Approximate Solution of Klein Gordon equation in Minimal Length effect for Hulthen potenial using AIM

The klein-Gordon equation is given

$$\left(E-V(r)\right)^2\psi(r,\theta,\varphi)=\left[P^2c^2+\left(M_oc^2+S(r)\right)^2\right]\psi(r,\theta,\varphi) \tag{1}$$

Momentum operator in minimal length formalism as follow

$$P=\left(1+\alpha_{ML}\hat{p}\right)\hat{p}_i \tag{2}$$

Square term of equation (2) is given

$$P^2=\left(1+2\alpha_{ML}p^2\right)p^2 \tag{3}$$

Here we have used $\alpha^2{}_{ML}\approx 0$. with,

$$p=-i\hbar\nabla \tag{4}$$

$$p^2=-\hbar^2\nabla^2=-\hbar^2\Delta \tag{5}$$

Then, equation (3) and (5) is inserted in equation (1), we get

$$\left(E-V(r)\right)^2\psi(r,\theta,\varphi)=\left[\left(1+2\alpha_{ML}p^2\right)p^2c^2+\left(M_oc^2+V(r)\right)^2\right]\psi(r,\theta,\varphi)$$

$$\left(E-V(r)\right)^2\psi(r,\theta,\varphi)=\left[\left(1+2\alpha_{ML}\left(-\hbar^2\Delta\right)\right)\left(-\hbar^2\Delta\right)c^2+\left(M_oc^2+S(r)\right)^2\right]\psi(r,\theta,\varphi)$$

$$\left(E-V(r)\right)^2\psi(r,\theta,\varphi)=\left[-\left(1-2\alpha_{ML}\hbar^2\Delta\right)\left(\hbar^2\Delta\right)c^2+\left(M_oc^2+S(r)\right)^2\right]\psi(r,\theta,\varphi) \tag{6}$$

By setting $\hbar=c=1$ (natural unit) , $S(r)=V(r)$ and by setting $V(r)\rightarrow\frac{1}{2}V(r)$, in Eq. (6) become,

$$\left(E-\frac{1}{2}V(r)\right)^2\psi(r,\theta,\varphi)=\left[-\left(\Delta-2\alpha_{ML}\Delta^2\right)+\left(M_o+\frac{1}{2}V(r)\right)^2\right]\psi(r,\theta,\varphi)$$

$$\left(E^2-EV(r)+\frac{1}{4}V(r)^2\right)\psi(r,\theta,\varphi)=\left[-\left(\Delta-2\alpha_{ML}\Delta^2\right)+\left(M_o{}^2+M_oV(r)+\frac{1}{4}V(r)^2\right)\right]\psi(r,\theta,\varphi)$$

$$\left(E^2-EV(r)+\frac{1}{4}V(r)^2-M_o{}^2-M_oV(r)-\frac{1}{4}V(r)^2\right)\psi(r,\theta,\varphi)=-\left(\Delta-2\alpha_{ML}\Delta^2\right)\psi(r,\theta,\varphi)$$

$$\left(E^2-M_o{}^2-(E+M_o)V(r)\right)\psi(r,\theta,\varphi)=-\left(\Delta-2\alpha_{ML}\Delta^2\right)\psi(r,\theta,\varphi)$$

$$\left[-\Delta+2\alpha_{ML}\Delta^2-\left(E^2-M_o{}^2-(E+M_o)V\right)\right]\psi(r,\theta,\varphi)=0 \tag{7}$$

By applying the modified approximate wave functionwhich is modification of approximate wave function proposed by Chabab *et al*. in Ref [11] (Chabab *et al.*, 2016),

$$\psi\left(r,\theta,\varphi\right)=\left(1+2\alpha_{ML}\Delta\right)\phi\left(r,\theta,\varphi\right) \tag{8}$$

in equation (7), we have

$$\left[-\Delta+2\alpha_{ML}\Delta^{2}-\left(E^{2}-M_{o}^{2}-\left(E+M_{o}\right)V\left(r\right)\right)\right]\left(1+2\alpha_{ML}\Delta\right)\phi\left(r,\theta,\varphi\right)=0$$

$$\begin{bmatrix}-\Delta-2\alpha_{ML}\Delta^{2}+2\alpha_{ML}\Delta^{2}+4\alpha^{2}{}_{ML}\Delta^{3}-\left(E^{2}-M_{o}^{2}-\left(E+M_{o}\right)V\left(r\right)\right)\\ -2\alpha_{ML}\Delta\left(E^{2}-M_{o}^{2}-\left(E+M_{o}\right)V\left(r\right)\right)\end{bmatrix}\phi\left(r,\theta,\varphi\right)=0$$

