Research Article

On the Second-Order Quantum (p, q) -Difference Equations with Separated Boundary Conditions

Chanon Promsakon,¹ Nattapong Kamsrisuk,¹
Sotiris K. Ntouyas ^{2,3} and Jessada Tariboon ¹

¹Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand

²Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece

³Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to Jessada Tariboon; jessada.t@sci.kmutnb.ac.th

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In this paper, we investigate the existence and uniqueness of solutions for a boundary value problem for second-order quantum (p, q) -difference equations with separated boundary conditions, by using classical fixed point theorems. Examples illustrating the main results are also presented.

1. Introduction

Quantum calculus or q -calculus is known as the study of calculus without limits. The study of q -calculus was initiated by Euler on studying infinite series. In 1910, Jackson [1, 2] was the first one, who established the q -derivative or q -difference operator for a function f on $[0, \infty)$ by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \neq 0, \quad (1)$$

$$\text{and } D_q f(0) = \lim_{t \rightarrow 0} D_q f(t), \quad t = 0,$$

and gave its properties. The q -integral of a function f on $[0, \infty)$ is defined by

$$\int_0^t f(s) d_q s = (1-q)t \sum_{n=0}^{\infty} q^n f(q^n t), \quad (2)$$

provided that infinite series converges. The general theory of linear quantum difference equations was published in 1912 by Carmichael [3]. Details of its basic notions, results, and

methods can be found in the text [4]. For other papers on the subject, see [5–7]. In recent years, the topic has been attracting the attention of several researchers and a variety of new results can be found in [8–13] and the references therein. In addition, the classical quantum calculus was generalized to q_k -calculus by Tariboon and Ntouyas [14]. For details, we refer to the recent monograph [15].

There are some applications of q -calculus and difference equations to molecular problems in physics. In 1967, Finkelstein [16] studied behaviors of hydrogen atoms by using Schrödinger equation and q -calculus. In [17], the author investigated the q -field theory. The q -Coulomb problem and q -hydrogen atom were studied by [18–21]. In addition, Yang-Mills theories and also q -Yang-Mills equation were developed by [22–24]. The theory of quantum group applied to vibration and rotation molecules with q -algebra and q -Heisenberg algebra technique was established in [25–27]. The subject of elementary particle physics and chemical physics using q -calculus was investigated in [28–31]. The string theory involving q -calculus was studied in [32].

The subject of (p, q) -calculus first appeared in quantum algebras which contained two quantum numbers p and

q introduced by [33]. For some recent results see [34–41] and the references cited therein. Recently in [42], the authors initiated the study on boundary value problems for (p, q) -difference equations by investigating the existence and uniqueness of solution for first-order quantum (p, q) -difference equation subject to a nonlocal condition of the form

$$D_{p,q}x(t) = f(t, x(pt)), \quad t \in \left[0, \frac{T}{p}\right], \quad (3)$$

$$x(0) = \alpha x(T) + \sum_{i=1}^m \beta_i \int_0^{\eta_i} x(s) d_{p_i, q_i} s, \quad (4)$$

where $0 < q < p \leq 1$, $0 < q_i < p_i \leq 1$, $i = 1, 2, \dots, m$, are quantum numbers, $D_{p,q}$ is (p, q) -difference operator, $f \in C([0, T/p] \times \mathbb{R}, \mathbb{R})$, $T > 0$, $\alpha, \beta_i, i = 1, 2, \dots, m$, are given constants, and the points $\eta_i \in [0, p_i T]$, $i = 1, 2, \dots, m$. They proved existence and uniqueness results for problem (3)-(4) by using the classical fixed point theorems, such as Banach's fixed point theorem, Boyd and Wong fixed point theorem for nonlinear contractions, and Leray-Schauder nonlinear alternative.

In this paper, we continue the study on (p, q) boundary value problems, by considering the following second-order quantum (p, q) -difference equation with separated boundary conditions

$$D_{p,q}^2 x(t) = f(t, x(p^2 t)), \quad t \in \left[0, \frac{T}{p^2}\right], \quad (5)$$

$$\alpha_1 x(0) + \alpha_2 D_{p,q} x(0) = \alpha_3,$$

$$\beta_1 x(T) + \beta_2 D_{p,q} x\left(\frac{T}{p}\right) = \beta_3,$$

where $0 < q < p \leq 1$ are two given quantum numbers, $D_{p,q}^2$ is the second-order (p, q) -difference operator, and $f \in C([0, T/p^2] \times \mathbb{R}, \mathbb{R})$, $T > 0$, $\alpha_i, \beta_i, i = 1, 2, 3$, are given real constants. Some new existence results are established by using Schaefer fixed point theorem, Krasnoselskii's fixed point theorem, and Lelay-Schauder's nonlinear alternative. In addition the uniqueness of solutions is established via Banach contraction mapping principle.

The paper is organized as follows: In Section 2, we recall some definitions and basic facts from (p, q) -calculus. The main existence and uniqueness results are given in Section 3. Examples illustrating the obtained results are presented in Section 4.

2. Preliminaries

In this section, we recall some basic concepts of (p, q) -calculus (see [41]). The (p, q) -number is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad (6)$$

where $0 < q < p \leq 1$. For each $k, n \in \mathbb{N}$, $n \geq k \geq 0$, the (p, q) -factorial and (p, q) -binomial are defined by

$$[n]_{p,q}! = \prod_{k=1}^n [k]_{p,q}, \quad n \geq 1, \quad [0]_{p,q}! = 1, \quad (7)$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!}, \quad (8)$$

respectively.

