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Research Article

Existence of Positive Ground State Solution for the Nonlinear Schrödinger-Poisson System with Potentials

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In the present paper we study the existence of positive ground state solutions for the nonautonomous Schrödinger-Poisson system with competing potentials. Under some assumptions for the potentials we prove the existence of positive ground state solution.

1. Introduction

In this paper we are concerned with the existence of solution of the nonautonomous Schrödinger-Poisson system

$$\begin{aligned} -\Delta u + A(x)u + B(x)\phi(x)u &= Q(x)|u|^{p-1}u, \\ & x \in \mathbb{R}^3, \\ -\Delta\phi &= B(x)u^2, \\ & u \in H^1(\mathbb{R}^3), \end{aligned} \tag{1}$$

where $p \in (3,5)$, $A(x)$, $B(x)$, and $Q(x)$ are positive functions such that $\lim_{|x|\longrightarrow\infty} A(x) = a_\infty > 0$, $\lim_{|x|\longrightarrow\infty} B(x) = b_\infty > 0$, and $\lim_{|x|\longrightarrow\infty} Q(x) = q_\infty > 0$. Here $B : \mathbb{R}^3 \longrightarrow \mathbb{R}$ denotes the nonnegative measurable function which represents a nonconstant charge corrector to the density u^2. A and Q are called the potentials of system (1). In the context of the so-called Density Functional Theory, variants of system (1) appear as mean field approximations of quantum many-body systems; see [1–3].

This kind of system also arises in many fields of physics. Indeed, one considers the following system:

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= -\frac{\hbar^2}{2m}\Delta\psi + k(x)\psi + B(x)\phi\psi \\ &\quad - Q(x)|\psi|^{p-2}\psi, \quad x \in \mathbb{R}^3,\ t \in \mathbb{R}, \\ -\frac{\hbar^2}{2m}\Delta\phi(x) &= B(x)|\psi|^2, \quad u \in H^1(\mathbb{R}^3), \end{aligned} \tag{2}$$

where i is the imaginary unit, Δ is the Laplacian operator, and $\hbar$ is the Planck constant. A standing wave is a solution of (2) of the form $\psi(x,t) = u(x)e^{i\omega t}$, $\omega > 0$ and $t \in \mathbb{R}$. It is clear that $\psi(x,t)$ solves (2) if and only if $u(x)$ solves the so-called stationary system

$$\begin{aligned} -\frac{\hbar^2}{2m}\Delta u + A(x)u + B(x)\phi(x)u &= Q(x)|u|^{p-2}u, \\ & x \in \mathbb{R}^3, \\ -\frac{\hbar^2}{2m}\Delta\phi &= B(x)u^2, \\ & x \in \mathbb{R}^3, \end{aligned} \tag{3}$$

where $A(x) = k(x) + \omega\hbar$. Here we consider that the case $\hbar$ is constant. Without loss of generality we assume that $\hbar^2/2m = 1$, then system (3) becomes (1). System (1) is also modeled in Abelian Gauge Theories; for instance, see [4–6] and reference therein. In fact, in order to describe the interaction of a nonlinear Schrödinger field with an electromagnetic field $\mathbb{E}-\mathbb{H}$, the gauge potentials $\mathbb{A}-\mathbb{F}$

$$\begin{aligned} \mathbb{A} &: \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}^3, \\ \mathbb{F} &: \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R} \end{aligned} \tag{4}$$

are related to $\mathbb{E}-\mathbb{H}$ by the Maxwell equations

$$\mathbb{E} = -\left(\nabla \mathbb{F} + \frac{\partial \mathbb{A}}{\partial t}\right), \quad \mathbb{H} = \nabla \times \mathbb{A}. \tag{5}$$

If we are interested in finding standing waves (solutions of a field equation whose energy travels as a localized packet preserving this localization in time) and we consider the electrostatic case (when $\mathbb{A} = 0$), the Schrödinger field is described by a real function $u : \mathbb{R}^3 \longrightarrow \mathbb{R}$, which represents the matter (see [4, 7]). In this situation we need to consider the stationary states of Schrödinger-Maxwell system (1). The system is also used in quantum electrodynamics, semiconductor theory, nonlinear optics, and plasma physics; for more information on this direction, one can refer to [7–9] and the references therein.

In recent years many papers focus on the existence, multiplicity, and concentration of positive solutions of (3) for the semiclassical case ($\hbar > 0$ sufficiently small). In this framework one is interested not only in existence of solutions but also in their asymptotic behavior as $\hbar \longrightarrow 0$. Typically, the solution tends to concentrate on critical points of $A(B$ or $Q)$. These solutions are called spikes. For more information on this direction. one can refer to [10–14] and the references therein.

In the present paper we are interested in studying the case when $\hbar$ is constant. In [15], the authors studied the existence and nonexistence of solutions of (1) when $A = B = Q = 1$. The existence of the multiple solutions of (1) has been found in the paper [16] in a radial setting. In the paper [17], the author considers that the case $B = 1, V, K$ is radial and satisfies

$$\frac{A_0}{1+|x|^{\alpha}} \le A(x) \le A_1, \quad 0 < Q(x) \le \frac{Q_0}{1+|x|^{\beta}}, \tag{6}$$

where $\alpha \in (0,2], \beta, A_0, A_1$, and Q_0 are positive constants. The author proved the existence of nontrivial positive classical mountain-pass solution of (1). Moreover, some generalizations of the last cases, with $Q(x)|u|^{p-1}u$ replaced by a more generic function $f(x,u)$, were considered in [18, 19]. It is well known that, dealing with system (1), one has to face different kinds of difficulties, which are related to the potential and to the unboundedness of the space $\mathbb{R}^3$. Many early studies were devoted to the autonomous case and to the case in which the coefficients are supposed to be radial (see [16, 20, 21]), just to overcome the lack of compactness-taking advantage of the compact embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ $(\forall q \in (2,6))$. More recently, many contributions to (1) have also been given looking at cases in which no symmetry assumptions are given on the coefficients appearing in (1); one can refer to the papers [22–24]. Furthermore, for more results on the existence of positive solutions, ground and bound states, one can see [18, 19, 21, 25–33] and references therein. Nearly, the paper [34] proves the existence of bound state solution of (1) under some decay condition on A, B, and Q. Precisely, assume that $A(x) = V_\infty + W(x)$ and $\int_{\mathbb{R}^3} W(x)(|x|e^{2\sqrt{V_\infty}|x|}) < \infty$, $Q(x) = Q_\infty - \beta(x)$ and $\int_{\mathbb{R}^3} \beta(x)(|x|e^{2\sqrt{V_\infty}|x|}) < \infty$, and