By setting $\alpha^{2}{}_{ML}\approx 0$, then we have

$$\left[-\Delta-\left(E^{2}-M_{o}^{2}-\left(E+M_{o}\right)V\left(r\right)\right)-2\alpha_{ML}\Delta\left(E^{2}-M_{o}^{2}-\left(E+M_{o}\right)V\left(r\right)\right)\right]\phi\left(r,\theta,\varphi\right)=0$$

$$\left[-\left(1+2\alpha_{ML}\left(E^{2}-M_{o}^{2}-\left(E+M_{o}\right)V\left(r\right)\right)\right)\Delta-\left(E^{2}-M_{o}^{2}-\left(E+M_{o}\right)V\left(r\right)\right)\right]\phi\left(r,\theta,\varphi\right)=0$$

$$\left[-\Delta-\frac{\left(E^{2}-M_{o}^{2}-\left(E+M_{o}\right)V\left(r\right)\right)}{\left(1+2\alpha_{ML}\left(E^{2}-M_{o}^{2}-\left(E+M_{o}\right)V\left(r\right)\right)\right)}\right]\phi\left(r,\theta,\varphi\right)=0 \tag{9}$$

Equation (9) is similar with equation (20) in Chabab *et al*. Then, we use Binomial expansion for equation (9), we get

$$\begin{aligned}\left[\frac{1}{\left(1+2\alpha_{ML}\left(E^{2}-M_{o}^{2}-2\left(E+M_{o}\right)V\right)\right)}\right]&=\left(1+2\alpha_{ML}\left(E^{2}-M_{o}^{2}-2\left(E+M_{o}\right)V\right)\right)^{-1}\\ &=1-\frac{2\alpha_{ML}\left(E^{2}-M_{o}^{2}-2\left(E+M_{o}\right)V\right)}{1!}+\frac{\left(2\alpha_{ML}\left(E^{2}-M_{o}^{2}-2\left(E+M_{o}\right)V\right)\right)^{2}}{2!}+\ldots\\ &=1-2\alpha_{ML}\left(E^{2}-M_{o}^{2}-2\left(E+M_{o}\right)V\right)\end{aligned}$$

We obtain,

$$\left[-\Delta-\left(E^{2}-M_{o}^{2}-\left(E+M_{o}\right)V\left(r\right)\right)\left(1-2\alpha_{ML}\left(E^{2}-M_{o}^{2}-2\left(E+M_{o}\right)V\left(r\right)\right)\right)\right]\phi\left(r,\theta,\varphi\right)=0$$

$$\left[\Delta+\left(E^{2}-M_{o}^{2}-\left(E+M_{o}\right)V\left(r\right)\right)\left(1-2\alpha_{ML}\left(E^{2}-M_{o}^{2}-2\left(E+M_{o}\right)V\left(r\right)\right)\right)\right]\phi\left(r,\theta,\varphi\right)=0$$

$$\left[\Delta+\begin{pmatrix}\left(E^{2}-M_{o}^{2}-\left(E+M_{o}\right)V\left(r\right)\right)\\ -2\alpha_{ML}\left(E^{2}-M_{o}^{2}-\left(E+M_{o}\right)V\left(r\right)\right)^{2}\end{pmatrix}\right]\phi\left(r,\theta,\varphi\right)=0 \tag{10}$$

Operator Laplacian is given,

$$\Delta=\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \tag{11}$$

Equation (11) is substituted in equation (10), is yields

$$\left[\begin{pmatrix}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\\+\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\\+\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\end{pmatrix}+\begin{pmatrix}\left(E^2-M_o{}^2-\left(E+M_o\right)V\left(r\right)\right)\\-2\alpha_{ML}\left(E^2-M_o{}^2-\left(E+M_o\right)V\left(r\right)\right)^2\end{pmatrix}\right]\phi\left(r,\theta,\varphi\right)=0$$

$$\left[\begin{pmatrix}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\\+\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\\+\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\end{pmatrix}+\begin{pmatrix}\left(E^2-M_o{}^2-\left(E+M_o\right)V\left(r\right)\right)\\-2\alpha_{ML}\begin{pmatrix}\left(E^2-M_o{}^2\right)^2\\-2\left(E^2-M_o{}^2\right)\left(E+M_o\right)V\left(r\right)\\+\left(E+M_o\right)^2V^2\left(r\right)\end{pmatrix}\end{pmatrix}\right]\phi\left(r,\theta,\varphi\right)=0$$

