

Research Article

Existence and Uniqueness of Positive Solution for *p***-Laplacian Kirchhoff-Schrödinger-Type Equation**

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We study the existence and uniqueness of positive solution for the following *p*-Laplacian-Kirchhoff-Schrödinger-type equation: $\{-(a + b \int_{\Omega} |\nabla u|^p) \triangle_p u + \lambda v(x)|u|^{p-2}u = hf(u) - \mu g(u)$, in Ω , u > 0, in Ω , u = 0, on $\partial\Omega\}$, where $\Omega \subset \mathbb{R}^N$ ($N \ge 3$), $\lambda, \mu \ge 0$ are parameters, v(x), f(u), g(u) and h are under some suitable assumptions. For the purpose of overcoming the difficulty caused by the appearance of the Schrödinger term and general singularity, we use the variational method and some mathematical skills to obtain the existence and uniqueness of the solution to this problem.

1. Introduction and Main Results

In this paper, we discuss the following problem:

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth bounded domain with boundary $\partial \Omega$, $\lambda, \mu \geq 0$ are parameters, $\Delta_p u =$ div $(|\nabla u|^{p-2} \nabla u)$ is the *p*-Laplacian operator, and $p \geq 2$, $a, b \geq 0$ with a + b > 0.

In recent years, a lot of scholars have studied the singular Kirchhoff problem (for more details, we refer the reader to [1-4]), the Schrödinger-Poisson system (we refer the reader to [5-8]), and the Kirchhoff-Schrödinger-Poisson system (we refer the reader to [9-12]). The authors use various methods to obtain the properties of the solution, which makes such problems very interesting. Inspired by the above papers, later scholars begin to make some expanding study about the above problems. For example, in [13],

Guo and Nie studied the existence and multiplicity of nontrivial solutions for p-Laplacian Schrödinger-Kirchhoff-type equations by variational methods. For a more complex situation, we refer the reader to [14]. The related studies on the elliptic equations also can be found in [15–26].

However, up to now, no paper has appeared in the literature which discusses the existence and uniqueness of the positive solution for the *p*-Laplacian-Kirchhoff-Schrödinger-type problem. This paper attempts to fill this gap in the literature. Inspired by the above works, in this paper, we try to study the existence and uniqueness of solution to the problem (1) by using the variational method.

 (f_0) $f \in C((0,\infty), R_+)$ satisfies that there exists $\sigma > 0$, such that f is nonincreasing on $(0,\sigma]$, $\int_0^{\sigma} f(s)ds < \infty$, and there exists $\alpha, \gamma \in (0, 1)$ such that

$$\lim_{s \to 0^+} f(s) s^{\alpha} = +\infty,$$

$$\lim_{s \to \infty} \frac{f(s)}{s^{\gamma}} = 0.$$
(2)

 (h_0) $h \in L^{p^*/(p^*-1-\gamma)}(\Omega)$ satisfies h(x) > 0, a.e. $x \in \Omega$.

(g) $g \in C(R_+, R_+)$ and there exists a constant c > 0, such that

$$g(s) \le c\left(s^{p-1} + s^{p^*-1}\right), \quad s \in R_+.$$
 (3)

 $(v_1) v(x) \in C(\Omega, R), v(x) > 0$ and the minimum of v(x) can be achieved in Ω . In other words, there exists a constant c', such that $c' = \inf_{x \in \Omega} v(x)$.

(h_1) h is bounded in Ω satisfies h(x) > 0, a.e. $x \in \Omega$.

 (f_1) There exists a constant $k \in (0, \lambda c'/\|h\|_{\infty})$ such that

$$f(s) - f(t) \le k(s-t)^{p-1}, \quad \sigma \le t \le s, \tag{4}$$

where $\|\cdot\|_{\infty}$ denotes the maximal value in Ω .

In this paper, we will make full use of the following definitions.

First, we define the space $E_{\lambda} = \{u \in W_0^{1,p}(\Omega) : \int_{\Omega} \lambda v(x) |u|^p < \infty\}$ and the norm

$$\|u\|_{E_{\lambda}}^{p} = \int_{\Omega} a |\nabla u|^{p} + \lambda v(x) |u|^{p}.$$
(5)

We denote the norm in $L^p(\Omega)$ by $||u||_p = (\int_{\Omega} |u|^p)^{1/p}$.