$$0 \le B(x) \le ce^{-\sigma|x|}, \quad \text{for } |x| \text{ large enough}, \tag{7}$$

where V_∞ and Q_∞ are positive constants. From this assumption one can easily deduce that $\lim_{|x|\longrightarrow\infty} B(x) = 0$. In order to get the compactness, the paper [34] studies the case when the limit equation is Scalar Schrödinger equation, i.e.,

$$-\Delta u + a_\infty u = q_\infty |u|^{p-1} u, \quad u \in H^1(\mathbb{R}^3). \tag{8}$$

According to [35, 36], (8) has unique positive solution, and the energy level of any sign-changing solution is strictly greater than $2c_\infty$, where c_∞ is the least energy level for the positive solution of (8). This information is very important for proving the existence of positive bound solution (high energy) of (1.1) in [34].

Motivated by [34, 37], in the present paper we shall study the case when the limit equation is the Schrödinger-Poisson system

$$-\Delta u + a_\infty u + b_\infty \phi(x) u = q_\infty |u|^{p-1} u, \quad \text{in } \mathbb{R}^3, \quad -\Delta\phi = b_\infty u^2, \quad u \in H^1(\mathbb{R}^3), \tag{9}$$

and, under some conditions for the A, B, and Q, we prove the existence of positive ground state solution of (1).

In order to state our main results, we shall give some assumptions. For $\lambda \in \mathbb{R}^+$, we define $A(x) = a_\infty + \lambda a(x)$, $B(x) = b_\infty - b(x)$, and $Q(x) = q_\infty + q(x)$. Throughout the paper we need the following conditions.

(A_1) $a_\infty > 1$, $a \in L^{3/2}(\mathbb{R}^3)$, $0 < a < 1$, $a \ne 0$, $\lim_{|x|\longrightarrow\infty} a(x) = 0$.

(A_2) $b_\infty > 1$, $b \in L^{3/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $0 \le b \le 1$, $b \ne 0$, $\lim_{|x|\longrightarrow\infty} b(x) = 0$.

(A_3) $q_\infty > 1$, $q \in L^\infty(\mathbb{R}^3)$, $0 < q < 1$, $q \ne 0$, $\lim_{|x|\longrightarrow\infty} q(x) = 0$.

Clearly, system (1) becomes

$$\begin{aligned} &-\Delta u + (a_\infty + \lambda a(x)) u + (b_\infty - b(x)) \phi(x) u \\ &\quad = (q_\infty + q(x)) |u|^{p-1} u, \quad \text{in } \mathbb{R}^3, \\ &-\Delta\phi = (b_\infty - b(x)) u^2, \quad u \in H^1(\mathbb{R}^3), \end{aligned} \tag{10}$$

where $\lambda \in \mathbb{R}^+$ and $p \in (3,5)$. In the following we shall focus on system (10). According to [11, 12], we know that (9) has a positive radial symmetric solution w. Moreover, w is decreasing when the radial coordinate increases. Precisely, there exists a constant $c^* > 0$ such that

$$\lim_{|x|\longrightarrow+\infty} |w(x)| e^{c^*|x|} = \text{Constant}. \tag{11}$$

Then we have the following main results.

Theorem 1. *Assume that $p \in (3,5)$ and (A_1)-(A_3) hold. Moreover, if one of the following conditions holds*

(1) $\lim_{|x|\to\infty} q(x)e^{(2\tau/(1+\tau))c^*|x|} = \infty$ *and* $\lim_{|x|\to\infty} q(x)e^{(2\tau/(1+2\tau))c^*|x|} \le c_0$, $\lim_{|x|\to\infty} a(x)|x|^{1+\tau}e^{(2\tau/(1-\tau))c^*|x|} \le c_1$, $\lim_{|x|\to\infty} b(x)e^{(2\tau(1-\tau)/(1+\tau)(1+2\tau))c^*|x|} \ge c_2$, *where* $\tau \in (0,1)$, c_0, c_1, *and* c_2 *are positive constants.*

(2) $\lim_{|x|\to\infty} q(x)e^{(2\tau/(1+\tau))c^*|x|} = \infty$ *and* $\lim_{|x|\to\infty} a(x)e^{((p-1)\tau/(1-\tau))c^*|x|} \le c_3$, *where* $\tau \in (0,1)$ *and* c_3 *is positive constant.*

Then system (10) has a positive ground state solution for each $\lambda \in \mathbb{R}^+$.

Remark 2. (i) From the assumptions on B, we know that $\lim_{|x|\to\infty} B(x) = b_\infty$. Hence we need to consider the limit equation (9) to recover the compactness. This is quite different from the recent work of [34]. The main novelty here is that we shall compare the decay rate of A, B, and Q to recover the compactness and prove the existence of positive ground state solution.

(ii) Note that in condition (1) of Theorem 1, we know that $a(x)$ is decaying faster than $b(x)$. In condition (2) of Theorem 1, we only need the decay condition for $a(x)$ (whatever the decay speed of $b(x)$ to 0 as $|x| \to \infty$) to prove the existence of positive solution. This is different phenomenon compared to [34].

2. Preliminary Results

We define the following notation:

(i) $\|\cdot\|$ is the norm of $H^1(\mathbb{R}^3)$ defined by $\|u\|^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + a_\infty u^2)$;

(ii) $\|\cdot\|_{D^{1,2}}$ is the norm of $D^{1,2}(\mathbb{R}^3)$ defined by $\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3}|\nabla u|^2$;

(iii) $(\cdot,\cdot)$ is the scalar product of $H^1(\mathbb{R}^3)$ defined by $(u,v) = \int_{\mathbb{R}^3}[\nabla u\nabla v + uv]$;

(iv) let C, c^*, c, c_i $(i = 1,\cdots)$ denote different positive constants.

