$

Similarly, the plunger's external contour is close to circular one and may be represented in the following form:

$$\begin{aligned} r &= \rho_1(\theta), \\ \rho_1 &= R'_0 + \varepsilon H_1(\theta), \\ H_1(\theta) &= \sum_{k=0}^{n} \left(a_k^1 \cos k\theta + b_k^1 \sin k\theta\right). \end{aligned} \tag{2}$$

The concentrated force

$$P = R_0 \int_{\theta_1}^{\theta_2} (\cos\theta - f \sin\theta)\, p(\theta)\, d\theta \tag{3}$$

pressing plunger to the boundary of the hub's internal contour is applied at the plunger's center.

The pressure $p(\theta)$ is asymmetrically distributed on the contact area and creates a moment with respect to the plunger

$$M = -R_0^2 \int_{\theta_1}^{\theta_2} \tau_{r_\theta} d\theta = f R_0^2 \int_{\theta_1}^{\theta_2} p(\theta)\, d\theta. \tag{4}$$

The contact pressure $p(\theta)$ is unknown beforehand.

It is assumed that the wear of friction pair is of abrasive character. It is required to determine the function of displacements of the points of the hub external contour L_0 under given functions $H(\theta)$ and $H_1(\theta)$ at which minimization of stress state in the hub occurs.

The condition relating the displacements of the plunger and hub is of the form [16–18]

$$v_1 + v_2 = \delta(\theta), \quad \theta_1 \le \theta \le \theta_2, \tag{5}$$

where v_1 and v_2 are the normal components of displacements; $\delta(\theta)$ is the penetration of the points of the hub and plunger surfaces determined by the form of the internal hub surface and the plunger surface and also by the value of the pressing force $\mathbf{P}$; $\theta_2 - \theta_1$ is the value of the angle (area) of contact.

In the contact area, in addition to normal pressure tangential stress $\tau_{r\theta}$ acts, connected with contact pressure $p(\theta, t)$ by the Amonton-Coulomb law. The tangential forces (friction forces) work toward heat release in the contact area. The total amount of heat per unit time is proportional to power of friction forces. The amount of heat flux released at the point of the contact area with the coordinate θ will be equal to $Q(\theta, t) = Vfp(\theta, t)$, where V is the average speed of displacement of the plunger with respect to the hub over a period; f is the friction pair coefficient. Total amount of heat

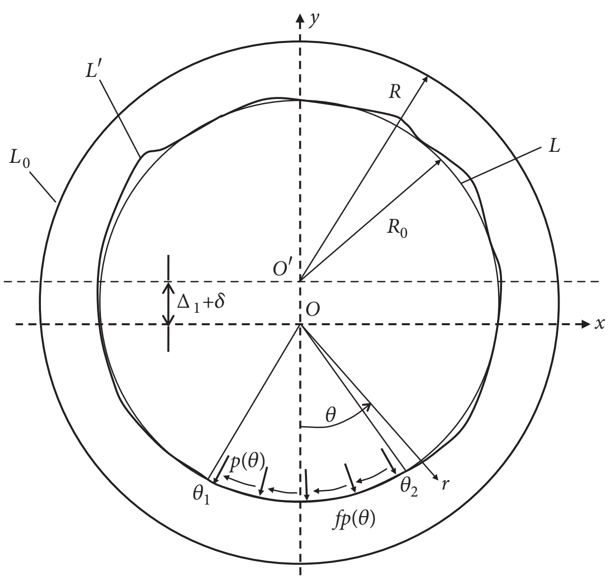

Figure 2: Design diagram of contact problem for hub.

$Q(\theta, t)$ will be consumed in the following way: $Q = Q_b + Q_1$, where Q_b is the heat entering the hub; Q_1 is the heat entering the plunger.

As the plunger's motion frequency is rather great, we consider the problem of determination of temperature stationary.

For displacements of the points of the hub's friction we have $v_1 = v_{1e} + v_{1r} + v_{1w}$, where v_{1e} are thermoelastic displacements of the hub's contact surface; v_{1r}, v_{1w} are displacements caused by crumpling of microregularities and wear of the hub's surface, respectively. Similarly, for the displacements of the plunger's surface we have $v_2 = v_{2e} + v_{2r} + v_{2w}$.

The rate of change of the displacements of the surface under hub's and plunger's wear will be [17]

$$\frac{dv_{kw}}{dt} = K^{(k)} p(\theta, t), \quad k = 1, 2, \tag{6}$$

where $K^{(k)}$ is the coefficient of wear of the hub and plunger's material ($k = 1, 2$), respectively.

To determine the displacements v_{1e}, v_{1r} and v_{2e}, v_{2r} it is necessary to solve the following thermoelasticity problems for the hub (Figure 2) and the plunger (Figure 3), respectively:

$$\begin{gathered} \Delta T = 0, \\ \text{for } r = \rho(\theta) \quad A_{T_1} \lambda \frac{\partial T}{\partial n} - A_{T_2} \alpha_1 T = -Q_*, \end{gathered} \tag{7}$$

$$\text{for } r = R \quad \lambda \frac{\partial T}{\partial r} + \alpha_2 T = 0, \tag{8}$$

$$\begin{aligned} &\text{for } r = \rho(\theta), \\ &\sigma_n = -p(\theta), \\ &\tau_{nt} = -fp(\theta) \\ &\qquad \text{on the contact area,} \\ &\sigma_n = 0, \\ &\tau_{nt} = 0 \\ &\qquad \text{out of the contact area,} \\ &\text{for } r = R, \quad v_r - i v_\theta = g(\theta); \\ &\Delta T_1 = 0, \\ &\text{for } r = \rho_1(\theta) \\ &\lambda_1 \frac{\partial T_1}{\partial n} = -Q(\theta) \\ &\qquad \text{on the contact area,} \end{aligned} \tag{9}$$

