Hindawi
Advances in Mathematical Physics
Volume 2018, Article ID 8919516, 9 pages
https://doi.org/10.1155/2018/8919516

Research Article

On the Regular Integral Solutions of a Generalized Bessel Differential Equation

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Received 15 July 2018; Accepted 28 October 2018; Published 4 November 2018

Academic Editor: Emilio Turco

The original Bessel differential equation that describes, among many others, cylindrical acoustic or vortical waves, is a particular case of zero degree of the generalized Bessel differential equation that describes coupled acoustic-vortical waves. The solutions of the generalized Bessel differential equation are obtained for all possible combinations of the two complex parameters, order and degree, and finite complex variable, as Frobenius-Fuchs series around the regular singularity at the origin; the series converge in the whole complex plane of the variable, except for the point-at-infinity, that is, the only other singularity of the differential equation. The regular integral solutions of the first and second kinds lead, respectively, to the generalized Bessel and Neumann functions; these reduce to the original Bessel and Neumann functions for zero degree and have alternative expressions for nonzero degree.

1. Introduction

The Bessel differential equation was first considered in connexion with the oscillations of a heavy chain [1] and vibrations of a circular membrane [2] and has had since [3] a vast number of applications supported by an extensive theory [4]. A substantial number of applications arise from the separation of variables in the Laplace operator in cylindrical and spherical coordinates that leads, respectively, to the cylindrical and spherical Bessel functions. The cases of specific interest include two types of cylindrical waves: (i) sound waves as compressible perturbations of a uniform flow; (ii) vortical waves as incompressible perturbations of a uniform flow with superimposed rigid body rotation. Whereas (i) and (ii) separately lead to the original Bessel differential equation, their coupling leads to a generalization. Thus, the consideration of coupled acoustic-vortical waves as rotational compressible perturbations of a uniform mean flow with rigid body rotation leads to the generalized Bessel equation that differs from the original in having an extra term involving a second parameter, namely, the degree μ, in addition to the order ν.

The generalized Bessel differential equation of order ν and degree μ may have other applications and deserves separate study as it leads to generalizations of the Bessel and Neumann functions. The generalized Bessel differential equation may also be obtained, aside from any physical or engineering motivations, by a purely mathematical argument, starting from the original Bessel differential equation and replacing the coefficients of the dependent variable and its derivative by polynomials of the independent variable; in this case the origin remains a regular singularity of the differential equation and the only other singularity is the point-at-infinity. Thus, solutions exist as Frobenius-Fuchs series [5, 6] with recurrence formula for the coefficients reducing to two terms only in the case of the generalized Bessel differential equations.

The generalized Bessel differential equation has singularities only at the origin and infinity. Since the singularity at the origin is regular, the Frobenius-Fuchs method leads to solutions valid for finite values of the variable. The solutions of the generalized Bessel differential equation around the regular singularity at the origin has (i) indices that are exponents of the leading power depending only on the order; (ii) recurrence relation for the coefficients of the power series expansion depending also on the degree. From (i) follows the familiar situation that generalized Bessel functions (Section 2) specify the general integral for noninteger order, and generalized Neumann functions (Section 3) are needed for integer order. From (ii) it follows that the series expansion for the generalized Bessel (Section 2.1) and Neumann (Section 3.1) functions differ from the original series in having finite products multiplying each term; these finite products can be expressed as ratios of Gamma functions, whose arguments become singular for zero degree. The Wronskians are used to select pairs of linearly independent particular integrals that lead to the general integral for noninteger (Section 2.2) and integer (Section 3.2) order.

2. Generalized Bessel Differential Equation in the Complex Plane

The origin is a regular singularity of the generalized Bessel equation with the same indices (Section 2.1) as the original, leading to generalized Bessel functions whose series expansion differs from the original in the coefficients following the leading term. A linear combination of (Section 2.1) generalized Bessel functions of order $\pm\nu$ and degree μ supplies the general integral (Section 2.2) of the generalized Bessel equation if the order ν is not an integer. For $\nu = n$ (i.e. an integer), the generalized Bessel functions of order $\pm n$ are linearly dependent for any degree μ, their Wronskian is zero, and generalized Neumann functions are needed (Section 3).

2.1. Generalized Bessel Functions of Arbitrary Order and Degree

Definition 1. The generalized Bessel differential equation is a linear second-order ordinary differential equation (1) in the complex plane, whose order ν and degree μ are complex numbers,

$$\mu, \nu, z \in \mathbb{C} : z^2 Q'' + z\left(1 - \frac{\mu}{2} z^2\right) Q' + \left(z^2 - \nu^2\right) Q = 0. \tag{1}$$

Remark 2. The original Bessel differential equation corresponds to zero degree $\mu = 0$. The choice of $\mu/2$ rather than μ in the coefficient of Q' simplifies subsequent expressions for the generalized Bessel and Neumann functions that are solutions of (1), much as the choice of ν^2 rather than ν in the coefficient of Q simplifies the expressions for the original Bessel and Neumann functions.

Theorem 3. *A solution of the generalized Bessel differential equation is the generalized Bessel function of order ν and degree μ specified by the power series with infinite radius of convergence*

$$|z| < \infty : J_\nu^\mu(z) = \left(\frac{z}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{\left(-z^2/4\right)^j}{j!\Gamma(1+j+\nu)} \prod_{l=0}^{j-1}\left[1 - \mu\left(l + \frac{\nu}{2}\right)\right] \tag{2}$$

valid for all μ.