In this part we mainly give some basic knowledge which will be used later. We first show that the second equation of (10) can be solved. We consider, for all $u \in H^1(\mathbb{R}^3)$, the linear functional J_u defined in $D^{1,2}(\mathbb{R}^3)$ by

$$J_u(v) = \int_{\mathbb{R}^3}(b_\infty - b(x))u^2 v. \tag{12}$$

We infer from condition (A_2) and Hölder inequality that

$$|J_u(v)| \le C|u|^2_{12/5}\|v\|_{D^{1,2}}. \tag{13}$$

By the Lax-Milgram theorem, we know that there exists unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3}\nabla\phi_u\nabla v = \int_{\mathbb{R}^3}(b_\infty - b(x))u^2 v \quad \forall v \in D^{1,2}(\mathbb{R}^3). \tag{14}$$

So, ϕ_u is a weak solution of $-\Delta\phi = u^2$ and the following formula holds

$$\begin{aligned}\phi_u(x) &= \int_{\mathbb{R}^3}\frac{(b_\infty - b(y))u^2(y)}{4\pi|x-y|}dy \\ &= \frac{1}{4\pi|x|} * \left((b_\infty - b)u^2\right).\end{aligned} \tag{15}$$

Moreover, $\phi_u > 0$ when $u \ne 0$.

We recall the following classical Hardy-Littlewood-Sobolev inequality (see [38, Theorem 4.3]). Assume that $f \in L^p(\mathbb{R}^3)$ and $g \in L^q(\mathbb{R}^3)$. Then one has

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f(x)g(y)}{|x-y|^t}dx\,dy \le c(p,q,t)|f|_p|g|_q, \tag{16}$$

where $1 < p,q < \infty$, $0 < t < 3$, and $1/p + 1/q + t/3 = 2$. By (16) we know that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|x-y|}dx\,dy \le c|u|^4_{12/5} \le c\|u\|^4. \tag{17}$$

It is well known that solutions of (10) correspond to critical points of the energy functional

$$\begin{aligned}\Phi_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + (a_\infty + \lambda a(x))u^2\right) \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3}(b_\infty - b(x))\phi_u u^2 \\ &\quad - \frac{1}{p+1}\int_{\mathbb{R}^3}(q_\infty + q(x))|u|^{p+1}.\end{aligned} \tag{18}$$

From (17), we know that Φ_λ is well defined and that

$$\begin{aligned}\Phi'_\lambda(u)[v] &= \int_{\mathbb{R}^3}(\nabla u\nabla v + (a_\infty + \lambda a(x))uv) \\ &\quad + \int_{\mathbb{R}^3}(b_\infty - b(x))\phi_u uv \\ &\quad - \int_{\mathbb{R}^3}(q_\infty + q(x))|u|^{p-2}uv.\end{aligned} \tag{19}$$

We define the operator $I : H^1(\mathbb{R}^3) \to D^{1,2}(\mathbb{R}^3)$ as

$$I[u] = \phi_u. \tag{20}$$

We infer from [34, Proposition 2.1-2.2] that I has the following properties.

Lemma 3. *(1) I is continuous.*

(2) I maps bounded sets into bounded sets.

(3) $I[tu] = t^2 I[u]$ *for all* $t \in \mathbb{R}$.

(4) If $u_n \rightharpoonup u \in H^1(\mathbb{R}^3)$, *then we have* $\int_{\mathbb{R}^3} b(x)\psi_{u_n}(x)u_n^2 \to \int_{\mathbb{R}^3} b(x)\psi_u(x)u^2$ *and* $\int_{\mathbb{R}^3} b(x)\psi_{u_n}(x)u_n\phi \to \int_{\mathbb{R}^3} b(x)\psi_u(x)u\phi$ *for each* $\phi \in H^1(\mathbb{R}^3)$, *as* $n \to \infty$, *where* $\psi_u(x) = \int_{\mathbb{R}^3}(b(y)u^2(y)/4\pi|x-y|)dy$.

(5) If $u_n \rightharpoonup u \in H^1(\mathbb{R}^3)$, *then we have* $I(u_n) \to I(u)$ *in* $D^{1,2}(\mathbb{R}^3)$ *and* $\int_{\mathbb{R}^3} b_\infty\overline{\phi}_{u_n}(x)u_n\phi \to \int_{\mathbb{R}^3} b_\infty\overline{\phi}_u(x)u\phi$ *for each* $\phi \in H^1(\mathbb{R}^3)$, *as* $n \to \infty$, *where* $\overline{\phi}_u(x) = \int_{\mathbb{R}^3}(b_\infty u^2(y)/4\pi|x-y|)dy$.

Proof. The conclusions (1)-(4) can be proved as in [34, Proposition 2.1-2.2]. Hence we only focus on the proof of (5). Since, by definition of I, for all $u \in D^{1,2}(\mathbb{R}^3)$ we have

$$\|I(u)\|_{D^{1,2}(\mathbb{R}^3)} = \|J_u\|_{\mathscr{L}(D^{1,2}(\mathbb{R}^3),\mathbb{R})} \tag{21}$$

then, in order to prove $I(u_n) \longrightarrow I(u)$ in $D^{1,2}(\mathbb{R}^3)$, it suffices to prove that

$$\|J_{u_n} - J_u\|_{\mathscr{L}(D^{1,2}(\mathbb{R}^3),\mathbb{R})} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{22}$$

Let $\varepsilon > 0$ be fixed arbitrarily. Then there exists a positive number $R_\varepsilon > 0$ so large that $|v|_{L^6(B^c_{R_\varepsilon})} < \varepsilon$ for $v \in D^{1,2}(\mathbb{R}^3)$, where $B_{R_\varepsilon} = \{x \in \mathbb{R}^3 : |x| \le R_\varepsilon\}$ and $B^c_{R_\varepsilon} = \mathbb{R}^3 \setminus B_{R_\varepsilon}$. Hence we deduce that

$$\begin{aligned}
\left|J_{u_n}(v) - J_u(v)\right| &= \int_{\mathbb{R}^3} (b_\infty - b(x))\left(u_n^2 - u^2\right) v\,dx \\
&= \int_{B_{R_\varepsilon}} (b_\infty - b(x))\left(u_n^2 - u^2\right) v\,dx \\
&\quad + \int_{B^c_{R_\varepsilon}} (b_\infty - b(x))\left(u_n^2 - u^2\right) v\,dx \\
&\le c|v|_6 |u_n - u|_{L^{12/5}(B_{R_\varepsilon})} |u_n + u|_{L^{12/5}(R_\varepsilon)} \\
&\quad + c|v|_{L^6(B_{R_\varepsilon})^c} \left|u_n^2 - u^2\right|_{6/5} \\
&\le \left(c\varepsilon + c|u_n - u|_{L^{12/5}(B_{R_\varepsilon})} |u_n + u|_{L^{12/5}(R_\varepsilon)}\right) \|v\|_{D^{1,2}}.
\end{aligned} \tag{23}$$