$$\left[\begin{pmatrix}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\\+\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\\+\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\end{pmatrix}+\begin{pmatrix}\left(E^2-M_o{}^2\right)-\left(E+M_o\right)V\left(r\right)-2\alpha_{ML}\left(E^2-M_o{}^2\right)^2\\+4\alpha_{ML}\left(E^2-M_o{}^2\right)\left(E+M_o\right)V\left(r\right)\\-2\alpha_{ML}\left(E+M_o\right)^2V^2\left(r\right)\end{pmatrix}\right]\phi\left(r,\theta,\varphi\right)=0 \tag{12}$$

By using separable variable method, we have new wave fuction

$$\phi\left(\mathrm{r},\theta,\varphi\right)=R\left(r\right)\Theta\left(\theta\right)\Phi\left(\varphi\right) \tag{13}$$

Equation (13) is inserted in equation (12) is given

$$\left[\begin{pmatrix}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\\+\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\\+\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\end{pmatrix}+\begin{pmatrix}\left(E^2-M_o{}^2\right)-\left(E+M_o\right)V\left(r\right)-2\alpha_{ML}\left(E^2-M_o{}^2\right)^2\\+4\alpha_{ML}\left(E^2-M_o{}^2\right)\left(E+M_o\right)V\left(r\right)\\-2\alpha_{ML}\left(E+M_o\right)^2V\left(r\right)^2\end{pmatrix}\right]R\left(\beta\right)\Theta\left(\theta\right)\Phi\left(\varphi\right)=0$$

$$\left[\begin{pmatrix} \frac{\Theta(\theta)\Phi(\varphi)}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R(r)}{\partial r} \\ +\frac{R(r)\Phi(\varphi)}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta} \\ +\frac{R(r)\Theta(\theta)\Phi(\varphi)}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2} \end{pmatrix} + \begin{pmatrix} (E^2-M_o{}^2)-(E+M_o)V(r)-2\alpha_{ML}(E^2-M_o{}^2)^2 \\ +4\alpha_{ML}(E^2-M_o{}^2)(E+M_o)V(r) \\ -2\alpha_{ML}(E+M_o)^2V(r)^2 \end{pmatrix} R(\beta)\Phi(\varphi)\Theta(\theta)\right]=0 \tag{14}$$

Equation (14) divided by $R(r)\Theta(\theta)\Phi(\varphi)$, so

$$\left[\begin{pmatrix} \frac{1}{R(r)r^2}\frac{\partial}{\partial r}r^2\frac{\partial R(r)}{\partial r} \\ +\frac{1}{\Theta(\theta)r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta} \\ +\frac{\Phi(\varphi)}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2} \end{pmatrix} + \begin{pmatrix} (E^2-M_o{}^2)-(E+M_o)V(r)-2\alpha_{ML}(E^2-M_o{}^2)^2 \\ +4\alpha_{ML}(E^2-M_o{}^2)(E+M_o)V(r)-2\alpha_{ML}(E+M_o)^2V(r)^2 \end{pmatrix}\right]=0 \tag{15}$$

Then equation (15) is multiplied by r^2, so

$$\left[\begin{pmatrix} \frac{1}{R(r)}\frac{\partial}{\partial r}r^2\frac{\partial R(r)}{\partial r} \\ +\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta} \\ +\frac{1}{\Phi(\varphi)}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2} \end{pmatrix} + \begin{pmatrix} (E^2-M_o{}^2)-(E+M_o)V(r)-2\alpha_{ML}(E^2-M_o{}^2)^2 \\ +4\alpha_{ML}(E^2-M_o{}^2)(E+M_o)V(r) \\ -2\alpha_{ML}(E+M_o)^2V(r)^2 \end{pmatrix} r^2\right]=0 \tag{16}$$

Next, equation (16) can be written

$$\left\{\begin{matrix} \left[\frac{1}{R(r)}\frac{\partial}{\partial r}r^2\frac{\partial R(r)}{\partial r} + \begin{pmatrix} (E^2-M_o{}^2)-(E+M_o)V(r)-2\alpha_{ML}(E^2-M_o{}^2)^2 \\ +4\alpha_{ML}(E^2-M_o{}^2)(E+M_o)V(r)-2\alpha_{ML}(E+M_o)^2V(r)^2 \end{pmatrix} r^2\right] \\ =-\left[\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}+\frac{1}{\Phi(\varphi)}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2}\right]=\lambda \end{matrix}\right\} \tag{17}$$