Let $f : [0, T] \rightarrow \mathbb{R}$. The (p, q) -derivative of function f is defined as

$$D_{p,q}f(t) = \frac{f(pt) - f(qt)}{(p-q)t}, \quad t \neq 0, \quad (9)$$

and $D_{p,q}f(0) = \lim_{t \rightarrow 0} D_{p,q}f(t)$. Observe that the function $D_{p,q}f(t)$ is defined on $[0, T/p]$ which is extended from $[0, T]$ of a function $f(t)$. In addition, we say that f is (p, q) -differentiable on $[0, T/p]$ provided that $D_{p,q}f(t)$ exists for all $t \in [0, T/p]$.

Let a function $f : [0, T] \rightarrow \mathbb{R}$. Then the (p, q) -integral of f is defined by

$$\int_0^t f(s) d_{p,q}s = (p-q)t \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}t\right), \quad (10)$$

provided that the right hand side converges. Note that the domain of function $\int_0^t f(s) d_{p,q}s$ is $[0, pT]$ which shrinks from $[0, T]$ of a function $f(t)$ since $0 < p \leq 1$.

In the following theorems we collect the basic properties of (p, q) -differentiation and (p, q) -integration, respectively. See [37].

Theorem 1 (see [37]). Suppose that $f, g : [0, T] \rightarrow \mathbb{R}$ are (p, q) -differentiable on $[0, T/p]$. Then

(a) $f + g : [0, T] \rightarrow \mathbb{R}$ is (p, q) -differentiable on $[0, T/p]$, and

$$D_{p,q}(f(t) + g(t)) = D_{p,q}f(t) + D_{p,q}g(t). \quad (11)$$

(b) $\lambda f : [0, T] \rightarrow \mathbb{R}$ is (p, q) -differentiable on $[0, T/p]$ for any constant λ , and

$$D_{p,q}(\lambda f)(t) = \lambda D_{p,q}f(t). \quad (12)$$

(c) $fg : [0, T] \rightarrow \mathbb{R}$ is (p, q) -differentiable on $[0, T/p]$, and

$$\begin{aligned} D_{p,q}(fg)(t) &= f(pt) D_{p,q}g(t) + g(qt) D_{p,q}f(t). \\ D_{p,q}(fg)(t) &= g(pt) D_{p,q}f(t) + f(qt) D_{p,q}g(t). \end{aligned} \quad (13)$$

(d) If $g(t) \neq 0$, then f/g is (p, q) -differentiable on $[0, T/p]$ with

$$\begin{aligned} D_{p,q}\left(\frac{f}{g}\right)(t) &= \frac{g(qt) D_{p,q}f(t) - f(qt) D_{p,q}g(t)}{g(pt)g(qt)}. \\ D_{p,q}\left(\frac{f}{g}\right)(t) &= \frac{g(pt) D_{p,q}f(t) - f(pt) D_{p,q}g(t)}{g(pt)g(qt)}. \end{aligned} \quad (14)$$

Theorem 2 (see [37]). Let $f, g : [0, T] \rightarrow \mathbb{R}$ be continuous functions and constants $0 < q < p \leq 1, a, b \in [0, T]$. The following formulas hold:

(a) The (p, q) -integration by parts is given by

$$\begin{aligned} \int_a^b f(pt) D_{p,q} g(t) d_{p,q}t \\ = f(t) g(t) \Big|_a^b - \int_a^b g(qt) D_{p,q} f(t) d_{p,q}t. \end{aligned} \quad (15)$$

$$(b) D_{p,q} \int_0^t f(s) d_{p,q}s = f(t).$$

$$(c) \int_0^t D_{p,q} f(s) d_{p,q}s = f(t) - f(0).$$

$$(d) \int_a^t D_{p,q} f(s) d_{p,q}s = f(t) - f(a) \text{ where } a \in (0, t).$$

Theorem 3 (see [42]). Let a function $f : [0, T] \rightarrow \mathbb{R}$ and constants $0 < q < p \leq 1$. Then for $t \in [0, p^2 T]$, we have

$$\int_0^t \int_0^s f(r) d_{p,q}r d_{p,q}s = \frac{1}{p} \int_0^t (t - qs) f\left(\frac{1}{p}s\right) d_{p,q}s. \quad (16)$$

The second-order (p, q) -difference of a function $f : [0, T] \rightarrow \mathbb{R}$ can be expressed as

$$\begin{aligned} D_{p,q}^2 f(t) &= D_{p,q} (D_{p,q} f)(t) \\ &= D_{p,q} \left(\frac{f(pt) - f(qt)}{(p - q)t} \right) \\ &= \frac{f(p^2t) - f(pqt)}{p(p - q)^2 t^2} - \frac{f(pqt) - f(q^2t)}{q(p - q)^2 t^2} \\ &= \frac{qf(p^2t) - qf(pqt) - pf(pqt) + pf(q^2t)}{pq(p - q)^2 t^2} \\ &= \frac{qf(p^2t) - (p + q)f(pqt) + pf(q^2t)}{pq(p - q)^2 t^2}, \\ &t \in \left[0, \frac{T}{p^2}\right]. \end{aligned} \quad (17)$$