Since $u_n \longrightarrow u$ in $L^p_{loc}(\mathbb{R}^3)$ $(p \in (2,6))$, we know that (22) holds.

Next we prove the later conclusion. For any fixed $R > 0$ large, we infer that for each $x \in \mathbb{R}$

$$\begin{aligned}
\left|\overline{\phi}_{u_n}(x) - \overline{\phi}_u(x)\right| &= \int_{\mathbb{R}^3} b_\infty \frac{u_n^2(y) - u^2(y)}{4\pi|x-y|} dy \\
&= \int_{|x-y|\le 1} b_\infty \frac{u_n^2(y) - u^2(y)}{4\pi|x-y|} dy \\
&\quad + \int_{1\le|x-y|\le R} b_\infty \frac{u_n^2(y) - u^2(y)}{4\pi|x-y|} dy \\
&\quad + \int_{|x-y|\ge R} b_\infty \frac{u_n^2(y) - u^2(y)}{4\pi|x-y|} dy \\
&\le c|u_n - u|_{L^{4/3}(B_1(x))} |u_n + u|_{L^4(B_1(x))} \\
&\quad \cdot \left(\int_{|x-y|\le 1} \frac{1}{|x-y|^2} dy\right) + c|u_n - u|_{L^{8/3}(B_R(x)\setminus B_1(x))} \\
&\quad \cdot |u_n + u|_{L^{8/3}(B_R(x)\setminus B_1(x))} \left(\int_{1\le|x-y|\le R} \frac{1}{|x-y|^4} dy\right) \\
&\quad + c|u_n - u|_{L^{8/3}(B^c_R(x))} |u_n + u|_{L^{8/3}(B^c_R(x))} \\
&\quad \cdot \left(\int_{|x-y|\ge R} \frac{1}{|x-y|^4} dy\right)
\end{aligned} \tag{24}$$

and, hence, we deduce that $\overline{\phi}_{u_n}(x) \longrightarrow \overline{\phi}_u(x)$ a.e., in $x \in \mathbb{R}^3$ and $|\overline{\phi}_{u_n} - \overline{\phi}_u|_{L^\infty(B_{R_0})} \longrightarrow 0$ for each $R_0 > 0$, as $n \longrightarrow \infty$. A direct computation shows that for $R > 0$ large

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} b_\infty \overline{\phi}_{u_n}(x) u_n \phi - \int_{\mathbb{R}^3} b_\infty \overline{\phi}_u(x) u\phi\right| \\
&\quad = \left|\int_{\mathbb{R}^3} b_\infty \left(\overline{\phi}_{u_n} - \overline{\phi}_u\right) u_n\phi + b_\infty \overline{\phi}_u(x)(u_n - u)\phi\right| \\
&\quad \le c\left|\int_{B_R} \left(\overline{\phi}_{u_n} - \overline{\phi}_u\right) u_n\phi\right| + c\left|\int_{B^c_R} \left(\overline{\phi}_{u_n} - \overline{\phi}_u\right) u_n\phi\right| \\
&\qquad + c\left|\int_{B_R} \overline{\phi}_u(x)(u_n - u)\phi\right| \\
&\qquad + c\left|\int_{B^c_R} \overline{\phi}_u(x)(u_n - u)\phi\right| \\
&\quad \le c\left|\overline{\phi}_{u_n} - \overline{\phi}\right|_{L^\infty(B_R)} |\phi|_2 |u_n|_2 \\
&\qquad + c\left|\overline{\phi}_{u_n} - \overline{\phi}\right|_6 |\phi|_{L^{(12/5)}(B^c_R)} |u_n|_{12/5} \\
&\qquad + c\left|\overline{\phi}_u\right|_6 |\phi|_{12/5} |u_n - u|_{L^{12/5}(B_R)} \\
&\qquad + c\left|\overline{\phi}_u\right|_{L^6(B^c_R)} |\phi|_{L^{12/5}(B^c_R)} |u_n - u|_{12/5}
\end{aligned} \tag{25}$$

which proves $\int_{\mathbb{R}^3} \overline{\phi}_{u_n}(x) u_n\phi \longrightarrow \int_{\mathbb{R}^3} \overline{\phi}_u(x) u\phi$ for each $\phi \in H^1(\mathbb{R}^3)$, as $n \longrightarrow \infty$. ◻

It is very easy to verify that, whatever $\lambda \in \mathbb{R}$ is, the function Φ_λ is bounded either from above or from below. Hence, it is convenient to consider Φ_λ restricted to a natural constraint, the Nehari manifold. We set

$$\mathscr{N}_\lambda := \left\{u \in H^1\left(\mathbb{R}^3\right) \setminus \{0\} : \Phi'_\lambda(u)[u] = 0\right\}. \tag{26}$$

Next lemma contains the statement of the main properties of $\mathscr{N}_\lambda$.

Lemma 4. *Assume that* (A_1)-(A_3) *hold. Then the following conclusions hold.*

(a) $\mathscr{N}_\lambda$ *is a* C^1 *regular manifold diffeomorphic to the sphere of* $H^1(\mathbb{R}^3)$.

(b) Φ_λ *is bounded from below on* $\mathscr{N}_\lambda$ *by a positive constant.*

(c) u *is a free critical point of* Φ_λ *if and only if* u *is a critical point of* Φ_λ *constrained on* $\mathscr{N}_\lambda$.