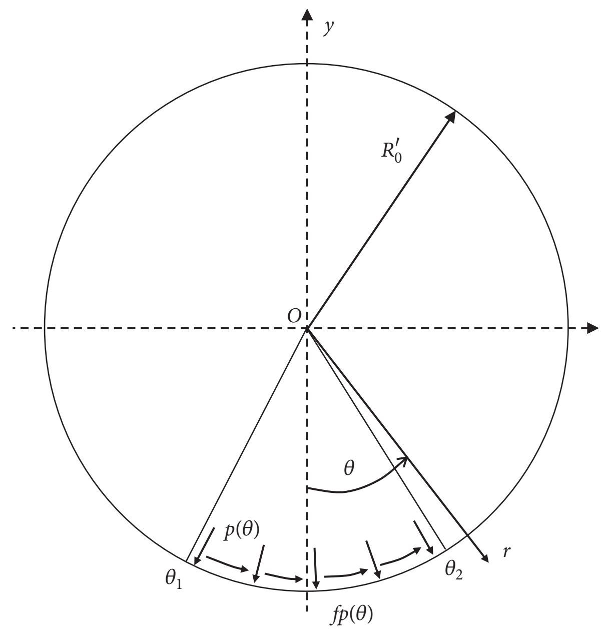

Figure 3: Design diagram of contact problem for plunger.

$$\lambda_1 \frac{\partial T_1}{\partial n} + \alpha_1 T_1 = 0 \quad \text{out of the contact area,} \tag{10}$$

$$\begin{aligned} &\text{for } r = \rho_1(\theta), \\ &\sigma_n = -p(\theta); \\ &\tau_{nt} = -fp(\theta) \\ &\qquad \text{on the contact area,} \\ &\sigma_n = 0, \\ &\tau_{nt} = 0 \\ &\qquad \text{out of the contact area.} \end{aligned} \tag{11}$$

Here Δ is the Laplace operator; T is overtemperature of the hub; A_{T_1} is heat-absorbing surface; A_{T_2} is cooling surface; $Q_*(\theta) = Q_b(\theta) = \alpha_{m.n.} V f p(\theta)$ on the contact area; $Q_* = 0$ out of the contact area; λ, λ_1 are the thermal conductivity of the hub's material and plunger's material, respectively; α_1, α_2 are the heat transfer coefficients between the internal and external surface of the hub and medium, respectively; n, t are natural coordinates; v_r, v_θ are radial and tangential components of displacements vector; σ_n, σ_t, τ_{nt} are stress tensor components; $g(\theta)$ is the sought-for function of displacements of the points of the hub's external contour: $i^2 = -1$.

For intensity of the surface heat source, on the friction area we have $Q_1(\theta) = \alpha_{m.n.1} f V p(\theta)$; $\alpha_{m.n.1}$ is the heat partition coefficient for the plunger.

The contact pressure $p(\theta)$ is to be determined in the process of solution of minimization problem. To solve the stated problem, the wear-contact problem on pressing the plunger into the surface of the hub and optimization problem should be solved jointly.

Without loss of generality of the optimization problem, we assume that the sought-for function $g(\theta)$ of displacements of the points of the external contour L_0 may be represented in the form of the Fourier series

$$g(\theta) = \sum_{k=0}^{\infty} \left(a_k^* \cos k\theta + b_k^* \sin k\theta\right). \quad (12)$$

The unknown quantities θ_1 and θ_2 are the ends of the contact area of the plunger and the hub. To find these quantities we use the condition [19] that pressure $p(\theta)$ continuously passes into zero, when the point θ leaves the contact area:

$$p(\theta_1) = 0, \quad p(\theta_2) = 0. \quad (13)$$

To find the functions of displacements of the points of the external contour $g(\theta)$, we should complement the problem statement with an additional condition (with a criterion that allows determining the function $g(\theta)$). As a criterion for determining the function of displacements of the points of the hub external contour (function $g(\theta)$) we adopt the condition of minimization of the maximum value of circumferential stresses in the hub's material. Minimization of the maximum value of the stresses σ_θ of the hub will boost the serviceability of the friction pair.

Consequently, the coefficients a_k^*, b_k^* of the function $g(\theta)$ should be managed so that minimization of the maximum value of the stress σ_θ in the hub is provided. This additional condition allows determining the sought-for function $g(\theta)$.

3. Solution Method

We look for temperature functions, stresses, displacements, and contact pressure in the hub and plunger in the form of expansions in small parameter wherein for simplification we ignore the members containing ε higher than first degree. We get the values of temperature, stress tensor components, and displacements for $r = \rho(\theta)$ (similarly for $r = \rho_1(\theta)$ as well) expanding in series the expressions for temperature, stresses, and displacements in the neighborhood of $r = R_0$. Each of approximations satisfies the system of differential equations of plane thermoelasticity. Using the perturbations method, in light of the foregoing we arrive at the sequence of boundary value problems of thermoelasticity for the hub.

In zero approximation,

$$\Delta T^{(0)} = 0, \quad (14)$$

$$\text{for } r = R_0\text{:} \quad A_{T_1}\lambda \frac{\partial T^{(0)}}{\partial r} - A_{T_2}\alpha_1 T^{(0)} = -Q_*^{(0)}, \quad (15)$$

$$\text{for } r = R\text{:} \quad \lambda \frac{\partial T^{(0)}}{\partial r} + \alpha_2 T^{(0)} = 0,$$

$$\text{for } r = R_0 : \quad \sigma_r^{(0)} = -p^{(0)}(\theta), \quad \tau_{r\theta}^{(0)} = -fp^{(0)}(\theta) \quad (16)$$

on the contact area,

$$\sigma_r^{(0)} = 0, \quad \tau_{r\theta}^{(0)} = 0 \quad (17)$$

out of the contact area,

$$\text{for } r = R\text{:} \quad v_r^{(0)} - iv_\theta^{(0)} = g^{(0)}(\theta); \quad (18)$$