By (ν_1) and the Poincaré inequality, we can deduce that the embedding $E_{\lambda} \hookrightarrow W_0^{1,p}$ is continuous. Thus, according to the continuity of the embedding $E_{\lambda} \hookrightarrow L^s(\Omega)(p \le s \le p^*)$, there are constants $c_s > 0$ such that

$$\|u\|_{s} \le c_{s} \|u\|_{E_{1}} . (6)$$

We make further assumptions for convenience. We assume f(s) = g(s) = 0 for all $s \in (-\infty, 0)$. Since $\lim_{s\to\infty} f(s)/s^{\gamma} = 0$ in (f_0) , we know there exists $c_0 > 0$, such that

$$f(s) \le c_0 s^{\gamma}, \quad s \in [\sigma/4, \infty),$$
 (7)

which implies

$$0 \le F(s) = \int_{0}^{s} f(t) dt \le \int_{0}^{s} c_{0} t^{\gamma} dt + \int_{0}^{\sigma} f(t) dt$$

$$= \frac{c_{0} s^{1+\gamma}}{1+\gamma} + c_{1}, \quad s \in R.$$
(8)

Also, from the fact that $\int_0^{\sigma} f(s)ds < \infty$, we can get that F is continuous on R. Thus for any $u \in E_{\lambda}$, by the conditions $(h_0), (8), (g), (6)$ and Hölder inequality, we have

$$\begin{split} \int_{\Omega} hF(u) &\leq \frac{c_0}{1+\gamma} \int_{\Omega} h |u|^{1+\gamma} dx + c_1 \int_{\Omega} h dx \\ &\leq \frac{c_0}{1+\gamma} \|u\|_{p^*}^{1+\gamma} \cdot \|h\|_{p^*/(p^*-1-\gamma)} + c_1 \|h\|_1 \qquad (9) \\ &\leq \frac{c_2}{1+\gamma} \|u\|_{E_{\lambda}}^{1+\gamma} \cdot \|h\|_{p^*/(p^*-1-\gamma)} + c_1 \|h\|_1 , \\ &0 &\leq \int_{\Omega} G(u) \leq c_3 \left(\|u\|_p^p + \|u\|_{p^*}^{p^*}\right) \\ &\leq c_4 \left(\|u\|_{E_{\lambda}}^p + \|u\|_{E_{\lambda}}^{p^*}\right), \end{split}$$

where $G(s) = \int_0^s g(t)dt$ for all $s \in R$ and c_0, c_1, c_2, c_3, c_4 are some positive constants. Next, we can define the energy functional corresponding to problem (1):

$$I(u) = \frac{a}{p} \int_{\Omega} |\nabla u|^{p} dx + \frac{b}{2p} \left(\int_{\Omega} |\nabla u|^{p} dx \right)^{2}$$

$$+ \frac{1}{p} \int_{\Omega} \lambda v(x) |u|^{p} + \int_{\Omega} \mu G(u) - \int_{\Omega} hF(u).$$
(11)

By a simple computation, we can get

$$\left\langle I'(u), \nu \right\rangle = a \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \nu$$

$$+ b \int_{\Omega} |\nabla u|^{p} \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \nu$$

$$+ \int_{\Omega} \lambda \nu(x) |u|^{p-2} u \cdot \nu - \int_{\Omega} hf(u) \nu$$

$$+ \int_{\Omega} \mu g(u) \nu.$$

$$(12)$$

It is clear that u with u(x) > 0, a.e. $x \in \Omega$, $u(x) \in E_{\lambda}$ is called a weak solution of the problem (1) if for any $\nu \in E_{\lambda}$ it holds

$$\left\langle I'\left(u\right),\nu\right\rangle = 0. \tag{13}$$

Finally, we will give the main results of the paper.

Theorem 1. If $a, b \ge 0$ with a + b > 0 and the assumptions $(f_0), (h_0), (g)$ and (v_1) hold, then the problem (1) possesses a positive solution for any $\lambda, \mu \in R_+$. Moreover, this solution is a global minimizer of I.

Theorem 2. If a > 0 and the assumptions $(f_0), (f_1), (h_1), (g)$ and (v_1) hold. Moreover, assume that g is nondecreasing on R_+ , then the solution for problem (1) is unique for any $\lambda, \mu \in R_+$.

Remark 3. The result obtained in the paper is an expanding study of the Kirchhoff-Schrödinger-type equation (p = 2); the difficulty is posed by the degenerate quasilinear elliptic operator. We mainly use the variational method to solve the problem.

This paper is organized as follows. In Section 2, we will give a preliminary. In Section 3, we will prove the main results.

In this paper, c, c_i denote various positive constants, which may vary from line to line.

2. Preliminary

To prove the main results in this paper, we will employ the following important lemma.

Lemma 4. If the assumptions (f_0) , (h_0) , (g), and (v_1) hold, then I attains the global minimum in E_{λ} ; that is, there exists $u_0 \in E_{\lambda}$ such that $I(u_0) = m := \inf_{E_{\lambda}} I$, and m < 0.