Now, for $h : [0, T/p^2] \rightarrow \mathbb{R}$, we transform the linear second-order (p, q) -difference equation with separated boundary condition

$$\begin{aligned} D_{p,q}^2 x(t) &= h(t), \quad t \in \left[0, \frac{T}{p^2}\right], \\ \alpha_1 x(0) + \alpha_2 D_{p,q} x(0) &= \alpha_3, \\ \beta_1 x(T) + \beta_2 D_{p,q} x\left(\frac{T}{p}\right) &= \beta_3, \end{aligned} \quad (18)$$

into an integral equation which will be used to define the solution of the boundary value problem (5). For convenience we put constants $\Lambda = \alpha_1(\beta_1 T + \beta_2) - \alpha_2 \beta_1 \neq 0$,

$$\begin{aligned} \Lambda_1 &= \frac{\alpha_3(\beta_1 T + \beta_2) - \alpha_2 \beta_3}{\Lambda}, \\ \Lambda_2 &= \frac{\alpha_1 \beta_3 - \alpha_3 \beta_1}{\Lambda}. \end{aligned} \quad (19)$$

Lemma 4. The unique solution of linear problem (18) is given by

$$\begin{aligned} x(t) &= \Lambda_1 + \Lambda_2 t \\ &+ \frac{(\alpha_2 - \alpha_1 t) \beta_1}{p \Lambda} \int_0^T (T - qs) h\left(\frac{s}{p}\right) d_{p,q}s \\ &+ \frac{(\alpha_2 - \alpha_1 t) \beta_2}{\Lambda} \int_0^{T/p} h(s) d_{p,q}s \\ &+ \frac{1}{p} \int_0^t (t - qs) h\left(\frac{s}{p}\right) d_{p,q}s. \end{aligned} \quad (20)$$

Proof. Applying (p, q) -integration to the first equation of (18), we have

$$D_{p,q} x(t) - D_{p,q} x(0) = \int_0^t h(s) d_{p,q}s, \quad t \in \left[0, \frac{T}{p}\right]. \quad (21)$$

Taking (p, q) -integration to (21) and using Theorem 3, we have

$$\begin{aligned} x(t) &= x(0) + D_{p,q} x(0) t \\ &+ \frac{1}{p} \int_0^t (t - qs) h\left(\frac{s}{p}\right) d_{p,q}s, \quad t \in [0, T]. \end{aligned} \quad (22)$$

For convenience, we set constants $x(0) = A, D_{p,q} x(0) = B$ and then

$$x(t) = A + Bt + \frac{1}{p} \int_0^t (t - qs) h\left(\frac{s}{p}\right) d_{p,q}s. \quad (23)$$

In particular for $t = T$ in (23), we have

$$x(T) = A + BT + \frac{1}{p} \int_0^T (T - qs) h\left(\frac{s}{p}\right) d_{p,q}s. \quad (24)$$

Using (21) and by replacing $t = T/p$, we obtain

$$D_{p,q} x\left(\frac{T}{p}\right) = B + \int_0^{T/p} h(s) d_{p,q}s. \quad (25)$$

From the first boundary condition of (18) and the above setting constants, we get

$$\alpha_1 A + \alpha_2 B = \alpha_3. \quad (26)$$

The second boundary condition of (18) with (24) and (25) yields

$$\begin{aligned} \beta_1 \left(A + BT + \frac{1}{p} \int_0^T (T - qs) h\left(\frac{s}{p}\right) d_{p,q}s \right) \\ + \beta_2 \left(B + \int_0^{T/p} h(s) d_{p,q}s \right) = \beta_3. \end{aligned} \quad (27)$$

Solving (26) and (27) for constants A and B , it follows that

$$A = \Lambda_1 + \frac{1}{\Lambda} \left(\frac{\alpha_2 \beta_1}{p} \int_0^T (T - qs) h\left(\frac{s}{p}\right) d_{p,q}s + \alpha_2 \beta_2 \int_0^{T/p} h(s) d_{p,q}s \right) \quad (28)$$

and

$$B = \Lambda_2 - \frac{1}{\Lambda} \left(\frac{\alpha_1 \beta_1}{p} \int_0^T (T - qs) h\left(\frac{s}{p}\right) d_{p,q}s + \alpha_1 \beta_2 \int_0^{T/p} h(s) d_{p,q}s \right). \quad (29)$$

Substituting constants A and B into (23), we obtain the unique solution in (20) of linear problem (18). The proof is completed. $\square$

3. Main Results

Let $\mathcal{C} := C([0, T], \mathbb{R})$ denote the Banach space of all continuous functions from $[0, T]$ to $\mathbb{R}$ endowed with sup-norm $\|x\| = \sup\{|x(t)| : t \in [0, T]\}$. In view of Lemma 4 we define an operator $\mathcal{A} : \mathcal{C} \rightarrow \mathcal{C}$ by

$$\begin{aligned} (\mathcal{A}x)(t) &= \Lambda_1 + \Lambda_2 t \\ &+ \frac{(\alpha_2 - \alpha_1 t) \beta_1}{p \Lambda} \int_0^T (T - qs) f(s, x(ps)) d_{p,q}s \\ &+ \frac{(\alpha_2 - \alpha_1 t) \beta_2}{\Lambda} \int_0^{T/p} f(s, x(p^2 s)) d_{p,q}s \\ &+ \frac{1}{p} \int_0^t (t - qs) f(s, x(ps)) d_{p,q}s. \end{aligned} \quad (30)$$

It should be noticed that problem (5) has solutions if and only if the operator $\mathcal{A}$ of equation $x = \mathcal{A}x$ has fixed points. Our first result is an existence theorem for separated boundary value problem of quantum (p, q) -difference equation (5) by using Schaefer's fixed point theorem.