Remark 4. In the case of the original Bessel function the degree is zero, $\mu = 0$, and the last factor in (2) is omitted, leading to the usual series expansion [7] that is alternating for real variable z and has fixed sign for imaginary variable. The last factor that distinguishes the generalized Bessel function for real positive $\nu > 0$ is (i) positive for $\mu < 0$ and does not change the sign of the terms of the series; (ii) for real $\mu > 0$ the sign of the terms of the series is fixed for $\mu(l + \nu/2) > 1$, and in particular if $\mu\nu/2 > 1$ terms of the series have the same sign. For example, if μ and ν are real such that $\mu\nu/2 > 1$ then the series for the generalized Bessel function (2) has fixed sign for real z and alternating sign for imaginary z; this is the reverse of the series for the original Bessel function when the last factor in (2) is omitted.

Corollary 5. *If the degree is not zero (3a) the last factor in the coefficients (2) may be written in (3b)*

$$\mu \neq 0 : \tag{3a}$$

$$\prod_{l=0}^{j-1}\left[1 - \mu\left(l + \frac{\nu}{2}\right)\right] = (-\mu)^j \prod_{l=0}^{j-1}\left(l + \frac{\nu}{2} - \frac{1}{\mu}\right) = (-\mu)^j \frac{\Gamma(j + \nu/2 - 1/\mu)}{\Gamma(\nu/2 - 1/\mu)}; \tag{3b}$$

substitution of (3b) in (2) leads to

$$\mu \neq 0 : \tag{4a}$$

$$J_\nu^\mu(z) = \frac{(z/2)^\nu}{\Gamma(\nu/2 - 1/\mu)} \sum_{j=0}^{\infty} \frac{\left(\mu z^2/4\right)^j}{j!} \frac{\Gamma(j + \nu/2 - 1/\mu)}{\Gamma(1 + j + \nu)}. \tag{4b}$$

This is an alternative expression (4b) for the generalized Bessel function (2) when the degree is not zero (4a) and involves powers of $(\mu z^2)/4$ rather than $(-z^2)/4$, hence omitting the alternating sign and inserting the factor μ.

Proof of Theorem 3. The origin $z = 0$ is a regular singularity, and the only other singularity is at infinity; thus [8] the solution as a Frobenius series [5, 9] by the Fuchs theorem [6, 10] converges in the whole finite complex z-plane

$$|z| < \infty : Q_\sigma(z) = \sum_{j=0}^{\infty} a_j(\sigma) z^{j+\sigma}. \tag{5}$$

Substituting (5) into (1) and equating the coefficients of equal powers of z lead to a two-term recurrence formula

$$\left[(\sigma + j)^2 - \nu^2\right] a_j(\sigma) = -\left[1 - \frac{\mu}{2}(\sigma + j - 2)\right] a_{j-2}(\sigma). \tag{6}$$

Setting (7a) leads to (7b); since $a_0 = 0$ would lead to $a_j = 0$ in (6) and $Q = 0$ in (5), a nontrivial solution requires (7c) leading to the indicial equation (7d) with roots (7e):

$$j = 0: \tag{7a}$$

$$\left(\sigma^2 - \nu^2\right) a_0(\sigma) = 0; \tag{7b}$$

$$a_0(\sigma) \neq 0; \tag{7c}$$

$$\sigma^2 = \nu^2; \tag{7d}$$

$$\sigma_\pm = \pm\nu; \tag{7e}$$

thus, the indices (7e) are specified by the order ν as in the original Bessel equation, but the recurrence formula for the coefficients (6),

$$\begin{aligned} a_{2j} &= -\frac{1 - (\mu/2)(\sigma + 2j - 2)}{(2j + \sigma)^2 - \nu^2} a_{2j-2}(\sigma) \\ &= -\frac{1 - \mu(j - 1 + \sigma/2)}{(2j + \sigma - \nu)(2j + \sigma + \nu)} a_{2j-2}(\sigma), \end{aligned} \tag{8}$$

shows that they depend also on the degree μ. Applying j times (8) leads to the explicit coefficients

$$\begin{aligned} a_{2j}(\nu) &= (-1)^j a_0(\nu) \prod_{l=1}^{j} \frac{1 - \mu(l - 1 + \nu/2)}{4l(l + \nu)} \\ &= a_0(\nu) \frac{(-1/4)^j}{j!} \frac{\Gamma(1+\nu)}{\Gamma(1+j+\nu)} \prod_{l=1}^{j} \left[1 - \mu\left(l - 1 + \frac{\nu}{2}\right)\right] \end{aligned} \tag{9}$$

where Γ is the Gamma function [11, 12]. For the original Bessel functions, $\mu = 0$, the last function is unity leading to the usual expression [4], using for consistency (10a):

$$a_0(\nu) = \frac{2^{-\nu}}{\Gamma(1+\nu)}. \tag{10a}$$

Thus, the generalized Bessel function of the first kind, order ν, and degree μ is given by (5) and (10b),

$$a_{2j}(\nu) = \frac{2^{-\nu}(-1/4)^j}{j!\,\Gamma(1+j+\nu)} \prod_{l=0}^{j-1} \left[1 - \mu\left(l + \frac{\nu}{2}\right)\right], \tag{10b}$$

leading to (2). □

2.2. Wronskian, Linear Independence of Solutions, and General Integral. The generalized Bessel differential equation (1) is satisfied by the generalized Bessel functions (2) of orders $\pm\nu$ as follows from the indices (7e). If the functions $J_{\pm\nu}$ are linearly independent, their linear combination specifies the general integral of the generalized Bessel equation (1). Next the Wronskian is calculated for two solutions of the generalized Bessel differential equation (1), in particular for the pair of solutions $J_{\pm\nu}$. It follows that the Wronskian is zero for $\nu = n$ (i.e., an integer), implying that $J^\mu_{\pm n}(z)$ are linearly dependent. The linear relation between $J^\mu_{-n}(z)$ and $J^\mu_n(z)$ is obtained for all integer values of n.