Proof. For any $u \in E_{\lambda}$, by (5), (9)–(11), we can get

$$I(u) = \frac{a}{p} \int_{\Omega} |\nabla u|^{p} dx + \frac{b}{2p} \left(\int_{\Omega} |\nabla u|^{p} dx \right)^{2} + \frac{1}{p} \int_{\Omega} \lambda v(x) |u|^{p} + \int_{\Omega} \mu G(u) - \int_{\Omega} hF(u) \geq \frac{a}{p} \int_{\Omega} |\nabla u|^{p} dx + \frac{1}{p} \int_{\Omega} \lambda v(x) |u|^{p} - \frac{c_{2}}{1+\gamma} ||u||_{E_{\lambda}}^{1+\gamma} \cdot ||h||_{p*/(p^{*}-1-\gamma)} - c_{1} ||h||_{1} = \frac{1}{p} ||u||_{E_{\lambda}}^{p} - \frac{c_{2} ||h||_{p*/(p^{*}-1-\gamma)}}{1+\gamma} \cdot ||u||_{E_{\lambda}}^{1+\gamma} - c_{1} ||h||_{1}.$$

Since $p \ge 2$ and $\gamma \in (0, 1)$, we can obtain that *I* is coercive and bounded from below on E_{λ} . The definition that $m = \inf_{E_{\lambda}} I$ makes some sense.

Since $\lim_{s\to 0^+}(f(s)/s^{-\alpha})\to\infty$ in the condition (f_0) , there exists $\sigma_1>0$ such that

$$f(s) \ge s^{-\alpha},$$

$$F(s) \ge \frac{s^{1-\alpha}}{1-\alpha},$$

$$s \in (0, \sigma_1].$$
(15)

Choosing a nonnegative function $\varphi \in C_0^{\infty}(\Omega) \setminus \{0\}$ with $\max_{\Omega} \varphi \leq \sigma_1$, then for any $t \in (0, 1]$, $t\varphi \in (0, \sigma_1]$, by (5), (10), (11), (15), we have

$$I(t\varphi) = \frac{at^{p}}{p} \int_{\Omega} |\nabla\varphi|^{p} dx + \frac{bt^{2p}}{2p} \left(\int_{\Omega} |\nabla\varphi|^{p} dx \right)^{2}$$
$$+ \frac{t^{p}}{p} \int_{\Omega} \lambda v(x) |\varphi|^{p} + \int_{\Omega} \mu G(t\varphi)$$
$$- \int_{\Omega} hF(t\varphi)$$
$$\leq \frac{t^{p}}{p} \|\varphi\|_{E_{\lambda}}^{p} + \frac{bt^{2p}}{2p} \left(\int_{\Omega} |\nabla\varphi|^{p} dx \right)^{2}$$

$$+ \mu c_4 \left(\left\| \varphi \right\|_{E_{\lambda}}^{p} t^{p} + \left\| \varphi \right\|_{E_{\lambda}}^{p^*} t^{p^*} \right) \\ - \frac{t^{1-\alpha}}{1-\alpha} \int_{\Omega} h \varphi^{1-\alpha}.$$
(16)

Since $1 - \alpha \in (0, 1)$ and h(x) > 0, a.e. $x \in \Omega$ in (h_0) , $p^* > p \ge 2$, we can get that $I(t\varphi) < 0$ for t > 0 small enough. That is m < 0.

On the basis of the definition of *m*, we can deduce that there exists a sequence $\{u_n\} \in E_\lambda$ such that $\lim_{n\to\infty} I(u_n) = m$. Since *I* is coercive and m < 0, $\{u_n\}$ is bounded in E_λ . Going if necessary to a subsequence, still denoted by $\{u_n\}$, there exists $u_0 \in E_\lambda$ such that

$$\begin{cases} u_n \to u_0, & \text{weakly in } E_\lambda, \\ u_n \to u_0, & \text{strongly in } L^s(\Omega), \ \left(1 \le s < p^*\right) \ (17) \\ u_n(x) \to u_0(x) & \text{a.e. in } \Omega, \end{cases}$$

as $n \to \infty$. It follows from (8) and Sobolev embedding theorem that $\{F(u_n)\}$ is bounded in $L^{p^*/(1+\gamma)}$. Moreover, from the continuity of *F*, we can get that $F(u_n(x)) \to$ $F(u_o(x))$, a.e. $x \in \Omega$. Thus, we obtain $F(u_n) \to F(u_0)$ in $L^{p^*/(1+\gamma)}(\Omega)$.

Since $h \in L^{p^*/(p^*-1-\gamma)}(\Omega)$, we have

$$\lim_{n \to \infty} \int_{\Omega} hF(u_n) = \int_{\Omega} hF(u_0).$$
 (18)

Moreover, by Fatou's lemma, we have

$$\liminf_{n \to \infty} \int_{\Omega} G(u_n) \ge \int_{\Omega} G(u_0).$$
(19)

According to the weakly lower semicontinuity of the norm, (18) and (19), we have

$$m = \liminf_{n \to \infty} I(u_n) = \liminf_{n \to \infty} \left\{ \frac{a}{p} \int_{\Omega} |\nabla u_n|^p + \frac{b}{2p} \left(\int_{\Omega} |\nabla u_n|^p \right)^2 + \frac{1}{p} \int_{\Omega} \lambda v(x) |u_n|^p + \int_{\Omega} \mu G(u_n) - \int_{\Omega} hF(u_n) \right\} \ge \frac{1}{p} \|u_0\|_{E_{\lambda}}^p$$

$$+ \frac{b}{2p} \left(\int_{\Omega} |\nabla u_0|^p \right)^2 + \int_{\Omega} \mu G(u_0) - \int_{\Omega} hF(u_o) = I(u_0) \ge m,$$

$$(20)$$

which yields $I(u_0) = m$. The proof is completed.