Theorem 5. Assume that

(H_1) $f : [0, T/p^2] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and there exists a constant $N > 0$ such that

$$|f(t, x(p^2 t))| \leq N, \quad (31)$$

for each $t \in [0, T/p^2]$ and all $x \in \mathbb{R}$.

Then the separated boundary value problem (5) has at least one solution on $[0, T]$.

Proof. Schaefer's fixed point theorem is used to prove that the operator $\mathcal{A}$ defined by (30) has at least one fixed point. So, we divide the proof into four steps.

Step 1 (continuity of $\mathcal{A}$). Let $\{x_n\}$ be a sequence of function such that $x_n \rightarrow x$ on $[0, T/p^2]$. Given that f is continuous function on $[0, T/p^2]$, we have

$$f(t, x_n(p^2 t)) \rightarrow f(t, x(p^2 t)). \quad (32)$$

Therefore, we obtain

$$\begin{aligned} |(\mathcal{A}x_n)(t) - (\mathcal{A}x)(t)| &\leq \frac{(|\alpha_2| + |\alpha_1|T)|\beta_1|}{p|\Lambda|} \\ &\cdot \int_0^T (T - qs) |f(s, x_n(ps)) - f(s, x(ps))| d_{p,q}s \\ &+ \frac{(|\alpha_2| + |\alpha_1|T)|\beta_2|}{|\Lambda|} \\ &\cdot \int_0^{T/p} |f(s, x_n(p^2 s)) - f(s, x(p^2 s))| d_{p,q}s + \frac{1}{p} \\ &\cdot \int_0^t (t - qs) |f(s, x_n(ps)) - f(s, x(ps))| d_{p,q}s, \end{aligned} \quad (33)$$

which implies that $\|\mathcal{A}x_n - \mathcal{A}x\| \rightarrow 0$ as $n \rightarrow \infty$. Hence the operator $\mathcal{A}$ is continuous.

Step 2 ($\mathcal{A}$ maps bounded sets into bounded sets in $[0, T]$). Choosing $r > 0$, we define a bounded ball B_r as $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. Then, for any $x \in B_r$, we have

$$\begin{aligned} |(\mathcal{A}x)(t)| &\leq |\Lambda_1| + |\Lambda_2 t| + \frac{(|\alpha_2| + |\alpha_1|T)|\beta_1|}{p|\Lambda|} \\ &\cdot \int_0^T (T - qs) |f(s, x(ps))| d_{p,q}s \\ &+ \frac{(|\alpha_2| + |\alpha_1|T)|\beta_2|}{|\Lambda|} \int_0^{T/p} |f(s, x(p^2 s))| d_{p,q}s \\ &+ \frac{1}{p} \int_0^t (t - qs) |f(s, x(ps))| d_{p,q}s \leq |\Lambda_1| + |\Lambda_2 T| \\ &+ \frac{(|\alpha_2| + |\alpha_1|T)|\beta_1|}{p|\Lambda|} \int_0^T (T - qs) N d_{p,q}s \\ &+ \frac{(|\alpha_2| + |\alpha_1|T)|\beta_2|}{|\Lambda|} \int_0^{T/p} N d_{p,q}s + \frac{1}{p} \\ &\cdot \int_0^T (T - qs) N d_{p,q}s = |\Lambda_1| + |\Lambda_2 T| \\ &+ \frac{(|\alpha_2| + |\alpha_1|T)T^2|\beta_1|}{(p+q)|\Lambda|} N + \frac{(|\alpha_2| + |\alpha_1|T)T|\beta_2|}{p|\Lambda|} \\ &\cdot N + \frac{T^2}{p+q} N := \rho \end{aligned} \quad (34)$$

This means that $\|\mathcal{A}x\| \leq \rho$. Therefore, the set $\mathcal{A}B_r$ is uniformly bounded.

Step 3 ($\mathcal{A}$ maps bounded sets into equicontinuous sets of $[0, T]$). Let $\tau_1, \tau_2 \in [0, T]$ with $\tau_1 < \tau_2$ be two points and B_r be a bounded ball in $\mathcal{C}$. Then for any $x \in B_r$, we get

$$\begin{aligned}
|(\mathcal{A}x)(\tau_2) - (\mathcal{A}x)(\tau_1)| &\leq (\tau_2 - \tau_1) |\Lambda_2| \\
&+ \frac{(\tau_2 - \tau_1) |\alpha_1 \beta_1|}{p |\Lambda|} \int_0^T (T - qs) |f(s, x(ps))| d_{p,q}s \\
&+ \frac{(\tau_2 - \tau_1) |\alpha_1 \beta_2|}{|\Lambda|} \int_0^{T/p} |f(s, x_n(p^2s))| d_{p,q}s \\
&+ \frac{1}{p} \left| \int_0^{\tau_2} (\tau_2 - qs) f(s, x(ps)) d_{p,q}s \right. \\
&\left. - \int_0^{\tau_1} (\tau_1 - qs) f(s, x(ps)) d_{p,q}s \right| \leq (\tau_2 - \tau_1) |\Lambda_2| \\
&+ \frac{(\tau_2 - \tau_1) |\alpha_1 \beta_1|}{p |\Lambda|} \int_0^T (T - qs) |f(s, x(ps))| d_{p,q}s \\
&+ \frac{(\tau_2 - \tau_1) |\alpha_1 \beta_2|}{|\Lambda|} \int_0^{T/p} |f(s, x_n(p^2s))| d_{p,q}s \\
&+ \frac{1}{p} \left| \int_0^{\tau_1} (\tau_2 - \tau_1) f(s, x(ps)) d_{p,q}s \right. \\
&\left. + \int_{\tau_1}^{\tau_2} (\tau_2 - qs) f(s, x(ps)) d_{p,q}s \right| \leq (\tau_2 - \tau_1) |\Lambda_2| \\
&+ \frac{(\tau_2 - \tau_1) |\alpha_1 \beta_1|}{p |\Lambda|} \int_0^T (T - qs) |f(s, x(ps))| d_{p,q}s \\
&+ \frac{(\tau_2 - \tau_1) |\alpha_1 \beta_2|}{|\Lambda|} \int_0^{T/p} |f(s, x_n(p^2s))| d_{p,q}s \\
&+ \frac{N}{p} \left\{ \int_0^{\tau_1} (\tau_2 - \tau_1) d_{p,q}s + \int_{\tau_1}^{\tau_2} (\tau_2 - qs) d_{p,q}s \right\}.
\end{aligned} \tag{35}$$