Lemma 6. *If Q_1, Q_2 are any two solutions of the generalized Bessel differential equation,*

$$z^2 Q_1'' + z\left(1 - \frac{\mu}{2}z^2\right) Q_1' + \left(z^2 - \nu^2\right) Q_1 = 0, \tag{11a}$$

$$z^2 Q_2'' + z\left(1 - \frac{\mu}{2}z^2\right) Q_2' + \left(z^2 - \nu^2\right) Q_2 = 0, \tag{11b}$$

their Wronskian is given by

$$\begin{aligned} W(z) &= Q_1(z) Q_2'(z) - Q_1'(z) Q_2(z) \\ &= \frac{W_0(\mu,\nu)}{z} \exp\left(\frac{1}{4}\mu z^2\right) \end{aligned} \tag{12a}$$

where $W_0(\mu,\nu)$: (i) is the residue of the Wronskian at its simple pole at the origin,

$$\lim_{z\longrightarrow 0} z W(z) = W_0(\mu,\nu); \tag{12b}$$

(ii) does not depend on the variable z but may depend on the order ν and degree μ; (iii) depends on the choice of particular integrals $\{Q_1(z), Q_2(z)\}$ in (11a) and (11b).

Proof. Multiplying the second equation (11b) by Q_1 and multiplying the first equation (11a) by Q_2 and subtracting lead to

$$\begin{aligned} &\left(Q_1 Q_2' - Q_2 Q_1'\right) z \left(1 - \frac{\mu}{2}z^2\right) = -z^2 \left(Q_1 Q_2'' - Q_2 Q_1''\right) \\ &\quad = -z^2 \left(Q_1 Q_2' - Q_2 Q_1'\right)'. \end{aligned} \tag{13}$$

Thus, the Wronskian (14a) satisfies a first-order differential equation (14b),

$$W(Q_1, Q_2) \equiv \left(Q_1 Q_2' - Q_2 Q_1'\right) \neq 0: \tag{14a}$$

$$\frac{W'}{W} = \frac{\mu z}{2} - \frac{1}{z}. \tag{14b}$$

The solution of (14b) is (12a) where (12b) is an arbitrary constant of integration $W_0(\mu,\nu)$ that is independent of z. □

Remark 7. The constant factor $W_0(\mu,\nu)$ in the Wronskian (12a) depends on the particular choice of linearly independent solutions.

Lemma 8. *The generalized Bessel function of order $\pm\nu$ and degree μ has Wronskian*

$$W\left(J^\mu_{+\nu}(z), J^\mu_{-\nu}(z)\right) = -\frac{2}{\pi z} \sin(\nu\pi) \exp\left(\frac{1}{4}\mu z^2\right). \tag{15}$$

Proof. In the case of the generalized Bessel functions (2),

$$J^\mu_{\pm\nu}(z) \sim \frac{(z/2)^{\pm\nu}}{\Gamma(1\pm\nu)} \left[1 + \mathcal{O}\left(z^2\right)\right], \tag{16a}$$

$$\frac{d}{dz}\left[J^\mu_{\pm\nu}(z)\right] \sim \pm\frac{\nu}{2} \frac{(z/2)^{\pm\nu-1}}{\Gamma(1\pm\nu)} \left[1 + \mathcal{O}\left(z^2\right)\right], \tag{16b}$$

the limit as $z \longrightarrow 0$ specifies the Wronskian

$$\begin{aligned} W(0) &= \lim_{z\longrightarrow 0}\left\{J_\nu^\mu(z)\frac{d}{dz}\left[J_{-\nu}^\mu(z)\right]-J_{-\nu}^\mu(z)\frac{d}{dz}\left[J_\nu^\mu(z)\right]\right\} \\ &= -2\frac{\nu}{2}\frac{2}{z}\frac{1}{\Gamma(1+\nu)\Gamma(1-\nu)}=-\frac{2}{z}\frac{1}{\Gamma(\nu)\Gamma(1-\nu)} \\ &= -\frac{2}{\pi z}\sin(\nu\pi) \end{aligned} \tag{17a}$$

using the symmetry formula [13] for the Gamma function in (17a). Comparison of (17a) with (12a) in the limit $z \longrightarrow 0$, or with (12b), shows that for the choice of particular integrals (16a) of the generalized Bessel differential equation the constant is given by

$$W_0(\mu,\nu)=-\frac{2}{\pi}\sin(\nu\pi) \tag{17b}$$

and thus depends only on the order ν but not on the degree μ. Substitution of (17b) in (12a) proves (15). □

Theorem 9. *The general integral of the generalized Bessel equation (1) is a linear combination (18b) with arbitrary constants $C_\pm$ of the functions of the first kind (2) for the indices (7e) provided that the indices (18a) are not integers,*

$$\nu \neq 0,\pm 1,\pm 2,\ldots=\pm n: \tag{18a}$$

$$Q(z)=C_+J_\nu^\mu(z)+C_-J_{-\nu}^\mu(z). \tag{18b}$$

Proof. The general integral (18b) holds provided that $J_{\pm\nu}^\mu(z)$ are linearly independent. This is the case if their Wronskian (15) is not zero. The Wronskian only vanishes for $\nu = n$, an integer, so the general integral (18b) holds (18a) only for noninteger values of ν. □

The Frobenius-Fuchs method [14] suggests that the two solutions may be linearly dependent and the general integral (18b) fails if the difference of indices (7e) is an integer, that is, if $\sigma_+ - \sigma_- = 2\nu$ corresponds to [15] order ν either (i) an integer (cylindrical Bessel functions) or (ii) an integer plus-one-half (spherical Bessel functions). In the case (ii) of order an integer plus-one-half, the two solutions in (18b) are linearly independent because the Wronskian (15) is not zero so the general integral (18b) holds. It remains to consider the case (i) of order an integer in (1) when the functions $J_{\pm n}^\mu$ must be linearly dependent. This is confirmed by the following relation.