3. Proof of Main Results

Proof of Theorem 1. Since $m \le I(u_0^+) \le I(u_0) = m$, then $I(u_0^+) = I(u_0)$. Thus we may assume $u_0 \ge 0$. Owing to m < 0, we know $u_0 \ne 0$. Next we will give the two-step proof.

(i) Firstly, we shall prove
$$u_0(x) > 0$$
, a.e. $x \in \Omega$

For any $\nu \in E_{\lambda}$ with $\nu(x) \ge 0$, a.e. $x \in \Omega$ and t > 0, we have

$$0 \leq \frac{I(u_{0} + t\nu) - I(u_{0})}{t}$$

$$= \frac{a}{pt} \left\{ \int_{\Omega} |\nabla(u_{0} + t\nu)|^{p} - \int_{\Omega} |\nabla u_{0}|^{p} \right\}$$

$$+ \frac{b}{2pt} \left\{ \left(\int_{\Omega} |\nabla(u_{0} + t\nu)|^{p} \right)^{2} - \left(\int_{\Omega} |\nabla u_{0}|^{p} \right)^{2} \right\}$$

$$+ \frac{\lambda}{pt} \left\{ \int_{\Omega} \nu(x) |u_{0} + t\nu|^{p} - \nu(x) |u_{0}|^{p} \right\}$$

$$+ \frac{\mu}{t} \left\{ \int_{\Omega} G(u_{0} + t\nu) - G(u_{0}) \right\}$$

$$- \frac{1}{t} \left\{ \int_{\Omega} hF(u_{0} + t\nu) - hF(u_{0}) \right\}.$$
(21)

Letting $t \to 0^+$, we can get

$$\liminf_{t \to 0^{+}} \int_{\Omega} \frac{h}{t} \left[F(u_{0} + t\nu) - F(u_{0}) \right]$$

$$\leq a \int_{\Omega} \left| \nabla u_{0} \right|^{p-2} \nabla u_{0} \nabla \nu$$

$$+ b \int_{\Omega} \left| \nabla u_{0} \right|^{p} \int_{\Omega} \left| \nabla u_{0} \right|^{p-2} \nabla u_{0} \nabla \nu$$

$$+ \lambda \int_{\Omega} \nu(x) \left| u_{0} \right|^{p-2} u_{0} \nu + \mu \int_{\Omega} g(u_{0}) \nu.$$
(22)

Thus, by Fatou's lemma and Lemma 2.3 in [27], we can get

$$\begin{split} \int_{\Omega} hf(u_0) \, \nu &\leq a \int_{\Omega} \left| \nabla u_0 \right|^{p-2} \nabla u_0 \nabla \nu \\ &+ b \int_{\Omega} \left| \nabla u_0 \right|^p \int_{\Omega} \left| \nabla u_0 \right|^{p-2} \nabla u_0 \nabla \nu \\ &+ \lambda \int_{\Omega} \nu(x) \left| u_0 \right|^{p-2} u_0 \nu \\ &+ \mu \int_{\Omega} g(u_0) \, \nu. \end{split}$$
(23)

Let $e_1 \in E_{\lambda}$ be the first eigenfunction of the operator $-\Delta_p$ with the Dirichlet boundary and $e_1(x) > 0$ for all $x \in \Omega$. Taking $\nu = e_1$ in (23), by (ν_1) and the condition (g), we have

$$\begin{split} \int_{\Omega} hf\left(u_{0}\right)e_{1} &\leq a\int_{\Omega}\left|\nabla u_{0}\right|^{p-2}\nabla u_{0}\nabla e_{1} \\ &+ b\int_{\Omega}\left|\nabla u_{0}\right|^{p}\int_{\Omega}\left|\nabla u_{0}\right|^{p-2}\nabla u_{0}\nabla e_{1} \\ &+ \lambda\int_{\Omega}v\left(x\right)\left|u_{0}\right|^{p-2}u_{0}e_{1} \\ &+ \mu\int_{\Omega}g\left(u_{0}\right)e_{1} \end{split}$$

$$\leq a \int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 \nabla e_1$$

+ $b \int_{\Omega} |\nabla u_0|^p \int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 \nabla e_1$
+ $\lambda \int_{\Omega} v(x) |u_0|^{p-2} u_0 e_1$
+ $\mu c \int_{\Omega} \left(u_0^{p-1} + u_0^{p^*-1} \right) e_1 < \infty,$
(24)

which implies $u_0(x) > 0$, a.e. $x \in \Omega$ by the condition (h_0) . If not, there exists $E \subset \Omega$ such that m(E) > 0 and $u_0(x) = 0$ for all $x \in E$. Then by Lemma 2.3 in [27], we can get

$$\int_{\Omega} hf(u_0) e_1 \ge \int_E hf(u_0) e_1 = \infty.$$
(25)

It is a contradiction. So the claim $u_0(x) > 0$ is true.