in the first approximation,

$$\Delta T^{(1)} = 0 \quad (19)$$

$$\text{for } r = R_0\text{:} \quad A_{T_1}\lambda \frac{\partial T^{(1)}}{\partial r} - A_{T_2}\alpha_1 T^{(1)} = \left[A_{T_2}\alpha_1 \frac{\partial T^{(0)}}{\partial r} - A_{T_1}\lambda \frac{\partial^2 T^{(0)}}{\partial r^2}\right] H(\theta) - Q_*^{(1)}(\theta), \quad (20)$$

$$\text{for } r = R\text{:} \quad \lambda \frac{\partial T^{(1)}}{\partial r} + \alpha_2 T^{(1)} = 0, \quad (21)$$

$$\text{for } r = R_0 : \quad \sigma_r^{(1)} = N - p^{(1)}(\theta),$$

$$\tau_{r\theta}^{(1)} = T_t - fp^{(1)}(\theta)$$

on the contact area,

(22)

$$\sigma_r^{(1)} = N$$
$$\tau_{r\theta}^{(1)} = T_t$$

out of the contact area, (23)

for $r = R$:

$$v_r^{(1)} - iv_\theta^{(1)} = g^{(1)}(\theta).$$

Hence for $r = R_0$

$$N = -H(\theta)\frac{\partial\sigma_r^{(0)}}{\partial r} + 2\tau_{r\theta}^{(0)}\cdot\frac{1}{R_0}\frac{dH(\theta)}{d\theta},$$
$$T_t = \left(\sigma_\theta^{(0)} - \sigma_r^{(0)}\right)\frac{1}{R_0}\frac{dH(\theta)}{d(\theta)} - H(\theta)\frac{\partial\tau_{r\theta}^{(0)}}{\partial r}. \quad (24)$$

The boundary conditions for the plunger in each approximation may be written in the same way.

The solution of a boundary value problem for conductivity theory is sought by the method of separation of variables. The temperature is sought in the form of a product of two functions, one of which depends on the variable r, and the other depends only on the polar angle θ. We find the distribution of hub's overtemperature $T = T^{(0)} + \varepsilon T^{(1)} + \ldots$ in the form

$$T^{(0)} = C_{10} + C_{20}\ln r + \sum_{k=1}^{\infty}\left(C_{10}^{(k)}r^k + C_{20}^{(k)}r^{-k}\right)\cos k\theta + \sum_{k=1}^{\infty}\left(A_{10}^{(k)}r^k + A_{20}^{(k)}r^{-k}\right)\sin k\theta,$$
$$T^{(1)} = C_{11} + C_{21}\ln r + \sum_{k=1}^{\infty}\left(C_{11}^{(k)}r^k + C_{21}^{(k)}r^{-k}\right)\cos k\theta + \sum_{k=1}^{\infty}\left(A_{11}^{(k)}r^k + A_{21}^{(k)}r^{-k}\right)\sin k\theta. \quad (25)$$

The constants C_{10}, C_{20}, $C_{10}^{(k)}$, $C_{20}^{(k)}$, $A_{10}^{(k)}$, $A_{20}^{(k)}$ are determined from boundary conditions (15) of the problem in a zero approximation. The coefficients $C_{11}, C_{21}, C_{11}^{(k)}, C_{21}^{(k)}, A_{11}^{(k)}$, $A_{21}^{(k)}$ are found, respectively, from boundary conditions (20) of the problem in the first approximation.

To solve the thermoelasticity problem the thermoelastic potential of displacements [20] is used in each approximation. In the present problem the thermoelastic potentials of displacements in zero and first approximations are determined by solving the following differential equations:

$$\Delta F^{(0)} = \beta T^{(0)},$$
$$\Delta F^{(1)} = \beta T^{(1)},$$
$$\beta = \frac{1+\mu}{1-\mu}\alpha, \quad (26)$$

where μ is Poisson's ratio of the plunger.

We look for the solution of (26) in the form

$$F^{(0)} = \sum_{n=0}^{\infty}\left[f_n^{(0)}(r)\cos n\theta + f_n^{*(0)}(r)\sin n\theta\right],$$
$$F^{(1)} = \sum_{n=0}^{\infty}\left[f_n^{(1)}(r)\cos n\theta + f_n^{*(1)}(r)\sin n\theta\right]. \quad (27)$$

Using the functions $F^{(0)}(r,\theta)$ and $F^{(1)}(r,\theta)$ we satisfy differential equations (26), respectively. Applying the method of separation of variables, we obtain ordinary differential equations for the functions $f_n^{(0)}(r)$, $f_n^{*(0)}(r)$, $f_n^{(1)}(r)$, and $f_n^{*(1)}(r)$.

$$\frac{d^2 f_n^0}{dr^2} + \frac{1}{2}\frac{df_n^0}{dr} - \frac{n^2}{r^2}f_n^0 = \beta F_n^0,$$
$$\frac{d^2 f_n^{0*}}{dr^2} + \frac{1}{2}\frac{df_n^{0*}}{dr} - \frac{n^2}{r^2}f_n^{0*} = \beta F_n^{0*}. \quad (28)$$

Particular solutions of differential equations are sought by the method of variation of constants

$$f_n^0 = \beta\left[-\ln r\int_{R_0}^{r}\rho F_0^0(\rho)\,d\rho + \int_r^R \rho F_0^0(\rho)\ln\rho\, d\rho\right],$$
$$f_n^0 = -\frac{\beta}{2n}\left[r^n\int_r^R F_n^0(\rho)\rho^{1-n}d\rho + r^{-n}\int_{R_0}^{r}\rho F_n^0(\rho)\rho^{1+n}d\rho\right],$$
$$f_n^{0*} = -\frac{\beta}{2n}\left[r^n\int_r^R F_n^{0*}(\rho)\rho^{1-n}d\rho + r^{-n}\int_{R_0}^{r}\rho F_n^{0*}(\rho)\rho^{1+n}d\rho\right]. \quad (29)$$