Theorem 10. *The generalized Bessel functions of integer order and nonzero degree (19a) are related by (19b),*

$$\mu \neq 0: \tag{19a}$$

$$J_{-n}^\mu(z)=\mu^n\frac{\Gamma(n/2-1/\mu)}{\Gamma(-n/2-1/\mu)}J_n^\mu(z). \tag{19b}$$

Corollary 11. *Using the recurrence formula for the Gamma function,*

$$\Gamma\left(\frac{n}{2}-\frac{1}{\mu}\right)=\Gamma\left(-\frac{n}{2}-\frac{1}{\mu}\right)\prod_{l=1}^{n}\left[\frac{n}{2}-\frac{1}{\mu}-l\right], \tag{20}$$

the relation (19b) between generalized Bessel functions of order $\pm n$ becomes

$$J_{-n}^\mu(z)=(-1)^nJ_n^\mu(z)\prod_{l=1}^{n}\left[1-\mu\left(\frac{n}{2}-l\right)\right]. \tag{21}$$

Remark 12. This result also holds for the original Bessel function $\mu = 0$ when the last factor in (21) is unity.

Proof of Theorem 9. If the order is a positive integer, $\nu = n$, in the second solution J_{-n}^μ the Gamma function $\Gamma(1-n+j)=\infty$ for $j = 0, 1, \ldots, n-1$ in the denominator of (4b) suppresses the first n terms, leading to (22b),

$$k=j-n: \tag{22a}$$

$$\begin{aligned} &\Gamma\left(-\frac{n}{2}-\frac{1}{\mu}\right)J_{-n}^\mu(z) \\ &\quad=\left(\frac{z}{2}\right)^{-n}\sum_{j=n}^{\infty}\frac{\left(\mu z^2/4\right)^j}{j!\,(j-n)!}\Gamma\left(j-\frac{n}{2}-\frac{1}{\mu}\right) \\ &\quad=\left(\frac{z}{2}\right)^{-n}\sum_{k=0}^{\infty}\frac{\left(\mu z^2/4\right)^{k+n}}{k!\,(k+n)!}\Gamma\left(k+\frac{n}{2}-\frac{1}{\mu}\right) \\ &\quad=\mu^n\left(\frac{z}{2}\right)^{n}\sum_{k=0}^{\infty}\frac{\left(\mu z^2/4\right)^{k}}{k!\,(k+n)!}\Gamma\left(k+\frac{n}{2}-\frac{1}{\mu}\right) \\ &\quad=\mu^n\Gamma\left(\frac{n}{2}-\frac{1}{\mu}\right)J_n^\mu(z) \end{aligned} \tag{22b}$$

where the substitution (22a) was made. □

3. Solutions of the Generalized Bessel Equation for Any Order and Degree

It has been shown that, for any degree μ and integer order, $\nu = n$, the general integral (18b) fails (18a), and the generalized Bessel function J_{-n}^μ that is a regular integral of the first kind of the generalized Bessel differential equation (1) must be replaced by a generalized Neumann function Y_n^μ that is a regular integral of the second kind (Section 3.1) and hence is linearly independent, leading by linear combination of $J_\nu^\mu(z)$ and $Y_\nu^\mu(z)$ to the general integral (Section 3.2).

3.1. Generalized Neumann Function of Arbitrary Degree and Integer Order

Theorem 13. *A solution of the generalized Bessel equation (1) with arbitrary degree μ and integer order n is the generalized Neumann function*

$$Y_n^\mu(z)=\frac{2}{\pi}\log\left(\frac{z}{2}\right)J_n^\mu(z)+X_n^\mu(z)+Z_n^\mu(z) \tag{23}$$

Consisting of the sum of (i) a constant $2/\pi$ multiplying the generalized Bessel function (2) multiplied by a logarithmic singularity; (ii) the preliminary function

$$X_n^\mu(z) = -\frac{1}{\pi}\left(\frac{z}{2}\right)^{-n} \cdot \sum_{j=0}^{n-1}\left(\frac{z}{2}\right)^{2j}\frac{(n-j-1)!}{j!}\left\{\prod_{l=j}^{n-1}\left[1-\mu\left(l-\frac{n}{2}\right)\right]\right\}^{-1} \quad (24)$$

that has a pole of order n for $n = 2,3,\ldots$; (iii) the complementary function

$$Z_n^\mu(z) = -\frac{1}{\pi}\left(\frac{z}{2}\right)^{n} \cdot \sum_{j=0}^{\infty}\frac{\left(-z^2/4\right)^j}{j!\,(j+n)!}\left\{\prod_{l=0}^{j-1}\left[1-\mu\left(l+\frac{n}{2}\right)\right]\right\} \cdot \left[\psi(1+j)+\psi(1+j+n)-\psi\left(j+\frac{n}{2}-\frac{1}{\mu}\right)\right] \quad (25)$$

involving the digamma function ψ that has no singularities and starts with the power n.