(ii) u_0 is exactly a solution of the problem (1); that is, u_0 satisfies (13):

$$a \int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla \nu + b \int_{\Omega} |\nabla u_0|^p \int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0$$
$$\cdot \nabla \nu + \int_{\Omega} \lambda \nu (x) |u_0|^{p-2} u_0 \cdot \nu - \int_{\Omega} hf(u_0) \nu \qquad (26)$$
$$+ \int_{\Omega} \mu g(u_0) \nu = 0.$$

To obtain the conclusion, we define a function $\Psi : R \rightarrow R, \Psi(t) = I(u_0 + tu_0)$; that is,

$$\Psi(t) = \frac{a(1+t)^{p}}{p} \int_{\Omega} |\nabla u_{0}|^{p} + \frac{b(1+t)^{2p}}{2p} \left(\int_{\Omega} |\nabla u_{0}|^{p} \right)^{2} + \frac{(1+t)^{p}}{p} \int_{\Omega} \lambda v(x) |u_{0}|^{p} + \int_{\Omega} \mu G(u_{0} + tu_{0}) - \int_{\Omega} hF(u_{0} + tu_{0}).$$
(27)

From the above discussion, we know $\Psi(t)$ attains its minimum at t = 0. By Lemma 2.4 in [27], we can get that $\Psi(t)$ is differentiable at t = 0 and $\Psi'(0) = 0$; that is,

$$a \int_{\Omega} |\nabla u_0|^p + b \left(\int_{\Omega} |\nabla u_0|^p \right)^2 + \int_{\Omega} \lambda v(x) |u_0|^p + \int_{\Omega} \mu g(u_0) u_0 - \int_{\Omega} h f(u_0) u_0 = 0.$$
(28)

For each $v \in E_{\lambda}$ and $\varepsilon > 0$, we define $v_{\varepsilon} = u_0 + \varepsilon v$ and

$$\Omega_{+} = \left\{ x \in \Omega : u_{0} + \varepsilon v \ge 0 \right\},$$

$$\Omega_{-} = \left\{ x \in \Omega : u_{0} + \varepsilon v < 0 \right\}.$$
(29)

Then $v_{\varepsilon}^{-}|_{\Omega_{+}} = 0$ and $v_{\varepsilon}^{-}|_{\Omega_{-}} = u_{0} + \varepsilon v$.