As $\tau_1 \rightarrow \tau_2$, the right-hand side of the above inequality (which is independent of x) tends to zero. This shows that the set $\mathcal{A}B_r$ is equicontinuous set. From a consequence of Steps 1 to 3, together with the Arzelà-Ascoli theorem, we deduce that the operator $\mathcal{A} : \mathcal{C} \rightarrow \mathcal{C}$ is completely continuous.

Step 4. Finally, we show that the set

$$E = \{x \in \mathcal{C} : x = \theta \mathcal{A}x \text{ for some } 0 < \theta < 1\} \tag{36}$$

is bounded.

Let $x \in E$ be a solution of problem (5). Then $x(t) = \theta(\mathcal{A}x)(t)$ for some $0 < \theta < 1$. Hence, for each $t \in [0, T]$, by the method of computation in Step 2, we obtain

$$\|\mathcal{A}x\| \leq \rho. \tag{37}$$

This gives that the set E is bounded. By applying Schaefer's fixed point theorem, we get that $\mathcal{A}$ has at least one fixed point which is a solution of the second-order quantum (p, q) -difference equation with separated boundary value problem (5) on $[0, T]$. The proof is completed. $\square$

The second existence of a unique solution is based on Banach's contraction mapping principle. For convenience we set a positive constant

$$\Phi = \frac{(|\alpha_2| + |\alpha_1|T)T}{|\Lambda|} \left\{ \frac{|\beta_1|T}{p+q} + \frac{|\beta_2|}{p} \right\} + \frac{T^2}{p+q}. \tag{38}$$

Theorem 6. Assume that the function $f : [0, T/p^2] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfying the assumption:

(H₂) There exists constant $L > 0$ such that $|f(t, x) - f(t, y)| \leq L|x - y|$ for each $t \in [0, T/p^2]$ and $x, y \in \mathbb{R}$.

If

$$L\Phi < 1, \tag{39}$$

where Φ is defined by (38), then the boundary value problem of quantum (p, q) -difference equation (5) has a unique solution on $[0, T]$.

Proof. The existence of a unique fixed point of operator equation $x = \mathcal{A}x$, where the operator $\mathcal{A}$ is defined by (30), is proved by using Banach's contraction mapping principle. Let us define a ball $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ with the value r satisfying

$$r \geq \frac{|\Lambda_1| + |\Lambda_2 T| + M\Phi}{1 - L\Phi}, \tag{40}$$

where $M = \sup_{t \in [0, T/p^2]} |f(t, 0)|$. Note that the above inequality is well defined as $L\Phi < 1$. Next, we will show that $\mathcal{A}B_r \subset B_r$. For any $x \in B_r$, we have

$$\begin{aligned}
|(\mathcal{A}x)(t)| &\leq \sup_{t \in [0, T/p^2]} \left\{ |\Lambda_1| + |\Lambda_2|t + \frac{(|\alpha_2 - \alpha_1 t| \beta_1)}{p |\Lambda|} \right. \\
&\cdot \int_0^T (T - qs) |f(s, x(ps))| d_{p,q}s \\
&+ \frac{(|\alpha_2 - \alpha_1 t| \beta_2)}{|\Lambda|} \int_0^{T/p} |f(s, x_n(p^2s))| d_{p,q}s + \frac{1}{p} \\
&\cdot \int_0^T (T - qs) |f(s, x(ps))| d_{p,q}s \left. \right\} \leq |\Lambda_1| \\
&+ |\Lambda_2|T + \frac{(|\alpha_2| + |\alpha_1|T) |\beta_1|}{|p\Lambda|} \int_0^T (T - qs) \\
&\cdot (|f(s, x(ps)) - f(s, 0)| + |f(s, 0)|) d_{p,q}s \\
&+ \frac{(|\alpha_2| + |\alpha_1|T) |\beta_2|}{|\Lambda|} \\
&\cdot \int_0^{T/p} (|f(s, x(p^2s)) - f(s, 0)| \\
&+ |f(s, 0)|) d_{p,q}s + \frac{1}{p} \int_0^T (T - qs) \\
&\cdot (|f(s, x(ps)) - f(s, 0)| + |f(s, 0)|) d_{p,q}s
\end{aligned}$$