Remark 14. Substitution of (2), (24), and (25) into (23) specifies explicitly the Neumann function of complex degree μ and integer order n:

$$\begin{aligned} Y_n^\mu(z) &= \frac{2}{\pi}\left(\frac{z}{2}\right)^n\log\left(\frac{z}{2}\right) \cdot \sum_{j=0}^{\infty}\frac{\left(-z^2/4\right)^j}{j!\,(n+j)!}\left\{\prod_{l=0}^{j-1}\left[1-\mu\left(l+\frac{n}{2}\right)\right]\right\} \\ &\quad - \frac{1}{\pi}\left(\frac{z}{2}\right)^{-n}\sum_{j=0}^{n-1}\left(\frac{z}{2}\right)^{2j} \cdot \frac{(n-j-1)!}{j!}\left\{\prod_{l=j}^{n-1}\left[1-\mu\left(l-\frac{n}{2}\right)\right]\right\}^{-1} \\ &\quad - \frac{1}{\pi}\left(\frac{z}{2}\right)^{n}\sum_{j=0}^{\infty}\frac{\left(-z^2/4\right)^j}{j!\,(n+j)!}\left\{\prod_{l=0}^{j-1}\left[1-\mu\left(l+\frac{n}{2}\right)\right]\right\} \cdot \left[\psi(1+j)+\psi(1+j+n)-\psi\left(j+\frac{n}{2}-\frac{1}{\mu}\right)\right] \end{aligned} \quad (26)$$

For $\mu = 0$ the terms in curly brackets reduce to unity, and the original Neumann [16] function is regained.

Corollary 15. *In the case of nonzero degree, $\mu \neq 0$, there are alternate expressions for the three terms whose sum is (23) the generalized Neumann function: (i) the logarithmic factor multiplies the generalized Bessel function (2) that has the alternate form (4b) for nonzero degree (4a); (ii) the complementary function (25), using (3b) has the alternate form (27b) for nonzero degree (27a),*

$$\mu \neq 0, \quad (27a)$$

$$Z_n^\mu(z) = -\frac{1}{\pi}\frac{(z/2)^n}{\Gamma(n/2-1/\mu)} \cdot \sum_{j=0}^{\infty}\frac{\left(\mu z^2/4\right)^j}{j!\,(n+j)!}\Gamma\left(j+\frac{n}{2}-\frac{1}{\mu}\right) \cdot \left[\psi(1+j)+\psi(1+j+n)-\psi\left(j+\frac{n}{2}-\frac{1}{\mu}\right)\right]; \quad (27b)$$

(iii) the preliminary function (24) that for nonzero degree (28a) has the alternate form (28b),

$$\mu \neq 0, \quad (28a)$$

$$X_n^\mu(z) = -\frac{1}{\pi}\left(\frac{z}{2}\right)^n\frac{\left(-\mu z^2/4\right)^{-n}}{\Gamma(n/2-1/\mu)} \cdot \sum_{j=0}^{n-1}\left(-\frac{\mu z^2}{4}\right)^j\frac{(n-j-1)!}{j!}\Gamma\left(j-\frac{n}{2}-\frac{1}{\mu}\right). \quad (28b)$$

Proof of Theorem 13. The Frobenius-Fuchs method [17] indicates how a linearly independent function of the second kind can be obtained in the case of coincident indices or indices differing by an integer. For $\nu = -n$ the terms of the series (2) starting with $j = n$ become infinite due to the zero in the denominator. This can be avoided going back to solution (5) for arbitrary index σ in (8)

$$Q_\sigma(z) = a_0(\sigma)\,z^\sigma\sum_{j=0}^{\infty}\left(-z^2\right)^j\prod_{l=1}^{j}\frac{1-\mu(l-1+\sigma/2)}{(2l+\sigma-\nu)(2l+\sigma+\nu)}. \quad (29)$$

Multiplying by $\sigma + n$ cancels, for $\nu = -n$, the zero in the denominator $2l+\sigma+\nu = 2l+\sigma-n$ when $l = n$, and the limit $\sigma \longrightarrow -n$ can be taken with a finite result. This leads to a constant multiple of the generalized Bessel function. The substitution of (29) with a factor $\sigma+n$ in the generalized Bessel equation (1) leads to

$$z^2Q_\sigma'' + z\left(1-\frac{\mu}{2}z^2\right)Q_\sigma' + \left(z^2-\nu^2\right)Q_\sigma = a_0(\sigma)(\sigma-n)(\sigma+n)^2; \quad (30)$$

since the recurrence formula (8) ensures that all terms vanish except the first, as $\sigma \longrightarrow \pm n$ the r.h.s. (right-hand side) of (30) vanishes, leading in both cases to the generalized Bessel function or a constant multiple. The r.h.s. of (30) also vanishes taking the limit $\sigma \longrightarrow -n$ after differentiation with regard to σ, leading to the solution

$$Y_n^\mu(z) = \frac{2}{\pi}\lim_{\sigma\longrightarrow -n}\frac{\partial}{\partial\sigma}\left[(\sigma+n)\,Q_\sigma(z)\right]; \quad (31)$$

this solution involves a logarithmic term arising from $\partial(z^\sigma)/\partial\sigma = z^\sigma\log z$ in (31) and thus is linearly independent of the generalized Bessel function (2). It is designated generalized Neumann function and the factor $2/\pi$ was inserted for consistency with the usual definition [18] of the original Neumann function for $\mu = 0$. When substituting the series (31) into (29) the first n terms are separated to specify a preliminary function

$$X_n^\mu(z) = \frac{2}{\pi} \lim_{\sigma \longrightarrow -n} \frac{\partial}{\partial \sigma} \left\{ (\sigma+n)\, a_0(\sigma) \cdot \sum_{j=0}^{n-1} z^{\sigma+2j} (-1)^j \prod_{l=1}^{j} \frac{1-\mu(l-1+\sigma/2)}{(2l+\sigma-n)(2l+\sigma+n)} \right\}. \tag{32}$$