After defining the thermoelastic potential of displacements in a zero approximation for the hub, by the known formulas [20], we calculate the approximate thermoelastic potential of the stresses $\overline{\sigma}_r^{(0)}, \overline{\sigma}_\theta^{(0)}, \overline{\tau}_{r\theta}^{(0)}$ and displacements $\overline{v}_r^{(0)}$, $\overline{v}_\theta^{(0)}$ in the hub

$$\begin{aligned}\overline{\sigma}_r^{(0)} &= -2G\left\{\frac{1}{r}\sum_{n=0}^{\infty}\left[\frac{\partial f_n^0}{\partial r}\cos n\theta + \frac{\partial f_n^{0*}}{\partial r}\sin n\theta\right]\right.\\ &\quad\left. + \frac{1}{r^2}\sum_{n=0}^{\infty}\left(-n^2\right)\left[f_n^0\cos n\theta + f_n^{0*}\sin n\theta\right]\right\},\\ \overline{\sigma}_\theta^{(0)} &= -2G\sum_{n=0}^{\infty}\left[\frac{\partial^2 f_n^0}{\partial r^2}\cos n\theta + \frac{\partial^2 f_n^{0*}}{\partial r^2}\sin n\theta\right],\\ \overline{\tau}_{r\theta}^{(0)} &= 2G\left\{\left(-\frac{1}{r^2}\right)\sum_{n=0}^{\infty} n\left[f_n^{0*}\cos n\theta - f_n^0\sin n\theta\right]\right.\\ &\quad\left. + \frac{1}{r}\sum_{n=0}^{\infty} n\left[\frac{\partial f_n^{0*}}{\partial r}\cos n\theta - \frac{\partial f_n^0}{\partial r}\sin n\theta\right]\right\},\\ \overline{v}_r^{(0)} &= \sum_{n=0}^{\infty}\left[\frac{\partial f_n^0}{\partial r}\cos n\theta + \frac{\partial f_n^{0*}}{\partial r}\sin n\theta\right],\\ \overline{v}_\theta^{(0)} &= \frac{1}{r}\sum_{n=0}^{\infty}\left[(-n)\left(f_n^0\sin n\theta - f_n^{0*}\cos n\theta\right)\right].\end{aligned} \tag{30}$$

Here G is shear modulus of the plunger.

The found stresses and displacements will not satisfy boundary conditions (16)-(18). Thus, it is necessary to determine for the hub the second stress-strain states $\overline{\overline{\sigma}}_r^{(0)}, \overline{\overline{\sigma}}_\theta^{(0)}, \overline{\overline{\tau}}_{r\theta}^{(0)}$, $\overline{\overline{v}}_r^{(0)}, \overline{\overline{v}}_\theta^{(0)}$ such that boundary conditions (16)-(18) are fulfilled. To obtain the second stress-strain state in the hub, we have the following conditions:

$$\text{for } r = R_0: \quad \overline{\overline{\sigma}}_r^{(0)} = -p^{(0)}(\theta) - \overline{\sigma}_r^{(0)}, \tag{31}$$

$$\overline{\overline{\tau}}_{r\theta}^{(0)} = -fp^{(0)}(\theta) - \overline{\tau}_{r\theta}^{(0)} \quad \text{on the contact area}, \tag{32}$$

$$\begin{aligned}\overline{\overline{\sigma}}_r^{(0)} &= -\overline{\sigma}_r^{(0)},\\ \overline{\overline{\tau}}_{r\theta}^{(0)} &= -\overline{\tau}_{r\theta}^{(0)}\\ &\quad \text{out of the contact area},\end{aligned} \tag{33}$$

$$\text{for } r = R: \quad \overline{\overline{v}}_r^{(0)} - i\overline{\overline{v}}_\theta^{(0)} = g^{(0)}(\theta) - \left(\overline{v}_r^{(0)} - i\overline{v}_\theta^{(0)}\right).$$

By means of the Kolosov-Muskhelishvili formulas [19], we can write the boundary conditions of problem (31)-(33) in the form of a boundary value problem for determining two complex functions $\Phi^{(0)}(z)$ and $\Psi^{(0)}(z)$ for the hub

$$\begin{aligned}&\Phi^{(0)}(\tau_0) + \overline{\Phi^{(0)}(\tau_0)} - e^{2i\theta}\left[\overline{\tau}_0\Phi^{(0)}(\tau_0) + \Psi^{(0)}(\tau_0)\right]\\ &\quad = X^{(0)}(\theta),\end{aligned} \tag{34}$$

$$\begin{aligned}&\Phi^{(0)}(\tau) - k_b\overline{\Phi^{(0)}(\tau)} - e^{2i\theta}\left[\overline{\tau}\Phi^{(0)}(\tau) + \Psi^{(0)}(\tau)\right]\\ &\quad = 2G\left\{g^{(0)'}(\theta) - \left(\overline{v}_r^{(0)} - i\overline{v}_\theta^{(0)}\right)'\right\}.\end{aligned} \tag{35}$$

Here

$$\begin{aligned}&\tau_0 = R_0\exp(i\theta);\\ &\tau = R\exp(i\theta);\\ &k_b = 3 - 4\mu,\\ &X^{(0)}(\theta)\\ &= \begin{cases} -(1 - if)\,p^{(0)}(\theta) - \left(\sigma_r^{(0)} - \sigma_{r\theta}^{(0)}\right) & \text{on the contact area}\\ -\left(\overline{\sigma}_r^{(0)} - i\overline{\tau}_{r\theta}^{(0)}\right) & \text{out of the contact area.}\end{cases}\end{aligned} \tag{36}$$

We look for complex potentials $\Phi^{(0)}(z)$ and $\Psi^{(0)}(z)$ in the form

$$\begin{aligned}\Phi^{(0)}(z) &= \sum_{k=-\infty}^{\infty} a_k^{(0)} z^k,\\ \Psi^{(0)}(z) &= \sum_{k=-\infty}^{\infty} b_k^{(0)} z^k.\end{aligned} \tag{37}$$