Proof. (a) Let $u \in H^1(\mathbb{R}^3)\setminus\{0\}$ be such that $\|u\| = 1$. We claim that there exists a unique $t \in (0,\infty)$ such that $tu \in \mathscr{N}_\lambda$. Let $f(t) = \Phi_\lambda(tu)$ for $t > 0$. It is easy to verify, by (A_1)-(A_3), that $f(0) = 0$, $f(t) > 0$ for $t > 0$ small and $f(t) < 0$ for

t large. Therefore, $\max_{t>0} f(t)$ is achieved at a $t = t(u) > 0$ so that $f(tu) = 0$ and $t(u)u \in \mathcal{N}_\lambda$. Assume that there exist $t_1 > t_2 > 0$ such that $t_1 u, t_2 u \in \mathcal{N}_\lambda$. Then we see

$$\left(\frac{\|w\|^2 + \int_{\mathbb{R}^3} \lambda a(x)) u^2}{t_1^2} - \frac{\|w\|^2 + \int_{\mathbb{R}^3} \lambda a(x)) u^2}{t_2^2} \right) = \left(t_1^{p-3} - t_2^{p-3} \right) \int_{\mathbb{R}^3} (q_\infty + q(x)) |u|^{p+1}. \tag{27}$$

This is a contradiction. So, we prove that $f(t)$ admits a unique positive solution $t(u) > 0$ and $t(u)u \in \mathcal{N}_\lambda$. We infer from (16) and (A_1)-(A_3) that

$$\begin{aligned} c_1 |u|_{p+1}^2 &\le \left(\|u\|^2 + \int_{\mathbb{R}^3} \lambda a(x) u^2 + \int_{\mathbb{R}^3} (b_\infty - b(x)) \phi_u u^2 \right) \\ &= \int_{\mathbb{R}^3} (q_\infty + q(x)) |u|^{p+1} \le c_1 |u|_{p+1}^2. \end{aligned} \tag{28}$$

This implies that

$$\|u\| \ge c > 0. \tag{29}$$

Let $G_\lambda(u) := \Phi_\lambda'(u)[u]$. Then $G_\lambda \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ by the regularity of Φ_λ. Moreover, we infer from (29) that

$$\begin{aligned} G_\lambda'(u)[u] &= -(p-1) \int_{\mathbb{R}^3} \left(|\nabla u|^2 + (a_\infty + \lambda a(x)) u^2 \right) \\ &\quad - (p-3) \int_{\mathbb{R}^3} (b_\infty - b(x)) \phi_u u^2 \le -(p-3) C \\ &< 0. \end{aligned} \tag{30}$$

(b) For all $u \in \mathcal{N}_\lambda$,

$$\begin{aligned} \Phi_\lambda(u) &= \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^3} \left(|\nabla u|^2 + (a_\infty + \lambda a(x)) u^2 \right) \\ &\quad + \left(\frac{1}{4} - \frac{1}{p+1} \right) \int_{\mathbb{R}^3} (b_\infty - b(x)) \phi_u u^2 \\ &\ge \left(\frac{1}{2} - \frac{1}{p+1} \right) \|u\|^2 \ge c > 0. \end{aligned} \tag{31}$$

Here we use the fact that $3 < p < 5$.

(c) If $u \neq 0$ is a critical point of Φ_λ, then we have $\Phi_\lambda'(u) = 0$, $u \in \mathcal{N}_\lambda$. On the other hand, if u is a critical point of $\Phi_\lambda(u)$ constrained on $\mathcal{N}_\lambda$, then there exists $k \in \mathbb{R}$ such that

$$0 = \Phi_\lambda'(u)[u] = G_\lambda(u) = k G_\lambda'(u)[u]. \tag{32}$$

One infers from (30) that $k = 0$. □

Next we consider the limit functional $\Phi_\infty : H^1(\mathbb{R}^3) \longrightarrow \mathbb{R}$, defined as

$$\begin{aligned} \Phi_\infty(u) &= \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + a_\infty u^2 \right) + \frac{1}{4} \int_{\mathbb{R}^3} b_\infty \overline{\phi}_u(x) u^2 \\ &\quad - \frac{1}{p+1} \int_{\mathbb{R}^3} q_\infty |u|^{p+1}. \end{aligned} \tag{33}$$

And we consider the corresponding natural constraint

$$\mathcal{N}_\infty := \left\{ u \in H^1\left(\mathbb{R}^3\right) \setminus \{0\} : \Phi_\infty'(u)[u] = 0 \right\}. \tag{34}$$

Critical points of Φ_∞ are solutions of the limit problem at infinity

$$\begin{aligned} -\Delta u + a_\infty u + b_\infty \overline{\phi}_u(x) u &= q_\infty |u|^{p-1} u, \quad x \in \mathbb{R}^3, \\ -\Delta \overline{\phi}_u &= b_\infty u^2, \quad u \in H^1\left(\mathbb{R}^3\right), \end{aligned} \tag{35}$$

and, clearly, the conclusions of Lemma 4 hold true for Φ_∞ and $\mathcal{N}_\infty$. Moreover, for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, it is easy to see that there exists unique $t(u) > 0$ such that $t(u)u \in \mathcal{N}_\infty$. Set

$$m_\infty := \inf \left\{ \Phi_\infty(u),\ u \in \mathcal{N}_\infty \right\}. \tag{36}$$

From [11, 12], we deduce that m_∞ is achieved by a positive radially symmetric function w satisfying (11). In what follows, for any $y \in \mathbb{R}^3$, we use the translation symbol $w_y := w(\cdot - y)$. Set

$$m_\lambda := \inf \left\{ \Phi_\lambda(u),\ u \in \mathcal{N}_\lambda \right\}. \tag{37}$$

Then the following relations of m_λ and m_∞ hold true.