If the derivative $\partial/\partial\sigma$ is not applied to $(\sigma + n)$ this factor remains and leads to zero in the limit $\sigma \longrightarrow -n$; thus the only nonzero terms arise differentiating $\partial(\sigma+n)/\partial\sigma = 1$, that is, suppressing the factor $(\sigma+n)$ and taking the limit $\sigma \longrightarrow -n$ in the remaining terms leading to

$$\begin{aligned} X_n^\mu(z) &= \frac{2}{\pi} a_0(-n) \sum_{j=0}^{n-1} z^{2j-n} (-1)^j \cdot \prod_{l=1}^{j} \frac{1-\mu(l-1-n/2)}{4l(l-n)} = \frac{2}{\pi} a_0(-n) \\ &\quad \cdot z^{-n} \sum_{j=0}^{n-1} \frac{(z/2)^{2j}}{j!} \frac{1}{(n-1)\dots(n-j)} \prod_{l=0}^{j-1} \left[1 - \mu\left(l - \frac{n}{2}\right)\right] \end{aligned} \tag{33}$$

Choosing the leading coefficient,

$$a_0(-n) = -(n-1)!\,2^{n-1} \left\{ \prod_{l=0}^{n-1} \left[1 - \mu\left(l - \frac{n}{2}\right)\right] \right\}^{-1}, \tag{34}$$

and substituting in (33) specifies the preliminary function (24). The first two factors in (34) were chosen in agreement with the preliminary function for the original Neumann function; the factor in curly brackets is relevant for $\mu \neq 0$ and affects only the generalized Neumann function. The second part of the generalized Neumann function (31) involves the remaining terms of the series (29) stating with $j = n$ for which the factor in curved brackets cancels the same factor in the denominator:

$$\frac{\pi}{2} \left[Y_n^\mu(z) - X_n^\mu(z)\right] = \lim_{\sigma \longrightarrow -n} \frac{\partial}{\partial \sigma} \left\{ (\sigma+n)\, a_0(\sigma) \cdot z^\sigma \sum_{j=n}^{\infty} \left(-z^2\right)^j \prod_{l=1}^{j} \frac{1-\mu(l-1+\sigma/2)}{(2l+\sigma-\nu)(2l+\sigma+\nu)} \right\}. \tag{35}$$

The leading coefficient is chosen (36) to suppress the first n factors in the last product in (35),

$$a_0(\sigma) = 2^{-2n} 2^{-\sigma} \frac{(-1)^n}{\sigma+n} \prod_{l=1}^{n} \frac{(2l+\sigma-\nu)(2l+\sigma+\nu)}{1-\mu(l-1+\sigma/2)}, \tag{36}$$

simplifying (35) to (37c),

$$k = j - n, \tag{37a}$$

$$\beta = l - n: \tag{37b}$$

$$\begin{aligned} \frac{\pi}{2} \left[Y_n^\mu(z) - X_n^\mu(z)\right] &= \lim_{\sigma \longrightarrow -n} \frac{\partial}{\partial \sigma} \left\{ (-1)^n 2^{-2n} \left(\frac{z}{2}\right)^\sigma \cdot \sum_{j=n}^{\infty} \left(-z^2\right)^j \prod_{l=n+1}^{j} \frac{1-\mu(l-1+\sigma/2)}{(2l+\sigma+n)(2l+\sigma-n)} \right\} \\ &= \lim_{\sigma \longrightarrow -n} \frac{\partial}{\partial \sigma} \left\{ \left(\frac{z}{2}\right)^{2n} \left(\frac{z}{2}\right)^\sigma \cdot \sum_{k=0}^{\infty} \left(-z^2\right)^k \prod_{\beta=1}^{k} \frac{1-\mu(\beta+n-1+\sigma/2)}{(2\beta+\sigma+3n)(2\beta+\sigma+n)} \right\} \end{aligned} \tag{37c}$$

where (37a) and (37b) were used. The differentiation with regard to σ of $(z/2)^\sigma$ inserts the factor $\log(z/2)$ multiplying the generalized Bessel function of the first kind (2),

$$\frac{\pi}{2} \left[Y_n^\mu(z) - X_n^\mu(z)\right] = \log\left(\frac{z}{2}\right) J_n^\mu(z) + \frac{\pi}{2} Z_n^\mu(z), \tag{38}$$

and the remaining term is a complementary function. The complementary function corresponds to differentiation with regard to σ after the factor $(z/2)^\sigma$ in (37a), (37b), and (37c) and replacing the dummy summation indices (k, β) by (j, l) leads to

$$\begin{aligned} Z_n^\mu(z) &= \frac{2}{\pi} \left(\frac{z}{2}\right)^n \\ &\quad \cdot \sum_{j=0}^{\infty} (-1)^j z^{2j} \left\{ \lim_{\sigma \longrightarrow -n} \left[\prod_{l=1}^{j} \frac{1-\mu(l+n-1+\sigma/2)}{(2l+3n+\sigma)(2l+\sigma+n)} \sum_{\alpha=1}^{j} \left(-\frac{1}{2\alpha+\sigma+3n} - \frac{1}{2\alpha+\sigma+n} + \frac{1}{2\alpha+\sigma+2n-2-2/\mu} \right) \right] \right\} \end{aligned} \tag{39}$$

and then (25) follows where ψ is the digamma function [8, 13]. Thus, the generalized Neumann function of order n and degree μ is given by (38)≡(23). □