Equation (17) consist of radial and polar part, the polar part as follows,

$$-\left[\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}+\frac{1}{\Phi(\varphi)}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2}\right]=\lambda \tag{18}$$

and radial part is given,

$$\left[\frac{1}{R(r)}\frac{\partial}{\partial r}r^2\frac{\partial R(r)}{\partial r}+\begin{pmatrix}\left(E^2-M_o{}^2\right)-\left(E+M_o\right)V(r)-2\alpha_{ML}\left(E^2-M_o{}^2\right)^2\\+4\alpha_{ML}\left(E^2-M_o{}^2\right)\left(E+M_o\right)V(r)-2\alpha_{ML}\left(E+M_o\right)^2V(r)^2\end{pmatrix}r^2\right]=\lambda \tag{19}$$

In this paper, we use a radial part to get energy of Klein Gordon equation in minimal length effect. The radial part of equation (19) is multiplied by $\frac{R(r)}{r^2}$, so we get

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R(r)}{\partial r}+\begin{pmatrix}\left(E^2-M_o{}^2\right)-\left(E+M_o\right)V(r)-2\alpha_{ML}\left(E^2-M_o{}^2\right)^2\\+4\alpha_{ML}\left(E^2-M_o{}^2\right)\left(E+M_o\right)V(r)-2\alpha_{ML}\left(E+M_o\right)^2V(r)^2\end{pmatrix}R(r)\right]=\frac{\lambda}{r^2}R(r) \tag{20}$$

By setting $R(r)=\frac{U_H(r)}{r}$ in equation (20), we have

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\frac{U_H(r)}{r}+\begin{pmatrix}\left(E^2-M_o{}^2\right)-\left(E+M_o\right)V(r)-2\alpha_{ML}\left(E^2-M_o{}^2\right)^2\\+4\alpha_{ML}\left(E^2-M_o{}^2\right)\left(E+M_o\right)V(r)-2\alpha_{ML}\left(E+M_o\right)^2V(r)^2\end{pmatrix}\frac{U_H(r)}{r}\right]=\frac{\lambda}{r^2}\frac{U_H(r)}{r} \tag{21}$$

where

$$\begin{aligned}\frac{1}{r^2R(r)}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\frac{U_H(r)}{r}&=\frac{1}{r^2}\frac{\partial}{\partial r}r^2\left[\frac{1}{r}\frac{dU_H(r)}{dr}-\frac{1}{r^2}U_H(r)\right]\\&=\frac{1}{r^2}\frac{\partial}{\partial r}\left[r\frac{dU_H(r)}{dr}-U_H(r)\right]\\&=\frac{1}{r^2}\left[r\frac{d^2v(r)}{dr^2}+\frac{dU_H(r)}{dr}-\frac{dU_H(r)}{dr}\right]\\&=\frac{1}{r^2}\left[r\frac{d^2U_H(r)}{dr^2}\right]\\&=\frac{1}{r}\frac{d^2U_H(r)}{dr^2}\end{aligned} \tag{22}$$

Equation (22) is substituted in equation (23) and multiplied *r*, is yields

$$\left[\frac{d^2U_H(r)}{dr^2}\begin{pmatrix}\left(E^2-M_o{}^2\right)-\left(E+M_o\right)V(r)-2\alpha_{ML}\left(E^2-M_o{}^2\right)^2\\+4\alpha_{ML}\left(E^2-M_o{}^2\right)\left(E+M_o\right)V(r)-2\alpha_{ML}\left(E+M_o\right)^2V(r)^2\end{pmatrix}U_H(r)\right]=\frac{\lambda}{r^2}U_H(r) \tag{23}$$

with,

$$\lambda=L(L+1) \tag{24}$$

So we have,

$$\frac{d^2U_H(r)}{dr^2}-\frac{L(L+1)}{r^2}U_H(r)+\begin{pmatrix}(E^2-M_o^{\ 2})-(E+M_o)V(r)-2\alpha_{ML}(E^2-M_o^{\ 2})^2\\ +4\alpha_{ML}(E^2-M_o^{\ 2})(E+M_o)V(r)-2\alpha_{ML}(E+M_o)^2V(r)^2\end{pmatrix}U_H(r)=0 \tag{25}$$