To solve boundary value problem (34) and (35) with regard to analytic functions $\Phi^{(0)}(z)$ and $\Psi^{(0)}(z)$, we use the method of power series. For that the right hand sides of conditions (34) and (35) expand into Fourier series

$$\begin{aligned}X^{(0)}(\theta) &= \sum_{k=-\infty}^{\infty} A_k^{(0)} e^{ik\theta},\\ g^{(0)'}(\theta) - \left(\overline{v}_r^{(0)} - i\overline{v}_\theta^{(0)}\right)' &= \sum_{k=-\infty}^{\infty} F_k^{(0)} e^{ik\theta}.\end{aligned} \tag{38}$$

Requiring that functions (37) should satisfy boundary conditions (34) and (35), after some transformations we get an infinite linear algebraic system with respect to unknown coefficients $a_k^{(0)}, b_k^{(0)}$ whose solution is written in the form

$$\begin{aligned}a_0^{(0)} &= \frac{A_0^{(0)}R_0^2 - F_0^{(0)}R^2}{2R_0^2 - (1 - k_b)R^2},\\ a_{-1}^{(0)} &= \frac{A_1^{(0)}R_0}{1 + k_b},\end{aligned} \tag{39}$$

$$\begin{aligned}b_{-2}^{(0)}R_0^{-2} &= 2a_0^{(0)} - A_0^{(0)},\\ b_{-1}^{(0)} &= -\frac{k_b\overline{A}_1^{(0)}R_0}{1 + k_b},\end{aligned} \tag{40}$$

$$\begin{aligned}&a_k^{(1)}\\ &= \frac{(1 + k)\left(R^2 - R_0^2\right)M_k - \overline{M}_k\left(R_0^{-2k+2} + k_bR^{-2k+2}\right)}{\left(1 - k^2\right)\left(R^2 - R_0^2\right)^2 - \left(R_0^{-2k+2} + k_bR^{-2k+2}\right)\left(R_0^{2k+2} + k_bR^{2k+2}\right)}\\ &\qquad (k = \pm 2, \pm 3, \ldots),\\ &M_k = F_k^{(0)}R^{-k+2} - A_k^{(0)}R_0^{-k+2},\end{aligned}$$

$$a_1^{(0)} = \frac{2A_1^{(0)}\left(R^2 - R_0^2\right)R_0}{\left(1-k_b\right)\left(R_0^4 + k_b R^4\right)} - \frac{\overline{M}_{-1}}{R_0^4 + k_b R^4},$$
$$b_{k+2}^{(0)} R_0^{k+2} = (1-k)\, a_k^{(0)} R_0^k + \overline{a}_k^{(0)} R_0^{-k} - A_k^{(0)}. \tag{41}$$

The right sides of these formulas (39)-(41) contain the expansion coefficients of the contact pressure $p^{(0)}(\theta)$ and the functions of displacement at the points of the external contour

$$g^{(0)'}(\theta) = a_0^{*(0)} + \sum_{k=1}^{\infty}\left(a_k^{*(0)}\cos k\theta + b_k^{*(0)}\sin k\theta\right). \tag{42}$$

Now by means of complex potentials $\Phi^{(0)}(z)$, $\Psi^{(0)}(z)$, the Kolosov-Muskhelishvili formulas, and integration of kinetic equation (6) of wear of the hub's material in a zero approximation, we find the displacement $v_1^{(0)}$ of the hub's contact surface. In the same way we find the solution of thermoelasticity problem for the plunger in a zero approximation. Using this solution and kinetic equation of wear of the plunger's material in a zero approximation, we find the displacement $v_2^{(0)}$ of the plunger's contact surface. We substitute the found values $v_1^{(0)}$ and $v_2^{(0)}$ in the main contact equation (5) in a zero approximation.

For algebraization of the main equation, we look for the unknown functions of contact pressure in a zero approximation in the form of expansions

$$p^{(0)}(\theta,t) = p_0^0(\theta) + t p_1^0(\theta) + \dots;$$
$$p_0^0(\theta) = A_0^0 + \sum_{k=1}^{\infty}\left(A_k^0\cos k\theta + B_k^0 \sin k\theta\right); \tag{43}$$

$$p_1^0(\theta) = A_0^1 + \sum_{k=1}^{\infty}\left(A_k^1\cos k\theta + B_k^1 \sin k\theta\right)\dots \tag{44}$$

Substituting relation (43) and (44) in the main contact equation in a zero approximation, we find functional equations for successive determination of $p_0^0(\theta)$, $p_1^0(\theta)$, etc. To construct an algebraic system and to find A_k, B_k we equate the coefficients under identical trigonometric functions in the left and right hand sides of the functional equation of the contact problem. We get an infinite algebraic system with respect to A_k^0 $(k = 0,1,2,\dots)$, B_k^0 $(k = 1,2,\dots)$ and A_k^1, B_k^1, etc. The quantities θ_1 and θ_2 are unknown. Because of this the system of equations is nonlinear. To determine the quantities θ_1 and θ_2 $(\theta_1 = \theta_1^0 + \varepsilon\theta_1^1 + \dots; \theta_2 = \theta_2^0 + \varepsilon\theta_2^1 + \dots)$ we have condition (13). We can represent these equations in the following form:

for zero approximation

$$p^{(0)}\left(\theta_1^0\right) = 0,$$
$$p^{(0)}\left(\theta_2^0\right) = 0,$$

for first approximation

$$p^{(1)}\left(\theta_1^1\right) = 0,$$
$$p^{(1)}\left(\theta_2^1\right) = 0. \tag{45}$$