Inserting v_{ε}^{+} into (23) and using (28), we can get that

$$\begin{split} 0 &\leq a \int_{\Omega} |\nabla u_{0}|^{p-2} \nabla u_{0} \cdot \nabla v_{\varepsilon}^{*} + b \int_{\Omega} |\nabla u_{0}|^{p} \\ &\cdot \int_{\Omega} |\nabla u_{0}|^{p-2} \nabla u_{0} \cdot \nabla v_{\varepsilon}^{*} + \lambda \int_{\Omega} v(x) |u_{0}|^{p-2} u_{0} v_{\varepsilon}^{*} \\ &+ \mu \int_{\Omega} g(u_{0}) v_{\varepsilon}^{*} - \int_{\Omega} hf(u_{0}) v_{\varepsilon}^{*} \\ &= a \int_{\Omega} |\nabla u_{0}|^{p-2} \nabla u_{0} \cdot \nabla v_{\varepsilon} + b \int_{\Omega} |\nabla u_{0}|^{p-2} u_{0} v_{\varepsilon} \\ &+ \mu \int_{\Omega} g(u_{0}) v_{\varepsilon} - \int_{\Omega} hf(u_{0}) v_{\varepsilon} \\ &- \left\{ a \int_{\Omega} |\nabla u_{0}|^{p-2} \nabla u_{0} \cdot \nabla v_{\varepsilon}^{-} \\ &+ b \int_{\Omega} |\nabla u_{0}|^{p} \int_{\Omega} |\nabla u_{0}|^{p-2} \nabla u_{0} \cdot \nabla v_{\varepsilon}^{-} \\ &+ \lambda \int_{\Omega} v(x) |u_{0}|^{p-2} u_{0} v_{\varepsilon}^{*} + \mu \int_{\Omega} g(u_{0}) v_{\varepsilon} \\ &- \int_{\Omega} hf(u_{0}) v_{\varepsilon}^{-} \right\} = \varepsilon \left\{ a \int_{\Omega} |\nabla u_{0}|^{p-2} \nabla u_{0} \cdot \nabla v \\ &+ b \int_{\Omega} |\nabla u_{0}|^{p} \int_{\Omega} |\nabla u_{0}|^{p-2} \nabla u_{0} \cdot \nabla v \\ &+ b \int_{\Omega} |\nabla u_{0}|^{p} \int_{\Omega} |\nabla u_{0}|^{p-2} \nabla u_{0} \cdot \nabla v \\ &+ b \int_{\Omega} |\nabla u_{0}|^{p} \int_{\Omega} |\nabla u_{0}|^{p-2} \nabla u_{0} \cdot \nabla v \\ &+ b \int_{\Omega} |\nabla u_{0}|^{p} \int_{\Omega} |\nabla u_{0}|^{p-2} \nabla u_{0} \cdot \nabla v \\ &+ h \int_{\Omega} |v| |u_{0}|^{p-2} u_{0} v + \mu \int_{\Omega} g(u_{0}) v \\ &- \int_{\Omega} hf(u_{0}) v \right\} - \left\{ a \int_{\Omega_{-}} |\nabla u_{0}|^{p-2} \nabla u_{0} \\ &\cdot \nabla (u_{0} + \varepsilon v) + b \int_{\Omega} |\nabla u_{0}|^{p} \int_{\Omega_{-}} |\nabla u_{0}|^{p-2} \nabla u_{0} \\ &\cdot \nabla (u_{0} + \varepsilon v) + \lambda \int_{\Omega_{-}} v(x) |u_{0}|^{p-2} u_{0} (u_{0} + \varepsilon v) \\ &+ \mu \int_{\Omega_{-}} g(u_{0}) (u_{0} + \varepsilon v) - \int_{\Omega_{-}} hf(u_{0}) (u_{0} + \varepsilon v) \\ &+ \lambda \int_{\Omega} v(x) |u_{0}|^{p-2} u_{0} v + \mu \int_{\Omega} g(u_{0}) v \\ &- \int_{\Omega} hf(u_{0}) v \\ &- \sum_{1} (hf(u_{0}) v) \\ &- \varepsilon \left\{ \left(a + b \int_{\Omega} |\nabla u_{0}|^{p} \right) \int_{\Omega_{-}} |\nabla u_{0}|^{p-2} \nabla u_{0} \cdot \nabla v \\ &+ \lambda \int_{\Omega_{-}} v(x) |u_{0}|^{p-2} u_{0} v + \mu \int_{\Omega_{-}} g(u_{0}) v \\ &- \int_{\Omega} hf(u_{0}) v \\ &+ \lambda \int_{\Omega_{-}} v(x) |u_{0}|^{p-2} u_{0} v + \mu \int_{\Omega_{-}} g(u_{0}) v \\ &+ \lambda \int_{\Omega_{-}} v(x) |u_{0}|^{p-2} u_{0} v + \mu \int_{\Omega_{-}} g(u_{0}) v \\ &+ \lambda \int_{\Omega_{-}} v(x) |u_{0}|^{p-2} u_{0} v + \mu \int_{\Omega_{-}} g(u_{0}) v \\ &+ \lambda \int_{\Omega_{-}} v(x) |u_{0}|^{p-2} u_{0} v + \mu \int_{\Omega_{-}} g(u_{0}) v \\ &+ \lambda \int_{\Omega_{-}} v(x) |u_{0}|^{p-2} u_{0} v + \mu \int_{\Omega_{-}} g(u_{0}) v \\ &+ \lambda \int_{\Omega_{-}} v(x) |u_{0}|^{p-2} u_{0} v + \mu \int_{\Omega_{-}} g(u_{0}) v \\ &+ \lambda \int_{\Omega_{-}} v(x) |u_{0}|^{p-2} u_{0} v + \mu \int_{\Omega_{-}} g(u_{0}) v \\ &+ \lambda \int_{\Omega_{-}} v(x) |u_{$$

which implies that

$$\left(a+b\int_{\Omega}\left|\nabla u_{0}\right|^{p}\right)\int_{\Omega_{-}}\left|\nabla u_{0}\right|^{p-2}\nabla u_{0}\cdot\nabla v +\lambda\int_{\Omega_{-}}v\left(x\right)\left|u_{0}\right|^{p-2}u_{0}v+\mu\int_{\Omega_{-}}g\left(u_{0}\right)v \leq \left(a+b\int_{\Omega}\left|\nabla u_{0}\right|^{p}\right)\int_{\Omega}\left|\nabla u_{0}\right|^{p-2}\nabla u_{0}\cdot\nabla v \qquad (31) +\lambda\int_{\Omega}v\left(x\right)\left|u_{0}\right|^{p-2}u_{0}v+\mu\int_{\Omega}g\left(u_{0}\right)v -\int_{\Omega}hf\left(u_{0}\right)v.$$