3.2. Generalized Integral for Arbitrary Degree and Integer Order. Using the Wronskian it is proven that the generalized Neumann function is linearly independent from the generalized Bessel function, and thus their linear combination specifies the general integral of the generalized Bessel equation (1) in the case missing from (18a) and (18b), namely, integer order, $\nu = n$, and arbitrary degree.

Lemma 16. *The Wronskian of the generalized Bessel and Neumann functions of complex degree μ and integer order n is*

$$W\left(J_n^\mu(z), Y_n^\mu(z)\right) = \frac{2}{\pi z} A(\mu, n) \exp\left(\frac{1}{4}\mu z^2\right) \tag{40}$$

where $A(\mu, n)$ is given by

$$\begin{aligned}\frac{1}{A(\mu,n)} &\equiv \prod_{l=0}^{n-1}\left[1-\mu\left(l-\frac{n}{2}\right)\right] \\ &= (-\mu)^n \prod_{l=0}^{n-1}\left(l-\frac{n}{2}-\frac{1}{\mu}\right) \\ &= (-\mu)^n \frac{\Gamma(\mathrm{n}/2-1/\mu)}{\Gamma(-n/2-1/\mu)}.\end{aligned} \tag{41}$$

Remark 17. Coefficient (41) appears in relation (19b) between the generalized Bessel functions of any degree μ and integer order n:

$$n \in \mathbb{N} : J_{-n}^\mu(z) = (-1)^n A(\mu, n) J_n^\mu(z). \tag{42}$$

Remark 18. In the case of the original Bessel functions of zero degree coefficient (41) is unity (43a) simplifying (42) to the known [4, 11] relation (43b) between Bessel coefficients,

$$A(0, n) = 1 : \tag{43a}$$

$$J_{-n}(z) = (-1)^n J_n(z). \tag{43b}$$

Proof of Lemma 16. The generalized Neumann function (26) has leading term as $z \longrightarrow 0$,

$$Y_n^\mu(z) = -A(\mu, n)\frac{(n-1)!}{\pi}\left(\frac{z}{2}\right)^{-n}\left[1+\mathcal{O}\left(z^2\right)\right], \tag{44a}$$

$$\frac{d}{dz}\left[Y_n^\mu(z)\right] = A(\mu, n)\frac{n!}{2\pi}\left(\frac{z}{2}\right)^{-n-1}\left[1+\mathcal{O}\left(z^2\right)\right], \tag{44b}$$

involving the constant coefficient (41); this leads to the Wronskian (45a) with the generalized Bessel function (16a) and (16b) and also of integer order:

$$W\left(J_n^\mu(z), Y_n^\mu(z)\right) = \frac{2}{\pi z} A(\mu, n)\left[1+\mathcal{O}\left(z^2\right)\right]; \tag{45a}$$

$$W_0(\mu, n) = \frac{2}{\pi} A(\mu, n), \tag{45b}$$

and the Wronskian (45a) agrees with (12a) for (45b), specifying the exact Wronskian (40). □

Theorem 19. *The general integral of the generalized Bessel equation (1) is (46b) with C_1 and C_2 arbitrary constants,*

$$n = 0, \pm 1, \pm 2, \ldots : \tag{46a}$$

$$Q(z) = C_1 J_n^\mu(z) + C_2 Y_n^\mu(z), \tag{46b}$$

for integer order (46a) and arbitrary degree μ.

Proof. Both the generalized Bessel (2) and Neumann (26) functions satisfy the generalized Bessel equation (1) for arbitrary degree and integer order (46a); their Wronskian is nonzero so they are linearly independent and their linear combination (46b) specifies the general integral. □

4. Conclusions

The generalized Bessel differential equation appears for coupled acoustic-vortical wave problems, which would have satisfied the original Bessel differential equation in the decoupled acoustic or vortical case. The generalized Bessel differential equation (1) with order ν and degree μ reduces to the original Bessel differential equation for zero degree μ. The origin is a regular singularity and the other singularity is at infinity, so the Frobenius-Fuchs method specifies power series solutions valid in the finite complex plane. The indices are specified by the order but the recurrence formula for the coefficients depends also on the degree. Thus, the general integral of the generalized Bessel differential equation is a linear combination of generalized Bessel functions $J_{\pm\nu}^\mu$ if the order is not an integer; the generalized Bessel functions of integer order $J_{\pm n}^\mu$ are linearly dependent, as shown by an explicit relation between them. It follows that the general integral of the generalized Bessel differential equation for integer order and arbitrary degree requires the introduction of a generalized Neumann function. The generalized Bessel and Neumann functions have expressions that reduce to the original Bessel and Neumann functions for zero degree and also alternate expressions valid only for nonzero degree. The Bessel differential equation can be generalized further (see the Appendix) but in that case the coefficients on the power series solutions would no longer satisfy double recurrence formulas; the latter would be replaced by triple or higher order recurrence formulas.