$$\begin{aligned}
&\leq |\Lambda_1| + |\Lambda_2| T + (L \|x\| + M) \\
&\cdot \left\{ \frac{(|\alpha_2| + |\alpha_1| T) |\beta_1|}{p |\Lambda|} \int_0^T (T - qs) d_{p,q}s \right. \\
&+ \frac{(|\alpha_2| + |\alpha_1| T) |\beta_2|}{|\Lambda|} \int_0^{T/p} d_{p,q}s + \frac{1}{p} \\
&\cdot \int_0^T (T - qs) d_{p,q}s \left. \right\} \leq |\Lambda_1| + |\Lambda_2| T + (Lr \\
&+ M) \left\{ \frac{(|\alpha_2| + |\alpha_1| T) T^2 |\beta_1|}{(p + q) |\Lambda|} \right. \\
&+ \frac{(|\alpha_2| + |\alpha_1| T) T |\beta_2|}{p |\Lambda|} + \frac{T^2}{p + q} \left. \right\} = |\Lambda_1| + |\Lambda_2| T \\
&+ (Lr + M) \Phi \leq r,
\end{aligned} \tag{41}$$

which implies that $\|\mathcal{A}x\| < r$. Therefore, we have $\mathcal{A}B_r \subset B_r$. Finally we shall show that $\mathcal{A}$ is a contraction. For any $x, y \in \mathcal{C}$ and for each $t \in [0, T]$, we obtain

$$\begin{aligned}
| \mathcal{A}x(t) - \mathcal{A}y(t) | &\leq \frac{(|\alpha_2| + |\alpha_1| T) |\beta_1|}{|p\Lambda|} \int_0^T (T - qs) \\
&\cdot (|f(s, x(ps)) - f(s, y(ps))|) d_{p,q}s \\
&+ \frac{(|\alpha_2| + |\alpha_1| T) |\beta_2|}{|\Lambda|} \int_0^{T/p} (|f(s, x(p^2s)) \\
&- f(s, y(p^2s))|) d_{p,q}s + \frac{1}{p} \int_0^T (T - qs) \\
&\cdot (|f(s, x(ps)) - f(s, y(ps))|) d_{p,q}s \\
&\leq \|x - y\| \left\{ \frac{(|\alpha_2| + |\alpha_1| T) T^2 |\beta_1|}{(p + q) |\Lambda|} \right. \\
&+ \frac{(|\alpha_2| + |\alpha_1| T) T |\beta_2|}{p |\Lambda|} + \frac{T^2}{p + q} \left. \right\} = L\Phi \|x - y\|.
\end{aligned} \tag{42}$$

Therefore, we have $\|\mathcal{A}x - \mathcal{A}y\| \leq L\Phi \|x - y\|$. As $L\Phi < 1$, then the operator $\mathcal{A}$ is a contraction. Hence, by the Banach contraction mapping principle, it follows that the operator $\mathcal{A}$ has a unique fixed point which is the unique solution of problem (5) on $[0, T]$. This completes the proof. $\square$

The third existence theorem is based on Lelay-Schauder's nonlinear alternative.

Lemma 7 (nonlinear alternative for single-value maps [43]). *Let E be a Banach space, C be a closed, convex subset of E , U be an open subset of C , and $0 \in U$. Suppose that $A : \overline{U} \rightarrow C$ is continuous and compact (that is, $A(\overline{U})$ is a relatively compact subset of C map). Then either*

- (i) A has a fixed point in $\overline{U}$, or

- (ii) there is a $x \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $x = \lambda A(x)$.

Theorem 8. Assume that $f : [0, T/p^2] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. In addition we suppose that

- (H₃) there exist a continuous nondecreasing function $\psi : [0, \infty) \rightarrow (0, \infty)$ and a function $z \in C([0, T/p^2], \mathbb{R}^+)$ such that

$$|f(t, x)| \leq z(t) \psi(\|x\|) \quad \text{for each } (t, x) \in \left[0, \frac{T}{p^2}\right] \times \mathbb{R}; \tag{43}$$

- (H₄) there exists a constant $K > 0$ such that

$$\frac{K}{|\Lambda_1| + |\Lambda_2| T + \|z\| \psi(K) \Phi} > 1. \tag{44}$$

Then the separated boundary value problem (5) has at least one solution on $[0, T]$.

Proof. As in the proof of Theorem 5, the operator $\mathcal{A} : \mathcal{C} \rightarrow \mathcal{C}$ is completely continuous. The result will follow from the Lelay-Schauder nonlinear alternative (Lemma 7) once we have proved the boundedness of the set of all solutions to equations $x = \theta \mathcal{A}x$ for $\theta \in [0, 1]$.

Let x be a solution of problem (5). Then, from (H₃), we have

$$\begin{aligned}
\|x\| &\leq |\Lambda_1| + |\Lambda_2 T| + \|z\| \psi(\|x\|) \\
&\cdot \left\{ \frac{(|\alpha_2| + |\alpha_1| T) T^2 |\beta_1|}{(p + q) |\Lambda|} + \frac{(|\alpha_2| + |\alpha_1| T) T |\beta_2|}{p |\Lambda|} \right. \\
&+ \frac{T^2}{p + q} \left. \right\} = |\Lambda_1| + |\Lambda_2 T| + \|z\| \psi(\|x\|) \Phi.
\end{aligned} \tag{45}$$

Thus, we obtain

$$\frac{\|x\|}{|\Lambda_1| + |\Lambda_2 T| + \|z\| \psi(\|x\|) \Phi} \leq 1. \tag{46}$$

In view of (H₄), there exists a positive constant K such that $\|x\| \neq K$. Let us define the set

$$U = \{x \in \mathcal{C} : \|x\| < K\}. \tag{47}$$

Note that the operator $\mathcal{A} : \overline{U} \rightarrow \mathcal{C}$ is continuous and completely continuous. From the choice of U , there is no $x \in \partial U$ such that $x = \theta \mathcal{A}x$ for some $\theta \in (0, 1)$. Therefore, by applying the nonlinear alternative of Leray-Schauder type, we can conclude that the operator $\mathcal{A}$ has at least one fixed point in $\overline{U}$, which is a solution of the quantum (p, q) -difference boundary value problem (5) on $[0, T]$. The proof is completed. $\square$

The final existence theorem is established by using Krasnoselskii's fixed point theorem.