The Hulthen potential is given by

$$\begin{aligned}V_H(r)&=-V_o\frac{e^{-2\omega_H r}}{1-e^{-2\omega_H r}}=-V_o\frac{e^{-\omega_H r}}{e^{\omega_H r}-e^{-\omega_H r}}\\ &=-V_o\frac{\cosh\omega_H r-\sinh\omega_H r}{2\sinh\omega_H r}\\ &=\frac{V_o}{2}(1-\coth\omega_H r)\\ &=\left(\frac{V_o}{2}-\frac{V_o}{2}\coth\omega_H r\right)\end{aligned} \tag{26}$$

The square term of Hulthen potential as follow,

$$\begin{aligned}V^2(r)=\left(-V_o\omega_H\frac{e^{-\omega_H r}}{e^{\omega_H r}-e^{-\omega_H r}}\right)^2&=\left(\frac{V_o\omega_H}{2}-\frac{V_o\omega_H}{2}\coth\omega_H r\right)^2\\ &=\frac{(V_o\omega_H)^2}{4}\coth^2\omega_H r-2\frac{(V_o\omega_H)^2}{4}\coth\omega_H r+\frac{(V_o\omega_H)^2}{4}\\ &=\frac{(V_o\omega_H)^2}{4}\coth^2\omega_H r-\frac{(V_o\omega_H)^2}{2}\coth\omega_H r+\frac{(V_o\omega_H)^2}{4}\end{aligned} \tag{27}$$

Equation (26) and (27) is inserting in equation (25), so we get

$$\frac{d^2U_H(r)}{dr^2}-\frac{L(L+1)}{r^2}U_H(r)+\begin{pmatrix}(E^2-M_o^{\ 2})-2\alpha_{ML}(E^2-M_o^{\ 2})^2\\ -(E+M_o)\left(\frac{(V_o\omega_H)}{2}-\frac{(V_o\omega_H)}{2}\coth\omega_H r\right)\\ +4\alpha_{ML}(E^2-M_o^{\ 2})(E+M_o)\left(\frac{(V_o\omega_H)}{2}-\frac{(V_o\omega_H)}{2}\coth\omega_H r\right)\\ -2\alpha_{ML}(E+M_o)^2\begin{pmatrix}\frac{(V_o\omega_H)^2}{4}\coth^2\omega_H r\\ -\frac{(V_o\omega_H)^2}{2}\coth\omega_H r+\frac{(V_o\omega_H)^2}{4}\end{pmatrix}\end{pmatrix}U_H(r)=0$$

$$\frac{d^2U_H(r)}{dr^2}-\frac{L(L+1)}{r^2}U_H(r)+\begin{pmatrix}(E^2-M_o{}^2)-2\alpha_{ML}(E^2-M_o{}^2)^2\\ +(E+M_o)\frac{(V_o\omega_H)}{2}\coth\omega_H r-(E+M_o)\frac{(V_o\omega_H)}{2}\\ -2\alpha_{ML}(E^2-M_o{}^2)(E+M_o)(V_o\omega_H)\coth\omega_H r\\ +2\alpha_{ML}(E^2-M_o{}^2)(E+M_o)(V_o\omega_H)\\ -\alpha_{ML}(E+M_o)^2\frac{(V_o\omega_H)^2}{2}(1+\operatorname{csch}^2\omega_H r)\\ +\alpha_{ML}(E+M_o)^2(V_o\omega_H)^2\coth\omega_H r-\alpha_{ML}(E+M_o)^2\frac{(V_o\omega_H)^2}{2}\end{pmatrix}U_H(r)=0$$

$$\frac{d^2U_H(r)}{dr^2}-\frac{L(L+1)}{r^2}U_H(r)+\begin{pmatrix}-\alpha_{ML}(E+M_o)^2\frac{(V_o\omega_H)^2}{2}\operatorname{csch}^2\omega_H r\\ +\begin{pmatrix}(E+M_o)\frac{(V_o\omega_H)}{2}+\alpha_{ML}(E+M_o)^2(V_o\omega_H)^2\\ -2\alpha_{ML}(E^2-M_o{}^2)(E+M_o)(V_o\omega_H)\end{pmatrix}\coth\omega_H r\\ +\begin{pmatrix}(E^2-M_o{}^2)-(E+M_o)\frac{(V_o\omega_H)}{2}\\ +2\alpha_{ML}(E^2-M_o{}^2)(E+M_o)(V_o\omega_H)\\ -\alpha_{ML}(E+M_o)^2(V_o\omega_H)^2-2\alpha_{ML}(E^2-M_o{}^2)^2\end{pmatrix}\end{pmatrix}U_H(r)=0 \tag{28}$$