The right hand sides of the infinite algebraic systems contain the unknown coefficients $a_k^{*(0)}$, $b_k^{*(0)}$ of expansions of functions $g^{(0)}(\theta, a_k^{*(0)}, b_k^{*(0)})$. Under the known function $g^{(0)}(\theta, a_k^{*(0)}, b_k^{*(0)})$ the obtained systems allow determining the contact pressure, temperature, stress-strain state, and wear of friction pair elements by numerical calculations. To construct the missing equations $g^{(0)}(\theta, a_k^{*(0)}, b_k^{*(0)})$ the minimization of the maximum value of the hub's circumferential stress on the friction surface is required:

$$\min_{\eta^0\in C}\ \max_{\theta\in[\theta_1,\theta_2]} \sigma_\theta^{(0)s}\left(\eta^0,\theta\right) \tag{46}$$

under restrictions associated with bearing capacity and heat-resistance of the friction pair

$$p_{\max} \le [p],$$
$$(pV)_{\max} \le [pV], \tag{47}$$

and also $\sigma_{\theta\max}^0 \le [\sigma]$, where $[p]$ is the admissible specific load on contact surface; $[pV]$ is the admissible heat-resistance of the pair; $[\sigma]$ is the admissible stress for material of the hub and is determined experimentally. C is the constraints set. The design parameters $\eta = (a_k^{*(0)}, b_k^{*(0)})$ are nonnegative.

For the function $\sigma_\theta^{(0)s}$ maximum value in the hub's inner surface is found:

$$\sigma_\theta^{(0)s} = \sigma_\theta^{(0)s}\left(\theta_{\max}\right) = \sigma_{\theta\max}^{(0)s}\left(a_k^{*(0)}, b_k^{*(0)}\right), \tag{48}$$

where value $\theta_{\max}$ is solution of equation

$$\frac{d\sigma_\theta^{(0)s}}{d\theta} = 0. \tag{49}$$

The maximum of the function $\sigma_\theta^{(0)s}$ is found by the usual methods of differential calculus.

As the circumferential stress $\sigma_\theta^{(0)s}$ (control quality index) and $\sigma_{\theta\max}^{(0)s}$ linearly depend on sought-for coefficients of functions $g^{(0)}(\theta, a_k^{*(0)}, b_k^{*(0)})$, the solution of the considered optimization problem in the zero approximation is reduced to a linear programming problem. Thus, finding such non-negative values of variables α_k^*, β_k^*, a_k, b_k and $a_k^{*(0)}$, $b_k^{*(0)}$ that satisfy the obtained system of equations (constraints) and also reduce to minimum the linear function $\sigma_{\theta\max}^{(0)s}$ is required.

After determining the sought-for quantities of zero approximation we can pass to construction of the solution of the problem in the first approximation. Based on the obtained solution for $r = R_0$ we determine the functions N and T_t.

By means of known formulas [20], by the found thermoelastic potential $F^{(1)}(r,\theta)$, we find the stresses $\overline{\sigma}_r^{(1)}$, $\overline{\sigma}_\theta^{(1)}$, $\overline{\tau}_{r\theta}^{(1)}$ and displacements $\overline{v}_r^{(1)}$, $\overline{v}_\theta^{(1)}$ for the hub. The found stress and displacement component do not satisfy the boundary conditions (22) and (23) in the first approximation. Consequently, it is necessary to find the second stress-strain state $\overline{\overline{\sigma}}_r^{(1)}$, $\overline{\overline{\sigma}}_\theta^{(1)}$, $\overline{\overline{\tau}}_{r\theta}^{(1)}$, $\overline{\overline{v}}_r^{(1)}$, $\overline{\overline{v}}_\theta^{(1)}$ for the hub in the first approximation. We can write the boundary conditions for finding $\overline{\overline{\sigma}}_r^{(1)}$, $\overline{\overline{\sigma}}_\theta^{(1)}$, $\overline{\overline{\tau}}_{r\theta}^{(1)}$, $\overline{\overline{v}}_r^{(1)}$,

Table 1: The values of Fourier coefficients for an optimal function of displacements of hub's external contour points that allows the minimization of the stress state (mm).

a_0^*	a_1^*	a_2^*	a_3^*	a_4^*	a_5^*	a_6^*	a_7^*	a_8^*	a_9^*
0.1123	0.0879	0.0745	0.0617	0.0528	0.0419	0.0403	0.0315	0.0284	0.0201
	b_1^*	b_2^*	b_3^*	b_4^*	b_5^*	b_6^*	b_7^*	b_8^*	b_9^*
	0.0892	0.0764	0.0583	0.0519	0.0423	0.0374	0.0285	0.0223	0.0154

$\overline{\overline{v}}_\theta^{(1)}$ in the form of a boundary value problem for finding complex potentials $\Phi^{(1)}(z)$ and $\Psi^{(1)}(z)$ that we look for in the form of power series as in a zero approximation with obvious changes.

The further course of solution is as in a zero approximation. The right hand side of the system for determining the quantities $a_k^{(1)}$, $b_k^{(1)}$ contains the sought-for $p^{(1)}(\theta)$ and the functions of displacements at the points of the external contour

$$g^{(1)'} = a_0^{*(1)} + \sum_{k=1}^{\infty} a_k^{*(1)} \cos k\theta + b_k^{*(1)} \sin k\theta. \quad (50)$$

As in the zero approximation, the obtained systems of equations permit expressing the coefficients $a_k^{(1)}$, $b_k^{(1)}$ by the coefficients $a_k^{*(1)}$, $b_k^{*(1)}$ of the function of displacements at the points of the hub's external contour in the first approximation.

A thermoelasticity problem for the plunger in the first approximation is solved in the same way.