Lemma 9 (Krasnoselskii's fixed point theorem [44]). *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X . Let A, B be the operators such that (a) $Ax + By \in M$ whenever $x, y \in M$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 10. *Assume that $f : [0, T/p^2] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying the assumption (H_2) . In addition we suppose that*

$(H_5) |f(t, x)| \leq \kappa(t), \forall (t, x) \in [0, T/p^2] \times \mathbb{R}$ and $\kappa \in C([0, T/p^2], \mathbb{R}^+)$.

If

$$\left(\frac{T^2}{p+q} \right) L < 1, \quad (48)$$

then the separated boundary value problem of quantum (p, q) -difference equation (5) has at least one solution on $[0, T]$.

Proof. Let us define $\sup_{t \in [0, T/p^2]} |\kappa(t)| = \|\kappa\|$ and choose a suitable constant $\bar{r}$ as

$$\bar{r} \geq |\Lambda_1| + |\Lambda_2| T + \|\kappa\| \Phi, \quad (49)$$

where Φ is defined by (38). Furthermore, we define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\bar{r}} = \{x \in \mathcal{C} : \|x\| \leq \bar{r}\}$ by

$$\begin{aligned} (\mathcal{P}x)(t) &= \frac{(\alpha_2 - \alpha_1 t) \beta_1}{p\Lambda} \int_0^T (T - qs) f(s, x(ps)) d_{p,q}s \\ &\quad + \frac{(\alpha_2 - \alpha_1 t) \beta_2}{\Lambda} \int_0^{T/p} f(s, x(p^2s)) d_{p,q}s, \end{aligned} \quad (50)$$

$$\begin{aligned} (\mathcal{Q}y)(t) &= \Lambda_1 + \Lambda_2 t + \frac{1}{p} \int_0^t (t - qs) f(s, y(ps)) d_{p,q}s. \end{aligned}$$

Observe that $\mathcal{P}x + \mathcal{Q}y = \mathcal{A}x$. For $x, y \in B_{\bar{r}}$, we have

$$\begin{aligned} \|\mathcal{P}x + \mathcal{Q}y\| &\leq |\Lambda_1| + |\Lambda_2 T| + \|\kappa\| \\ &\quad \cdot \left\{ \frac{(|\alpha_2| + |\alpha_1| T) T^2 |\beta_1|}{(p+q) |\Lambda|} + \frac{(|\alpha_2| + |\alpha_1| T) T |\beta_2|}{p |\Lambda|} \right. \\ &\quad \left. + \frac{T^2}{p+q} \right\} = |\Lambda_1| + |\Lambda_2 T| + \|\kappa\| \Phi \leq \bar{r}, \end{aligned} \quad (51)$$

which yields $\mathcal{P}x + \mathcal{Q}y \in B_{\bar{r}}$. Therefore, the condition (a) of Lemma 9 is satisfied. It follows from the assumption (H_2) together with (48) that $\mathcal{Q}$ is a contraction mapping. Hence the condition (c) of Lemma 9 is fulfilled. Now we will show that the condition (b) of Lemma 9 is satisfied. Given that function

f is continuous, we get that the operator $\mathcal{P}$ is continuous. It is easy to verify that

$$\begin{aligned} \|\mathcal{P}x\| &\leq \|\kappa\| \left\{ \frac{(|\alpha_2| + |\alpha_1| T) T^2 |\beta_1|}{(p+q) |\Lambda|} \right. \\ &\quad \left. + \frac{(|\alpha_2| + |\alpha_1| T) T |\beta_2|}{p |\Lambda|} \right\}. \end{aligned} \quad (52)$$

Thus, the set $\mathcal{P}(B_{\bar{r}})$ is uniformly bounded. Next we prove the compactness of the operator $\mathcal{P}$. Give $\sup_{(t,x) \in [0, T/p^2] \times B_{\bar{r}}} |f(t, x)| = \bar{f} < \infty$ and let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then we get

$$\begin{aligned} |(\mathcal{P}x)(t_2) - (\mathcal{P}x)(t_1)| &\leq \frac{|\alpha_1 \beta_1| (t_2 - t_1)}{p |\Lambda|} \int_0^T (T - qs) |f(s, x(ps))| d_{p,q}s \\ &\quad + \frac{|\alpha_1 \beta_2| (t_2 - t_1)}{|\Lambda|} \int_0^{T/p} |f(s, x(p^2s))| d_{p,q}s \\ &\leq \bar{f} \left\{ \frac{|\alpha_1 \beta_1| T^2}{(p+q) |\Lambda|} + \frac{|\alpha_1 \beta_2| T}{|\Lambda|} \right\} (t_2 - t_1), \end{aligned} \quad (53)$$

which is independent of x and tends to zero as $t_1 \rightarrow t_2$. So, the set $\mathcal{P}(B_{\bar{r}})$ is equicontinuous. By the Arzelà-Ascoli theorem, $\mathcal{P}$ is compact on $B_{\bar{r}}$. Thus all the assumptions of Lemma 9 are satisfied. Therefore the separated boundary value problem of quantum (p, q) -difference equation (5) has at least one solution on $[0, T]$. The proof is completed. $\square$