The approximation of function $1/r^2$ is given as

$$\frac{1}{r^2}=\frac{\omega_H{}^2}{\sinh^2\omega_H r} \tag{29}$$

By inserting equation (29) in equation (28), we have

$$\left\{\begin{array}{l}\dfrac{d^2U_H(r)}{dr^2}-\left(\dfrac{\omega_H^2L(L+1)+\alpha_{ML}(E+M_o)^2\dfrac{(V_o\omega_H)^2}{2}}{\sinh^2\omega_H r}\right)U_H(r)\\ +\left(\begin{array}{l}+\left(\begin{array}{l}(E+M_o)\dfrac{(V_o\omega_H)}{2}+\alpha_{ML}(E+M_o)^2(V_o\omega_H)^2\\ -2\alpha_{ML}(E^2-M_o{}^2)(E+M_o)(V_o\omega_H)\end{array}\right)\coth\omega_H r\\ +\left(\begin{array}{l}(E^2-M_o{}^2)-(E+M_o)\dfrac{(V_o\omega_H)}{2}\\ +2\alpha_{ML}(E^2-M_o{}^2)(E+M_o)(V_o\omega_H)\\ -\alpha_{ML}(E+M_o)^2(V_o\omega_H)^2-2\alpha_{ML}(E^2-M_o{}^2)^2\end{array}\right)\end{array}\right)U_H(r)\end{array}\right\}=0 \tag{30}$$

To get simple equation, the equation (30) is reduce become

$$\frac{d^2U_H(r)}{dr^2}-\left[\frac{v_{PH}(v_{PH}-1)}{\sinh^2\omega_H r}-2q_{PH}\coth\omega_H r+\kappa_{PH}^2\right]U_H(r)=0 \tag{31}$$

where

$$v_{PH}(v_{PH}-1)=\left(\omega_H^2L(L+1)+\alpha_{ML}(E+M_o)^2\frac{(V_o\omega_H)^2}{2}\right) \tag{32}$$

$$2q_{PH}=\left((E+M_o)\frac{(V_o\omega_H)}{2}+\alpha_{ML}(E+M_o)^2(V_o\omega_H)^2-2\alpha_{ML}(E^2-M_o{}^2)(E+M_o)(V_o\omega_H)\right) \tag{33}$$

$$-\kappa_{PH}^2=\left(\begin{array}{l}(E^2-M_o{}^2)-(E+M_o)\dfrac{(V_o\omega_H)}{2}+2\alpha_{ML}(E^2-M_o{}^2)(E+M_o)(V_o\omega_H)\\ -\alpha_{ML}(E+M_o)^2(V_o\omega_H)^2-2\alpha_{ML}(E^2-M_o{}^2)^2\end{array}\right) \tag{34}$$

Equation (34) is second order differential equation that will be reduced to hypergeometric differential equation type, by letting

$$\coth\omega_H r=1-2z \tag{35}$$

$$-\omega_H\,\mathrm{csch}^2\,\omega_H r dr=-2dz \tag{36}$$

$$\frac{1}{\sinh^2\omega_H r}=\csc h^2\omega_H r=\coth^2\omega_H r-1=(1-2z)^2-1=(1-4z+4z^2)-1=-4z+4z^2=4z^2-4z=4z(z-1) \tag{37}$$

$$\frac{dz}{dr}=\frac{-\omega_H\,\mathrm{csch}^2\,\omega_H r}{-2}=\frac{\omega_H\,\mathrm{csch}^2\,\omega_H r}{2}=\frac{4\omega_H z(z-1)}{2}=2\omega_H z(z-1) \tag{38}$$

$$\frac{d}{dr}=\frac{d}{dz}\frac{dz}{dr}=2\omega_H(z^2-z)\frac{d}{dz} \tag{39}$$

$$\begin{aligned}\frac{d^2}{d\beta^2}&=2\omega_H z(z-1)\frac{d}{dz}\left(2\omega_H z(z-1)\frac{d}{dz}\right)\\&=2\omega_H z(z-1)\left(2\omega_H z(z-1)\frac{d^2}{dz^2}+2\omega_H(2z-1)\frac{d}{dz}\right)\\&=4\omega_H{}^2\,\mathrm{z}^2(z-1)^2\frac{d^2}{dz^2}+4\omega_H{}^2 z(z-1)(2z-1)\frac{d}{dz}\end{aligned} \tag{40}$$