Lemma 5. *Suppose that (A_1)-(A_3) hold. Then for each $\lambda \ge 0$ one has*

$$0 < m_\lambda \le m_\infty. \tag{38}$$

Proof. Let $\lambda \ge 0$ be fixed. The first inequality of (38) is a straight consequence of (31). In order to show the second inequality we should construct a sequence $\{u_n\} \subset \mathcal{N}_\lambda$ and $\lim_n \Phi_\lambda(u_n) = m_\infty$. To accomplish this, we take $\{y_n\}$, with $y_n \in \mathbb{R}^3$, $|y_n| \longrightarrow +\infty$, as $n \longrightarrow +\infty$ and set $u_n = t_n w_{y_n}$, where $w_{y_n} = w(x - y_n)$ and $t_n = t_\lambda(w_{y_n}) > 0$ such that $u_n = t_n w_{y_n} \in \mathcal{N}_\lambda$. Here we recall that w is a radial solution of (35). A direct computation shows that

$$\begin{aligned}\Phi_\lambda(u_n) &= \frac{t_n^2}{2}\left[\|w\|^2+\int_{\mathbb{R}^3}\lambda a(x)w_{y_n}^2\right]+\frac{t_n^4}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(b_\infty-b(x))w_{y_n}^2(x)(b_\infty-b(y))w_{y_n}^2(y)}{4\pi|x-y|}dx\,dy-\frac{q_\infty t_n^{p+1}}{p+1}\left|w_{y_n}\right|_{p+1}^{p+1}\\ &\quad-\frac{t_n^{p+1}}{p+1}\int_{\mathbb{R}^3}q(x)\left|w_{y_n}\right|^{p+1}\\ &=\frac{t_n^2}{2}\left[\|w\|^2+\int_{\mathbb{R}^3}\lambda a(x+y_n)w^2\right]-\frac{t_n^{p+1}q_\infty}{p+1}|w|_{p+1}^{p+1}-\frac{t_n^{p+1}}{p+1}\int_{\mathbb{R}^3}q(x+y_n)|w|^{p+1}\\ &\quad+\frac{t_n^4}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(b_\infty-b(x+y_n))w^2(x)(b_\infty-b(y+y_n))w^2(y)}{4\pi|x-y|}dx\,dy.\end{aligned}\tag{39}$$

It is clear that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}a(x+y_n)w^2=0,\qquad \lim_{n\to\infty}\int_{\mathbb{R}^3}q(x+y_n)|w|^{p+1}=0.\tag{40}$$

We claim that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{b(x+y_n)w^2(x)b(y+y_n)w^2(y)}{|x-y|}dx\,dy=0.\tag{41}$$

In fact, we infer from Hardy-Littlewood-Sobolev inequality (17) that for $R>0$ large

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{b(x+y_n)w^2(x)b(y+y_n)w^2(y)}{|x-y|}dx\,dy &\le c\left(\int_{\mathbb{R}^3}b^{6/5}(x+y_n)w^{12/5}(x)\,dx\right)\\ &\le c|w|_6\int_{B_R}b^2(x+y_n)\,dx+c|b|_2|w|_{L^6(B_R^c)}\end{aligned}\tag{42}$$

and, from condition (A_2), we know that the claim holds. On the other hand, we infer from (40)-(41) and $t_nw_{y_n}\in\mathcal{N}_\lambda$ that

$$\begin{aligned}\|w\|^2+t_n^2\int_{\mathbb{R}^3}b_\infty\overline{\phi}_w w^2+o(t_n^2)=t_n^{p-1}q_\infty|w|_{p+1}^{p+1}+o(t_n^{p-1}).\end{aligned}\tag{43}$$

Thus, we get $\sigma_0\le t_n\le\sigma_1$ for $\sigma_0,\sigma_1>0$. Moreover, we deduce from $w\in\mathcal{N}_\infty$ that

$$\left(\frac{1}{t_n^2}-1\right)\|w\|^2=\left(t_n^{p-3}-1\right)|w|_{p+1}^{p+1}+o(1).\tag{44}$$

Since $p\in(3,5)$, we infer that $t_n\longrightarrow1$ as $n\longrightarrow\infty$. Finally, we let $n\longrightarrow\infty$ in (39) and obtain

$$t_n\longrightarrow1,\qquad \Phi_\lambda(u_n)\longrightarrow m_\infty.\tag{45}$$

This finishes the proof. □

By applying the well-known concentration-compactness principle [39] and maximum principle [40], we have the following splitting lemma results. For the details of the proof, one can refer to [23, Lemma 4.1 and Corollary 4.2]

Lemma 6. *If the strict inequality*

$$m_\lambda<m_\infty\tag{46}$$

holds, then m_λ is achieved by a positive function. Furthermore, all the minimizing sequences are relatively compact.

Next we consider the special case $\lambda=0$.

Lemma 7. *Assume that (A_1)-(A_3) and $\lambda=0$ hold. Then (10) has a ground state positive solution.*

Proof. Note that

$$\begin{aligned}\Phi_0(u)&=\frac12\|u\|^2+\frac14\int_{\mathbb{R}^3}(b_\infty-b(x))\phi_u(x)u^2\\&\quad-\frac{1}{p+1}\int_{\mathbb{R}^3}(q_\infty+q(x))|u|^{p+1},\end{aligned}\tag{47}$$

and

$$\begin{aligned}\Phi_\infty(u)&=\frac12\|u\|^2+\frac14\int_{\mathbb{R}^3}b_\infty\overline{\phi}_u(x)u^2\\&\quad-\frac{1}{p+1}\int_{\mathbb{R}^3}q_\infty|u|^{p+1},\end{aligned}\tag{48}$$

where $\phi_u(x)=\int_{\mathbb{R}^3}((b_\infty-b(y))u^2(y)/4\pi|x-y|)dy$ and $\overline{\phi}_u(x)=\int_{\mathbb{R}^3}(b_\infty u^2(y)/4\pi|x-y|)dy$. Therefore, we have that

$$m_0\le m_\infty.\tag{49}$$

From (36) we know that $\Phi_\infty(w)=m_\infty$. Similar to the proof of Lemma 4 (a), we infer that $g(t)=\Phi_0(tw)$ has unique maximum $t_0>0$ such that $t_0w\in\mathcal{N}_0$. Then we infer the following from condition (A_2).

$$\begin{aligned}\Phi_0(t_0w)&=\frac{t_0^2}{2}\|w\|^2-\frac{t_0^{p+1}}{p+1}\int_{\mathbb{R}^3}(q_\infty+q(x))|u|^{p+1}+\frac{t_0^4}{4}\\&\quad\cdot\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(b_\infty-b(x))w^2(x)(b_\infty-b(y))w^2(y)}{4\pi|x-y|}dx\,dy\\&<\Phi_\infty(t_0w)\end{aligned}\tag{50}$$

Moreover, since $w \in \mathcal{N}_\infty$ is the unique maximum point of $g_\infty(t) = \Phi_\infty(tw)$ for $t > 0$, it follows that $\Phi_\infty(t_0 w) \le \Phi_\infty(w)$. Combining the above arguments, we infer that

$$m_0 \le \Phi_0(t_0 w) < \Phi_\infty(t_0 w) \le \Phi_\infty(w) = m_\infty. \tag{51}$$

From Lemma 6, we know that the conclusion holds. □

3. Proof of Theorem 1

In this section we shall give the proof of Theorem 1. This can be accomplished by the following Lemma.