4. Examples

Example 1. Consider the following separated boundary value problem for second-order quantum (p, q) -difference equation of the form

$$D_{1/2, 1/5}^2 x(t) = f\left(t, x\left(\frac{t}{4}\right)\right), \quad t \in [0, 4],$$

$$\frac{5}{6} x(0) + \frac{4}{5} D_{p,q} x(0) = \frac{1}{3}, \quad (54)$$

$$\frac{1}{4} x(1) + \frac{5}{6} D_{p,q} x(2) = \frac{3}{5}.$$

Set constants $p = 1/2, q = 1/5, T = 1, \alpha_1 = 5/6, \alpha_2 = 4/5, \alpha_3 = 1/3, \beta_1 = 1/4, \beta_2 = 5/6$, and $\beta_3 = 3/5$. Then we can find constants as $\Lambda = 0.7027777778, \Lambda_1 = -0.1691699605, \Lambda_2 = 0.5928853755$, and $\Phi = 6.132128741$.

(i) Let $f : [0, 4] \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$f\left(t, x\left(\frac{t}{4}\right)\right) = \frac{5t^2}{(1 + x^2(t/4))} + \frac{t^2 \sin t}{3t + 1}. \quad (55)$$

It is easy to see that $|f(t, x)| \leq 5.25 = N$ which satisfies (H_1) . Thus, by Theorem 5, the separated boundary value problem (54) with f given by (55) has at least one solution on $[0, 1]$.

(ii) Given $f : [0, 4] \times \mathbb{R} \rightarrow \mathbb{R}$ with

$$f\left(t, x\left(\frac{t}{4}\right)\right) = \frac{|\cos t|}{18} \left(\frac{x^2(t/4) + 2|x(t/4)|}{1 + |x(t/4)|} \right). \quad (56)$$

it follows that

$$|f(t, x) - f(t, y)| \leq \frac{1}{9} |x - y|. \quad (57)$$

Therefore, the condition (H_2) is satisfied with $L = 1/9$. Indeed, we have $L\Phi = 0.6813476379 < 1$. Hence, by Theorem 6, the separated boundary value problem of quantum (p, q) -difference equation (54) has a unique solution on $[0, 1]$.

Example 2. Consider the following separated boundary value problem for second-order quantum (p, q) -difference equation of the form

$$D_{1/3, 1/6}^2 x(t) = g\left(t, x\left(\frac{t}{9}\right)\right), \quad t \in \left[0, \frac{9}{2}\right], \quad (58)$$

$$\frac{7}{3}x(0) + \frac{5}{6}D_{p,q}x(0) = \frac{4}{7},$$

$$\frac{2}{5}x\left(\frac{1}{2}\right) + \frac{7}{2}D_{p,q}x\left(\frac{3}{2}\right) = \frac{6}{5}.$$

Set constants $p = 1/3$, $q = 1/6$, $T = 1/2$, $\alpha_1 = 7/3$, $\alpha_2 = 5/6$, $\alpha_3 = 4/7$, $\beta_1 = 2/5$, $\beta_2 = 7/2$, and $\beta_3 = 6/5$. Then we can compute that $\Lambda = 8.3$, $\Lambda_1 = 0.1342512908$, $\Lambda_2 = 0.3098106712$, and $\Phi = 1.813253012$.

(i) Let $g : [0, 9/2] \times \mathbb{R} \rightarrow \mathbb{R}$ be a nonlinear function defined by

$$g\left(t, x\left(\frac{t}{9}\right)\right) = \frac{1}{(t+2)^3} \left(\frac{x^4(t/9)}{1+x^2(t/9)} + 2 \right). \quad (59)$$

Choosing $z(t) = 1/(t+2)^3$ and $\psi(x) = x^2 + 2$, we notice that the condition (H_3) of Theorem 8 is satisfied. Furthermore, we can find that $\|z\| = 1/8$. Then there exists a constant K such that $K \in (0.9447899357, 3.467170197)$. Therefore, the separated boundary value problem (58) with g given by (59) has at least one solution on $[0, 1/2]$.

(ii) Let a function $g : [0, 9/2] \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$g\left(t, x\left(\frac{t}{9}\right)\right) = \left(\frac{3e^{-t}}{2+t^2} \cdot \frac{|x(t/9)|}{|x(t/9)|+1} \right) + \frac{t}{2}. \quad (60)$$

It follows that

$$|g(t, x)| \leq \frac{3e^{-t}}{2+t^2} + \frac{t}{2}. \quad (61)$$

Then the function g defined in (60) satisfies condition (H_5) with $\kappa(t) = ((3e^{-t})/(2+t^2)) + (t/2)$. In addition, the condition (H_2) is satisfied with $L = 3/2$ and

$$\left(\frac{T^2}{p+q} \right) L = 0.75 < 1. \quad (62)$$

Hence, by applying Theorem 10, the separated boundary value problem for second-order quantum (p, q) -difference equation (58) with g defined in (60) has at least one solution on $[0, 1/2]$.

Remark 3. In Example 2, Case (ii), although the function $g(t, x)$ satisfies condition (H_2) , Theorem 6 cannot be applied, since $L\Phi = 2.719879518 > 1$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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