Then, equation (39) and (40) is substituted in equation (34), we get

$$4\omega^2\,\mathrm{z}^2(\mathrm{z}-1)^2\frac{d^2U_H(r)}{dz^2}+4\omega_H{}^2\,\mathrm{z}(\mathrm{z}-1)(2z-1)\frac{dU_H(r)}{dz}-\left[4v_{PH}(v_{PH}-1)\mathrm{z}(\mathrm{z}-1)-2q_{PH}(1-2z)+\kappa_{PH}^2\right]U_H(r)=0$$

$$z(1-z)\frac{d^2U_H(r)}{dz^2}+(1-2z)\frac{dU_H(r)}{dz}+\left[\frac{v_{PH}(v_{PH}-1)}{\omega_H{}^2}+\frac{2q(1-2z)}{4\omega_H{}^2 z(1-z)}-\frac{\kappa_{PH}{}^2}{4\omega_H{}^2 z(1-z)}\right]U_H(r)=0$$

$$z(1-z)\frac{d^2U_H(r)}{dz^2}+(1-2z)\frac{dU_H(r)}{dz}+\left[v'_{PH}(v'_{PH}-1)+\frac{2q'_{PH}(1-2z)}{4z(1-z)}-\frac{\kappa'^2_{PH}}{4z(1-z)}\right]U_H(r)=0 \tag{41}$$

with

$$v'_{PH}(v'_{PH}-1)=\frac{v_{PH}(v_{PH}-1)}{\omega_H^2} \tag{42}$$

$$2q'_{PH}=\frac{2q_{PH}}{\omega_H^2} \tag{43}$$

$$\kappa'^2_{PH}=\frac{\kappa_{PH}^2}{\omega_H^2} \tag{44}$$

To solve a part of equation (41) is use

$$\frac{1}{4z(1-z)}=\frac{1}{4z}+\frac{1}{4(1-z)} \tag{45}$$

Then equation (45) is substituted in equation (41), we get

$$v'_{PH}\left(v'_{PH}-1\right)+\frac{2q'_{PH}\left(1-2z\right)}{4z\left(1-z\right)}-\frac{\kappa'^{2}_{PH}}{4z\left(1-z\right)}$$

$$v'_{PH}\left(v'_{PH}-1\right)+\frac{2q'_{PH}}{4z(1-z)}-\frac{4q'_{PH}z}{4z(1-z)}-\frac{\kappa'^{2}_{PH}}{4z(1-z)}$$

$$v'_{PH}\left(v'_{PH}-1\right)+\frac{2q'_{PH}}{4z}+\frac{2q'_{PH}}{4(1-z)}-\frac{4q'_{PH}}{4(1-z)}-\frac{\kappa'^{2}_{PH}}{4z}-\frac{\kappa'^{2}_{PH}}{4(1-z)}$$

$$v'_{PH}\left(v'_{PH}-1\right)+\frac{2q'_{PH}}{4z}-\frac{2\,q'_{PH}}{4(1-z)}-\frac{\kappa'^{2}_{PH}}{4z}-\frac{\kappa'^{2}_{PH}}{4(1-z)}$$

$$v'_{PH}\left(v'_{PH}-1\right)-\frac{-2\,q'_{PH}+\kappa'^{2}_{PH}}{4z}-\frac{2q'_{PH}+\kappa'^{2}_{PH}}{4(1-z)}$$

So equation (41) become,

$$z\left(1-z\right)\frac{d^{2}U_{H}\left(r\right)}{dz^{2}}+\left(1-2z\right)\frac{dU_{H}\left(r\right)}{dz}+\left[v'_{PH}\left(v'_{PH}-1\right)-\frac{-2q'_{PH}+\kappa'^{2}_{PH}}{4z}-\frac{2q'_{PH}+\kappa'^{2}_{PH}}{4(1-z)}\right]U_{H}\left(r\right)=0 \tag{46}$$

where,

$$\begin{aligned}&-2q'_{PH}+\kappa'^{2}_{PH}=4\alpha^{2}_{PH}\\&\textit{and}\\&2q'_{PH}+\kappa'^{2}_{PH}=4\beta^{2}_{PH}\end{aligned} \tag{47}$$