The present paper is, as far as the authors know, the first concerning the generalized Bessel differential equation (1), and its regular integral solutions of the first and second kind in the finite complex plane that specify, respectively, the generalized Bessel and Neumann functions. There is a vast literature on the original Bessel differential equation and its solutions in terms of the original Bessel and Neumann functions: (i) starting with the first developments [1–3, 16]; (ii) detailed in several monographs [4, 7, 9–11, 13–15, 17, 18]; (iii) continuing with research papers up to the present time on various properties [19–26]. Several generalizations of the original Bessel function have been proposed, usually [26, 27] based on extensions of (2) without the last factor; these generalized Bessel functions are not derived from an explicit differential equation and are not associated with Neumann functions. The present approach is different in that it starts from a generalized Bessel differential equation (1) and leads to generalized Bessel (2) and Neumann (26) functions. The vast literature [1–26, 28] on the original Bessel differential equation and related Bessel and Neumann functions includes many properties that could potentially be extended to the generalized Bessel differential equation and generalized Bessel and Neumann functions and may be the subject of future work.

Appendix

Further Generalizations of the Bessel Differential Equation

The original Bessel differential equation

$$z^2 Q'' + zQ' + \left(z^2 - \nu^2\right) Q = 0. \tag{A.1}$$

can be generalized replacing the coefficients of Q' and Q by polynomials of the independent variable,

$$\begin{aligned} z^2 Q'' &+ z\left(1 + \sum_{n=1}^{N} \alpha_n z^n\right) Q' \\ &+ \left(z^2 - \nu^2 + \beta_1 z + \sum_{m=3}^{M} \beta_m z^m\right) Q = 0, \end{aligned} \tag{A.2}$$

where $(\alpha_1, \ldots, \alpha_N)$ and $(\beta_1, \beta_3, \ldots, \beta_M)$ are constants. The origin $z = 0$ remains a regular singularity of the differential equation (A.2) and the only other singularity is the point-at-infinity. Thus, there exists a solution as a Frobenius-Fuchs series (5), with coefficients satisfying the recurrence formula:

$$\begin{aligned} &\left[(\sigma + j)^2 - \nu^2\right] a_j(\sigma) \\ &\quad = -\left[\alpha_1 (\sigma + j - 1) + \beta_1\right] a_{j-1}(\sigma) \\ &\qquad - \left[1 + \alpha_2 (\sigma + j - 2)\right] a_{j-2}(\sigma) \\ &\qquad - \sum_{n=3}^{N} (\sigma + j - n)\, \alpha_n a_{j-n}(\sigma) - \sum_{m=3}^{M} \beta_m a_{j-m}(\sigma) \end{aligned} \tag{A.3}$$

In order to have a two-term recurrence formula, involving only $a_j(\sigma)$ and $a_{j-2}(\sigma)$ that can be solved explicitly, all coefficients must vanish (A.4a),

$$\alpha_1 = \alpha_3 = \ldots = \alpha_N = 0 = \beta_1 = \beta_3 = \ldots = \beta_M, \tag{A.4a}$$

$$\alpha_2 = -\frac{\mu}{2}, \tag{A.4b}$$

except (A.4b). Substitution of (A.4a) and (A.4b) into (A.2) leads to the generalized Bessel differential equation (1).

Note that the more general differential equation (A.2) still has the indices (7a)–(7e). Also, the recurrence formula (A.3) with three terms could be solved using continued fractions. For example, relaxing conditions (A.4a) to

$$\alpha_3 = \ldots = \alpha_N = 0 = \beta_3 = \ldots = \beta_M, \tag{A.5}$$

the differential equation (A.2) becomes

$$\alpha \equiv \alpha_1,$$

$$\beta \equiv \beta_1,$$

$$\alpha_2 = -\frac{\mu}{2},$$

$$\begin{aligned} &z^2 Q'' + z\left(1 + \alpha z - \frac{\mu}{2} z^2\right) Q' + \left(z^2 - \nu^2 + \beta z\right) Q \\ &\quad = 0, \end{aligned} \tag{A.6}$$

which reduces to the generalized Bessel equation (1) for $\alpha = 0 = \beta$. If $\alpha \neq 0$ or $\beta \neq 0$ the solution of (A.6) is still a Frobenius-Fuchs series (5) with recurrence formula for the coefficients,

$$\begin{aligned} &\left[(\sigma + j)^2 - \nu^2\right] a_j(\sigma) \\ &\quad = -\left[\alpha(\sigma + j - 1) + \beta\right] a_{j-1}(\sigma) \\ &\qquad - \left[1 - \frac{\mu}{2}(\sigma + j - 2)\right] a_{j-2}(\sigma) \end{aligned} \tag{A.7}$$

that can be solved [28] as the finite descending continued fraction,

$$\begin{aligned} \frac{a_j(\sigma)}{a_{j-1}(\sigma)} = &-\frac{\beta + \alpha(\sigma + j - 1)}{(\sigma + j)^2 - \nu^2} \\ &- \frac{1 - (\mu/2)(\sigma + j - 2)}{(\sigma + j)^2 - \nu^2} \\ &\cdot \frac{1}{a_{j-1}(\sigma) / a_{j-2}(\sigma)}, \end{aligned} \tag{A.8}$$

with (a_0, a_1) that are arbitrary constants as the starting values to calculate $a_2, \alpha_3, \ldots$. The method of continued fractions would no longer apply if the recurrence formula for the coefficients would have more than 3 terms.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by FCT (Foundation for Science and Technology) through IDMEC (Institute of Mechanical Engineering), under LAETA Pest, OE/EME/LA0022.

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