Next we define $\Lambda_n = \{x \in \Omega : u_0(x) > 0, v(x) > -\infty, u_0(x) + v(x)/n < 0\}$ for all *n*. By simple computation, we can deduce that $\{\Lambda_n\}$ is a nonincreasing sequence of measurable sets and $\lim_{n\to\infty} \Lambda_n = \bigcap_{n=1}^{\infty} \Lambda_n = \emptyset$. Thus we have

$$\lim_{n \to \infty} m\left(\Lambda_n\right) = m\left(\lim_{n \to \infty} \Lambda_n\right) = 0.$$
(32)

Let $\varepsilon = 1/n$; then $\Omega_{-} \subset \{x \in \Omega : u_0(x) \le 0\} \cup \{x \in \Omega : v(x) = -\infty\} \cup \Lambda_n$ and $m(\Omega_{-}) = m(\Lambda_n) \to 0$ as $n \to \infty$. Selecting $\varepsilon = 1/n \to 0$ in (31), we have

$$0 \leq \left(a + b \int_{\Omega} |\nabla u_0|^p\right) \int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla v$$
$$+ \lambda \int_{\Omega} v(x) |u_0|^{p-2} u_0 v + \mu \int_{\Omega} g(u_0) v \qquad (33)$$
$$- \int_{\Omega} hf(u_0) v.$$

According to the arbitrariness of $v \in E_{\lambda}$, this inequality also holds for -v. Combining (33), we can get that, for any $v \in E_{\lambda}$,

$$\left(a+b\int_{\Omega}|\nabla u_{0}|^{p}\right)\int_{\Omega}|\nabla u_{0}|^{p-2}\nabla u_{0}\cdot\nabla v$$
$$+\lambda\int_{\Omega}v(x)|u_{0}|^{p-2}u_{0}v+\mu\int_{\Omega}g(u_{0})v\qquad(34)$$
$$-\int_{\Omega}hf(u_{0})v=0.$$

Thus, u_0 is exactly a weak solution of the problem (1). By Lemma 4, we know $I(u_0) = \inf_{E_{\lambda}} I$. Therefore, u_0 is exactly a global minimizer solution.

Proof of Theorem 2. Assume that u_1 is also a solution of problem (1). Letting $v = u_0 - u_1$, according to the definition of the weak solution and (26), we can get

$$0 = \left(a + b \int_{\Omega} |\nabla u_0|^p\right) \int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla (u_0 - u_1)$$
$$+ \int_{\Omega} \lambda v(x) |u_0|^{p-2} u_0 (u_0 - u_1)$$

$$+ \int_{\Omega} \mu g(u_0) (u_0 - u_1) - \int_{\Omega} h f(u_0) (u_0 - u_1),$$
(35)

$$0 = \left(a + b \int_{\Omega} |\nabla u_{1}|^{p}\right) \int_{\Omega} |\nabla u_{1}|^{p-2} \nabla u_{1} \cdot \nabla (u_{0} - u_{1}) + \int_{\Omega} \lambda v (x) |u_{1}|^{p-2} u_{1} (u_{0} - u_{1}) + \int_{\Omega} \mu g (u_{1}) (u_{0} - u_{1}) - \int_{\Omega} h f (u_{1}) (u_{0} - u_{1}),$$
(36)

which implies

$$\int_{\Omega} h(u_{0} - u_{1}) [f(u_{0}) - f(u_{1})] = a \int_{\Omega} (\nabla u_{0} - \nabla u_{1}) \\
\cdot (|\nabla u_{0}|^{p-2} \nabla u_{0} - |\nabla u_{1}|^{p-2} \nabla u_{1}) dx \\
+ b \int_{\Omega} \nabla (u_{0} - u_{1}) \{ \nabla u_{0} |\nabla u_{0}|^{p-2} \int_{\Omega} |\nabla u_{0}|^{p} \\
- \nabla u_{1} |\nabla u_{1}|^{p-2} \int_{\Omega} |\nabla u_{1}|^{p} \} + \int_{\Omega} \lambda v(x) (u_{0} - u_{1}) \\
\cdot \{ u_{0} |u_{0}|^{p-2} - u_{1} |u_{1}|^{p-2} \} + \int_{\Omega} \mu (u_{0} - u_{1}) \\
\cdot [g(u_{0}) - g(u_{1})].$$
(37)

Next, we will make some estimates for the equation. (i)

$$\int_{\Omega} \nabla (u_0 - u_1) \left\{ \nabla u_0 \left| \nabla u_0 \right|^{p-2} \int_{\Omega} \left| \nabla u_0 \right|^p - \nabla u_1 \left| \nabla u_1 \right|^{p-2} \int_{\Omega} \left| \nabla u_1 \right|^p \right\} \ge 0.$$
(38)

In fact, we estimate as follows.