In the first approximation the resolving equation of the contact problem is reduced to the algebraic form as in the zero approximation. For that we represent the sought-for functions of contact pressure in the form

$$\begin{aligned} p^{(1)}(\theta,t) &= p_0^1(\theta) + t p_1^1(\theta) + \dots; \\ p_0^1(\theta) &= A_{0,0}^1 + \sum_{k=1}^{\infty} \left(A_{k,0}^1 \cos k\theta + B_{k,0}^1 \sin k\theta \right); \end{aligned} \quad (51)$$

$$p_1^1(\theta) = A_{0,1}^1 + \sum_{k=1}^{\infty} \left(A_{k,1}^1 \cos k\theta + B_{k,1}^1 \sin k\theta \right) \dots \quad (52)$$

Substituting relation (51) and (52) in the main contact equation in a first approximation, we find functional equations for successive determination of $p_0^1(\theta)$, $p_1^1(\theta)$, etc. To construct an algebraic system and to find A_k^1, B_k^1 we equate the coefficients under identical trigonometric functions in the left and right hand sides of the functional equation of the contact problem. We get infinite linear algebraic systems with respect to $A_{0,0}^1$, $A_{k,0}^1$, $B_{k,0}^1$ and $A_{0,1}^1$, $A_{k,1}^1$, $B_{k,1}^1$ $(k = 1, 2, \dots)$, etc. The quantities θ_1^1, θ_2^1 are unknown and the system of equations becomes nonlinear. The obtained system of equations in the first approximation is not closed because of the unknown function $g^{(1)}(\theta)$. The obtained system of equations in zero and first approximations allows finding contact pressure, temperature distribution, stress-strain state, and wear of the hub and plunger of friction pair by numerical calculations under the given functions $g^{(0)}(\theta)$ and $g^{(1)}(\theta)$.

To construct the missing equations we require minimizing the maximum value of the stress σ_θ on the hub's friction surface

$$\sigma_{\theta\max} \longrightarrow \min. \quad (53)$$

with constraints involving load bearing capacity and heat-resistance of the pair and also $\sigma_{\theta\max} \le [\sigma]$, where σ is admissible stress for the hub's material and is determined experimentally.

For circumferential stress in the hub, for $r = \rho(\theta)$, we have

$$\sigma_\theta = \sigma_{\theta|r=R_0}^{(0)} + \varepsilon \left[H(\theta) \frac{\partial \sigma^{(0)}}{\partial r} + \sigma_\theta^{(1)} \right]_{|r=R_0} \quad (54)$$

For the function $\sigma_\theta(\theta)$ we find its maximum value. The coefficients a_k^*, b_k^* of the sought-for function $g(\theta)$ should be managed so that minimization $\sigma_{\theta\max}$ (minimax criterion) is provided. As the stress $\sigma_{\theta\max}$ (control quality index) linearly depends on sought-for coefficients of the function $g(\theta)$, then the stated minimization problem is reduced to a linear programming problem.

4. Analysis of Simulation Results

To realize the obtained mathematical model, numerical methods for solving linear programming problems can be used. In the minimization problem, the simplex algorithm is found to be the most effective method. In the considered problem there are many free parameters. These are different thermophysical and mechanical characteristics of materials, quality parameters of the surface of the hub's internal contour and plunger's external contour, geometrical sizes of the plunger and hub, and movement rate of the plunger. The numerical calculation is performed by the method of successive approximations [21] and simplex algorithm.

As a numerical example of the application of the mathematical model obtained in Section 3 the minimization was carried out for friction pair of oil-well sucker-rod pump. The calculations were performed for the hub of the pump.

The results of calculations of the expansion coefficients of displacements functions $g(\theta)$ at initial time $t = 0$ are cited in Table 1 for a rough internal contour described by a stationary random function with a zero mean value and known variance. As an example we accepted the following: $2R_0 = 57$ mm; $2R = 73$ mm; $2R_0' = 56.7$ mm; $E = 1.8 \cdot 10^5$ MPa; $E_1 = 2.1 \cdot 10^5$ MPa; $f = 0.2$; $\mu = 0.25$; $\mu = 0.3$; $K^{(1)} = 2 \cdot 10^{-8}$; $K^{(2)} = 2.5 \cdot 10^{-9}$; $\Delta_1 = R_0 - R_0' = 0.15$ mm; $V = 0.4$ m/s.

After solving of the algebraic systems the contact pressure was calculated as a function of polar angle for different values

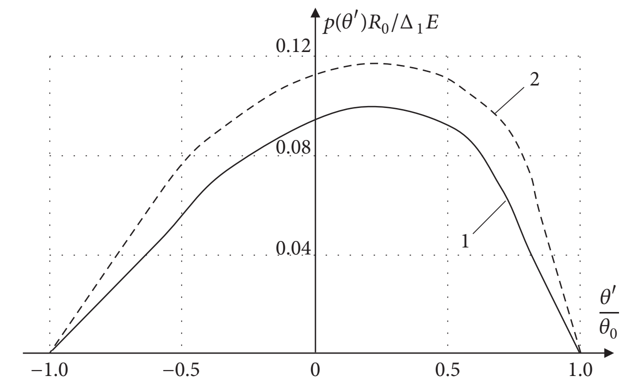

Figure 4: Dependence of the contact pressure on the values of polar angle for $V = 0.5$ m/s.

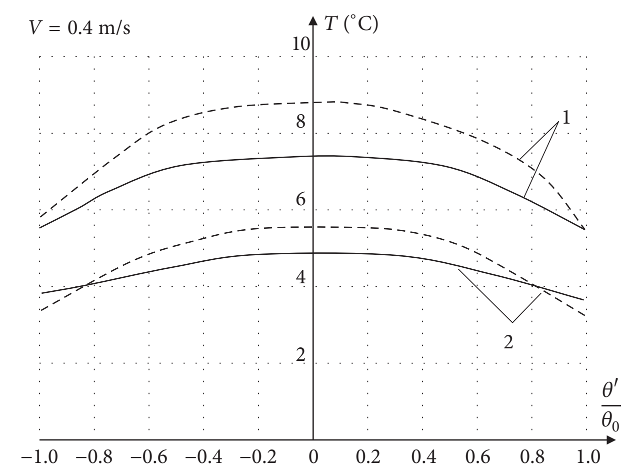

Figure 5: Temperature change in the hub for one stroke of the plunger for velocity of plunger motion $V = 0.4$ m/s.