Lemma 8. *Assume that* (A_1)*-*(A_3) *hold. Then for each* $\lambda \in \mathbb{R}^+$*, we have* $m_\lambda < m_\infty$.

Proof. We first observe that, by Lemma 7, we know that $m_0 < m_\infty$. So, we consider the case $\lambda > 0$ in the sequel. For fixed $\lambda > 0$, we choose t_n such that $u_n = t_n w_{y_n} \in \mathcal{N}_\lambda$, where y_n and t_n are chosen as in the proof of Lemma 5. Moreover, as in (45), we infer that $0 < \sigma_0 \le t_n \le \sigma_1$. Similar to (51), we know that $\Phi_\infty(t_n w) \le m_\infty$. Thus, we infer that

$$\begin{aligned} m_\lambda &\le \Phi_\lambda(u_n) = \Phi_\lambda(t_n w_{y_n}) = \frac{t_n^2}{2}\left[\|w\| + \lambda\int_{\mathbb{R}^3} a(x+y_n)w^2\right] - \frac{t_n^{p+1}}{p+1}\int_{\mathbb{R}^3}(q_\infty + q(x+y_n))|w|^{p+1} + \frac{t_n^4}{4} \\ &\cdot \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(b_\infty - b(x+y_n))w^2(x)(b_\infty - b(y+y_n))w^2(y)}{4\pi|x-y|}dx\,dy = \Phi_\infty(t_n w) + \frac{t_n^2}{2}\left[\lambda\int_{\mathbb{R}^3} a(x+y_n)w^2 + \frac{t_n^2}{2}\right. \\ &\cdot \left.\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{b(x+y_n)w^2(x)b(y+y_n)w^2(y)}{4\pi|x-y|}dx\,dy - t_n^2\int_{\mathbb{R}^3} b(x+y_n)\overline{\phi}_w w^2 - \frac{2t_n^{p-1}}{p+1}\int_{\mathbb{R}^3} q(x+y_n)|w|^{p+1}\right] \le m_\infty \\ &+ \frac{t_n^2}{2}\left[\lambda\int_{\mathbb{R}^3} a(x+y_n)w^2 + \frac{t_n^2}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{b(x+y_n)w^2(x)b(y+y_n)w^2(y)}{4\pi|x-y|}dx\,dy - t_n^2\int_{\mathbb{R}^3} b(x+y_n)\overline{\phi}_w w^2 - \frac{2t_n^{p-1}}{p+1}\right. \\ &\cdot \left.\int_{\mathbb{R}^3} q(x+y_n)|w|^{p+1}\right]. \end{aligned} \tag{52}$$

Since $b \in L^\infty(\mathbb{R}^3)$, we prove the conclusion if we show that, for large n,

$$\begin{aligned} &\lambda\int_{\mathbb{R}^3} a(x+y_n)w^2 \\ &\quad + c\int_{\mathbb{R}^3} b(x+y_n)w^2(x)\varphi_w(x)\,dx \\ &\quad - c\int_{\mathbb{R}^3} b(x+y_n)\overline{\phi}_w w^2 \\ &\quad - c\int_{\mathbb{R}^3} q(x+y_n)|w|^{p+1} < 0, \end{aligned} \tag{53}$$

where $\varphi_w(x) = \int_{\mathbb{R}^3}(b(y+y_n)w^2(y)/4\pi|x-y|)dy$. This is equal to proving that, for large n,

$$\begin{aligned} A_1 &= \lambda\int_{\mathbb{R}^3\setminus B_{\tau|y_n|}} a(x+y_n)w^2 \\ &\quad + c\int_{\mathbb{R}^3\setminus B_{\tau|y_n|}} b(x+y_n)w^2(x)\varphi_w(x)\,dx \\ &\quad - c\int_{\mathbb{R}^3\setminus B_{\tau|y_n|}} b(x+y_n)\overline{\phi}_w w^2 \\ &\quad - c\int_{\mathbb{R}^3\setminus B_{\tau|y_n|}} q(x+y_n)|w|^{p+1} \\ &< -\lambda\int_{B_{\tau|y_n|}} a(x+y_n)w^2 \\ &\quad - c\int_{B_{\tau|y_n|}} b(x+y_n)w^2(x)\varphi_w(x)\,dx \\ &\quad + c\int_{B_{\tau|y_n|}} b(x+y_n)\overline{\phi}_w w^2 \\ &\quad + c\int_{B_{\tau|y_n|}} q(x+y_n)|w|^{p+1} = A_2. \end{aligned} \tag{54}$$

A direct computation shows that for $R > 0$ large

$$\begin{aligned} \varphi_w(x) &= \int_{|x-y|\le 1}\frac{b(y+y_n)w^2(y)}{4\pi|x-y|}dy \\ &\quad + \int_{1\le|x-y|\le R}\frac{b(y+y_n)w^2(y)}{4\pi|x-y|}dy \\ &\quad + \int_{|x-y|\ge R}\frac{b(x+y_n)w^2(y)}{4\pi|x-y|}dy \\ &\le c|w|_4^2\left(\int_{|x-y|\le 1}\frac{1}{|x-y|}dy\right) \end{aligned}$$

$$+ c|w|_{8/3}^2 \left(\int_{1 \le |x-y| \le R} \frac{1}{|x-y|} dy \right)^{1/4}$$
$$+ c|w|_{8/3}^2 \left(\int_{|x-y| \ge R} \frac{1}{|x-y|} dy \right)^{1/4} \le c. \quad (55)$$