$$\begin{split} &\int_{\Omega} \nabla \left(u_{0} - u_{1} \right) \left\{ \nabla u_{0} \left| \nabla u_{0} \right|^{p-2} \int_{\Omega} \left| \nabla u_{0} \right|^{p} \\ &- \nabla u_{1} \left| \nabla u_{1} \right|^{p-2} \int_{\Omega} \left| \nabla u_{1} \right|^{p} \right\} = \left(\int_{\Omega} \left| \nabla u_{0} \right|^{p} \right)^{2} \\ &- \int_{\Omega} \left| \nabla u_{0} \right|^{p-2} \nabla u_{0} \cdot \nabla u_{1} \int_{\Omega} \left| \nabla u_{0} \right|^{p} + \left(\int_{\Omega} \left| \nabla u_{1} \right|^{p} \right)^{2} \\ &- \int_{\Omega} \left| \nabla u_{1} \right|^{p} \int_{\Omega} \left| \nabla u_{1} \right|^{p-2} \nabla u_{0} \cdot \nabla u_{1} \ge \left(\int_{\Omega} \left| \nabla u_{0} \right|^{p} \right)^{2} \\ &- \int_{\Omega} \left| \nabla u_{0} \right|^{p-1} \left| \nabla u_{1} \right| \int_{\Omega} \left| \nabla u_{0} \right|^{p} + \left(\int_{\Omega} \left| \nabla u_{1} \right|^{p} \right)^{2} \\ &- \int_{\Omega} \left| \nabla u_{1} \right|^{p} \int_{\Omega} \left| \nabla u_{1} \right|^{p-1} \left| \nabla u_{0} \right| \end{split}$$

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$$\geq \max\left\{\int_{\Omega} |\nabla u_{0}|^{p}, \int_{\Omega} |\nabla u_{1}|^{p}\right\}$$
$$\cdot \left(\int_{\Omega} (|\nabla u_{0}| - |\nabla u_{1}|) \left(|\nabla u_{0}|^{p-1} - |\nabla u_{1}|^{p-1}\right)\right) \geq 0.$$
(39)

(ii)

$$\int_{\Omega} v(x) (u_0 - u_1) \left\{ u_0 |u_0|^{p-2} - u_1 |u_1|^{p-2} \right\}$$

$$\geq \int_{\Omega} v(x) |u_0 - u_1|^p.$$
(40)

(iii)

$$\begin{split} &\int_{\Omega} \left(\nabla u_{0} - \nabla u_{1} \right) \left(\left| \nabla u_{0} \right|^{p-2} \nabla u_{0} - \left| \nabla u_{1} \right|^{p-2} \nabla u_{1} \right) \\ &= \int_{\Omega} \left| \nabla u_{0} \right|^{p-2} \left(\left| \nabla u_{0} \right|^{2} - \nabla u_{0} \cdot \nabla u_{1} \right) \\ &- \int_{\Omega} \left| \nabla u_{1} \right|^{p-2} \left(\nabla u_{0} \cdot \nabla u_{1} - \left| \nabla u_{1} \right|^{2} \right) \\ &\geq \int_{\Omega} \left| \nabla u_{0} \right|^{p-2} \left(\left| \nabla u_{0} \right|^{2} - \left| \nabla u_{0} \right| \cdot \left| \nabla u_{1} \right| \right) \\ &- \int_{\Omega} \left| \nabla u_{1} \right|^{p-2} \left(\left| \nabla u_{0} \right| \cdot \left| \nabla u_{1} \right| - \left| \nabla u_{1} \right|^{2} \right) \\ &= \int_{\Omega} \left(\left| \nabla u_{0} \right| - \left| \nabla u_{1} \right| \right) \left(\left| \nabla u_{0} \right|^{p-1} - \left| \nabla u_{1} \right|^{p-1} \right) \geq 0. \end{split}$$

(iv) Since g is nondecreasing on R_+ , we have

$$\int_{\Omega} \left[g(u_0) - g(u_1) \right] (u_0 - u_1) \ge 0.$$
(42)

(v) Next, we estimate the left side of (37), according to the conditions (f_0) and (f_1) , we can prove that

$$f(s) - f(t) \le k(s-t)^{p-1}, \quad 0 < t \le s < \infty.$$
 (43)

Thus by a simple deduction and (h_1) , (f_1) , one has

$$\int_{\Omega} h \left[f(u_0) - f(u_1) \right] (u_0 - u_1) \le k \int_{\Omega} h (u_0 - u_1)^p$$

$$\le k \| h(x) \|_{\infty} \int_{\Omega} |u_0 - u_1|^p.$$
(44)

It follows from (37), (38), (40)–(42), and (44) and we have

$$(\lambda c' - k \|h\|_{\infty}) \int_{\Omega} |u_0 - u_1|^p \le 0,$$
 (45)

where $c' = \inf_{x \in \Omega} v(x)$. Since $k \in (0, \lambda c' / ||h||_{\infty})$ in the condition (f_1) , we can get that $\int_{\Omega} |u_0 - u_1|^p = 0$, that is, $u_0 = u_1$. Therefore, the solution of the problem (1) is unique.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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