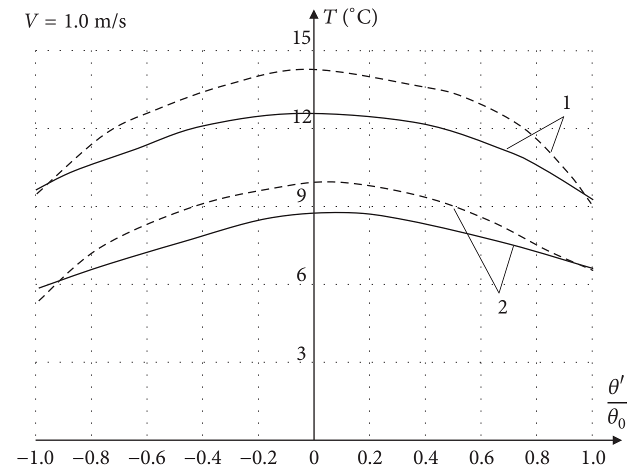

Figure 6: Temperature change in the hub for one stroke of the plunger for velocity of plunger motion $V = 1.0$ m/s.

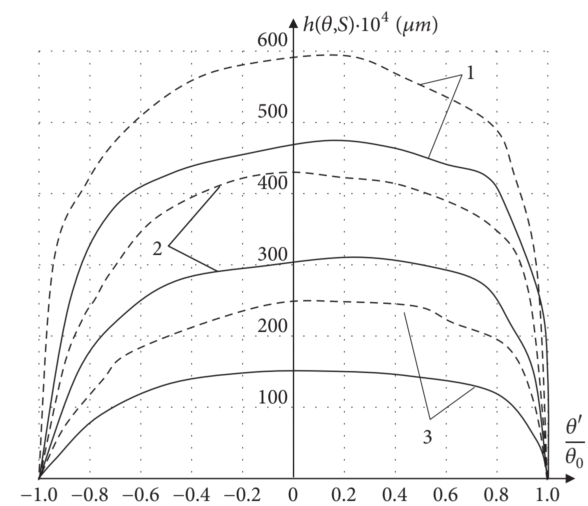

Figure 7: Abrasive wear of the hub surface for one stroke of the plunger.

of the free parameters. The contact pressure for the borehole sucker-rod oil pumps depending on the values of polar angle $\theta' = \theta - \theta_+$ ($\theta_+ = (1/2)(\theta_2 + \theta_1)$, $\theta_0 = (1/2)(\theta_2 - \theta_1)$) are represented in Figure 4 for the velocity of plunger motion $V = 0.5$ m/s. Here curve 1 is corresponding to the optimal solution and curve 2 is corresponding to case when the function of the displacements of points of the hub external contour is equal to zero. The large values of contact pressure, as a rule, are in the middle of the contact surface depending on the friction coefficient and the contact angle. The existence of friction forces in the contact area results in displacement of the diagram of contact pressure distribution to direction opposite to the action of the moment.

With known contact pressure it is possible to calculate the temperature distribution and abrasive wear of the friction pair details. The temperature change for different depths on the hub thickness was studied. The temperature change in the hub for one plunger stroke for different velocity of plunger motion is represented in Figures 5-6. Here solid and dashed lines, respectively, correspond to the optimal solution and to the case when the function of displacements of the points of the external contour of the hub is equal to zero. Curve 1 is corresponding to the surface temperature of the hub and curve 2 is corresponding to temperature on depth of 2 mm.

The results of calculations of the abrasive wear of the hub surface for one plunger stroke are shown in Figure 7 for different velocity of the plunger motion. Curves 1-3 correspond to the velocity $V = 0.2$, 0.5, and 1.0 m/s, respectively.

The calculation results show that at low values of the contact pressure the wear of the hub along the contact zone has uneven character. With increasing of the contact pressure the hub wear along the length of the contact zone tends to leveling and mainly depends on wear (friction) path. Because the friction coefficient of the pair has significant effect on the wear of the hub the dependence of maximum wear on the friction coefficient was calculated.

In Figure 8 the dependence of maximum wear is shown for friction path corresponding to ten work hours of the pump. Curve 1 corresponds to the optimal solution and curve

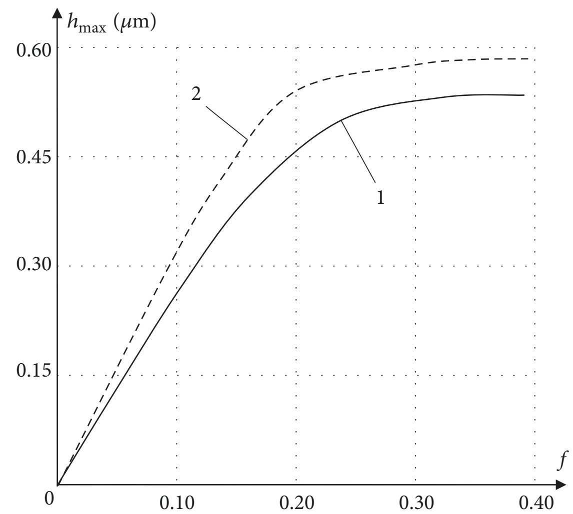

FIGURE 8: Dependence of maximum wear for friction path corresponding to ten work hours of the pump.

2 corresponds to case when the function of displacements of the points of the hub external contour is equal to zero.

Due to the wear the surface microgeometry of the plunger and hub will vary. The relations for the radial wear obtained in the study allow determination of the variation of friction surface for a given moment of time.

5. Conclusions

It is shown that by deriving the function of displacements of the external contour points of a hub of friction pair, one can control (minimize) the stress state distribution in the hub. This will increase serviceability of the pair "a hub-a plunger". The suggested method of minimization of stress state of a friction pair may be extended to other constructions of friction units.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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