Now we are ready to give the estimate of the term A_1. We infer from condition (A_2), (11), and (55) that

$$\begin{aligned} A_1 &\le \int_{\mathbb{R}^3 \setminus B_{\tau|y_n|}} \lambda a(x+y_n) w^2 \\ &+ c \int_{\mathbb{R}^3 \setminus B_{\tau|y_n|}} b(x+y_n) \varphi_w w^2 \\ &\le c \left(\left(\left(\int_{\mathbb{R}^3 \setminus B_{\tau|y_n|}} |a(x+y_n)|^{3/2} \right)^{2/3} \right. \right. \\ &+ \left. \int_{\mathbb{R}^3 \setminus B_{\tau|y_n|}} |b(x+y_n)|^{3/2} \right) \left[\int_{\mathbb{R}^3 \setminus B_{\tau|y_n|}} |w|^6 \right]^{1/3} \\ &\le ce^{-2c^* \tau |y_n|} \end{aligned} \quad (56)$$

holds for n large.

Next we give the estimate for the second part A_2. We first consider that the case $a(x)$ decays faster than $b(x)$. By condition (1) of Theorem 1, we know that, for all $M > 0$, there exists $n_0 \ge 1$ such that, for all $n \ge n_0$ and for all $x \in B_{\tau|y_n|}$,

$$\begin{aligned} &Me^{-2\tau c^* |y_n|} \le q(x+y_n) \le ce^{-(2\tau(1-\tau)/(1+2\tau))c^* |y_n|}, \\ &a(x+y_n) \le c(1-\tau)^{-(1+\tau)} |y_n|^{-(1+\tau)} e^{-2\tau c^* |y_n|} \\ &b(x+y_n) \ge ce^{-(2\tau(1-\tau)/(1+2\tau))c^* |y_n|}. \end{aligned} \quad (57)$$

Moreover, by [41, Lemmas 2.3 and 2.6], we know that

$$\overline{\phi}_w(x) = \int_{\mathbb{R}^3} \frac{b_\infty w^2(y)}{4\pi|x-y|} dy \sim \frac{1}{|x|}, \quad \text{as } |x| \longrightarrow \infty. \quad (58)$$

Since $|x+y_n| \ge (1-\tau)|y_n| \longrightarrow \infty$ as $n \longrightarrow \infty$, we infer from (58) and (A_2) that

$$\varphi_w(x) = \int_{\mathbb{R}^3} \frac{b(x+y_n) w^2(y)}{4\pi|x-y|} dy \sim o(\overline{\phi}_w) \quad \text{as } n \longrightarrow \infty. \quad (59)$$

Thus, we infer that, for n sufficiently large and for all $x \in B_{\tau|y_n|}$,

$$\frac{cb(x+y_n)\overline{\phi}_w}{q(x+y_n)} - \frac{cb(x+y_n)\varphi_w}{q(x+y_n)} - \frac{\lambda a(x+y_n)}{q(x+y_n)} > 0. \quad (60)$$

Hence we deduce that the inequality

$$\begin{aligned} A_2 &\ge \int_{B_{\tau|y_n|}} q(x+y_n) w^2 \left[\frac{cb(x+y_n)\overline{\phi}_w}{q(x+y_n)} + cw^{p-1} \right. \\ &- \left. \frac{cb(x+y_n)\varphi_w}{q(x+y_n)} - \frac{\lambda a(x+y_n)}{q(x+y_n)} \right] > c \int_{B_{\tau|y_n|}} q(x \\ &+ y_n) w^{p+1} > cMe^{-2\tau c^* |y_n|} \int_{B_1} w^{p+1} > cMe^{-2\tau c^* |y_n|} \end{aligned} \quad (61)$$

holds for n sufficiently large. Thus, by the arbitrariness of M, we can conclude that $A_1 < A_2$, as desired.

Finally, we consider the remaining case in Theorem 1. By (2) of Theorem 1, for all $\varepsilon > 0$ and $M > 0$, there exists $n_0 \ge 1$ such that, for all $n \ge n_0$ and for all $x \in B_{\tau|y_n|}$,

$$\begin{aligned} Me^{-2\tau c^* |y_n|} &\le q(x+y_n), \\ a(x+y_n) &\le ce^{-(p-1)\tau c^* |y_n|}, \end{aligned} \quad (62)$$

where $p \in (3,5)$. Thus we get that, for n sufficiently large and for all $x \in B_{\tau|y_n|}$,

$$cw^{p-1} - \frac{\lambda a(x+y_n)}{q(x+y_n)} > \frac{c}{2} w^{p-1}. \quad (63)$$

Hence, we infer from (59) that

$$\begin{aligned} &A_2 \\ &\ge c \int_{B_{\tau|y_n|}} \left(b(x+y_n) w^2 (\overline{\phi}_w - \varphi_w) \right) \\ &+ \int_{B_{\tau|y_n|}} \left(q(x+y_n) w^2 \left(cw^{p-1} - \frac{\lambda a(x+y_n)}{q(x+y_n)} \right) \right) \\ &> \frac{c}{2} \int_{B_{\tau|y_n|}} q(x+y_n) w^{p+1} > \frac{c}{2} Me^{-2\tau c^* |y_n|} \int_{B_1} w^{p+1} \\ &> \frac{c}{2} Me^{-2\tau c^* |y_n|}. \end{aligned} \quad (64)$$

Hence, by the arbitrariness of M, we can conclude that $A_1 < A_2$. This finishes the proof. □

Proof of Theorem 1. By Lemmas 6 and 8, we know that the conclusions of Theorem 1 hold. □

4. Conclusion

In this paper, the authors prove the existence of positive ground state solutions for the nonautonomous Schrödinger-Poisson system. In condition (1) of Theorem 1, we know that if $a(x)$ decays faster than $b(x)$, we find the existence of positive ground state solution of (10). In condition (2) of Theorem 1, we only need the decay condition for $a(x)$(whatever the decay speed of $b(x)$ to 0 as $|x| \longrightarrow \infty$) to prove the existence of positive solution. This is different phenomenon compared to the previous paper [34].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors contributed equally to this article. They have all read and approved the final manuscript.

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