By inserting equation (47) in equation (46), so we have

$$z\left(1-z\right)\frac{d^{2}U_{H}\left(r\right)}{dz^{2}}+\left(1-2z\right)\frac{dU_{H}\left(r\right)}{dz}+\left[v'_{PH}\left(v'_{PH}-1\right)-\frac{4\alpha^{2}_{PH}}{4z}-\frac{4\beta^{2}_{PH}}{4\left(1-z\right)}\right]U_{H}\left(r\right)=0 \tag{48}$$

To transform AIM type, we use new wave function is given

$$U_{H}\left(r\right)=z^{\alpha_{PH}}\left(1-z\right)^{\beta_{PH}}f\left(z\right) \tag{49}$$

The equation (48) is differential of z, the first differential equation (48) is

$$\frac{dU_{H}\left(r\right)}{dz}=\frac{\alpha_{PH}}{z}z^{\alpha_{PH}}\left(1-z\right)^{\beta_{PH}}f_{H}\left(z\right)-\frac{\beta_{PH}}{\left(1-z\right)}z^{\alpha_{PH}}\left(1-z\right)^{\beta_{PH}}f_{H}\left(z\right)+z^{\alpha_{PH}}\left(1-z\right)^{\beta_{PH}}f'_{H}\left(z\right) \tag{50}$$

The second differential equation is

$$\frac{d^{2}U_{H}\left(r\right)}{dz^{2}}=\left[\begin{array}{l}\frac{\alpha_{PH}\left(\alpha_{PH}-1\right)}{z^{2}}z^{\alpha_{PH}}\left(1-z\right)^{\beta_{PH}}f_{H}\left(z\right)+\frac{\beta_{PH}\left(\beta_{PH}-1\right)}{\left(1-z\right)^{2}}z^{\alpha_{PH}}\left(1-z\right)^{\beta_{PH}}f_{H}\left(z\right)\\+z^{\alpha_{PH}}\left(1-z\right)^{\beta_{PH}}f''_{H}\left(z\right)-\frac{2\alpha_{PH}\beta_{PH}}{z\left(1-z\right)}z^{\alpha_{PH}}\left(1-z\right)^{\beta_{PH}}f_{H}\left(z\right)\\+\frac{2\alpha_{PH}}{z}z^{\alpha_{PH}}\left(1-z\right)^{\beta_{PH}}f'_{H}\left(z\right)-\frac{2\beta_{PH}}{\left(1-Z\right)}z^{\alpha_{PH}}\left(1-z\right)^{\beta_{PH}}f'_{H}\left(z\right)\end{array}\right] \tag{51}$$

Then equation (49) (50) and (51) is substituted in equation (48), so we obtain

$$\left\{\begin{array}{l} z(1-z)\left[\begin{array}{l}\frac{\alpha_{PH}(\alpha_{PH}-1)}{z^2}z^{\alpha_{PH}}(1-z)^{\beta_{PH}}\, f_H(z)+\frac{\beta_{PH}(\beta_{PH}-1)}{(1-z)^2}z^{\alpha_{PH}}(1-z)^{\beta_{PH}}\, f_H(z)\\ +z^{\alpha_{PH}}(1-z)^{\beta_{PH}}\, f_H^{''}(z)-\frac{2\alpha_{PH}\beta_{PH}}{z(1-z)}z^{\alpha_{PH}}(1-z)^{\beta_{PH}}\, f_H(z)\\ +\frac{2\alpha_{PH}}{z}z^{\alpha_{PH}}(1-z)^{\beta_{PH}}\, f_H^{'}(z)-\frac{2\beta_{LC}}{(1-y)}y^{\alpha_{LC}}(1-y)^{\beta_{LC}}\, g_C^{'}(y)\end{array}\right]\\ +(1-2z)\left[\begin{array}{l}\frac{\alpha_{PH}}{z}z^{\alpha_{PH}}(1-z)^{\beta_{PH}}\, f_H(z)-\frac{\beta_{LC}}{(1-z)}z^{\alpha_{PH}}(1-z)^{\beta_{PH}}\, f_H(z)\\ +z^{\alpha_{PH}}(1-z)^{\beta_{PH}}\, f_H^{'}(z)\end{array}\right]\\ +\left[v_{PH}^{'}\left(v_{PH}^{'}-1\right)-\frac{4\alpha_{PH}^2}{4z}-\frac{4\beta_{PH}^2}{4(1-z)}\right]\left[z^{\alpha_{PH}}(1-z)^{\beta_{PH}}\, f_H(z)\right] \end{array}\right\}=0 \tag{52}$$

$$\begin{array}{l}\left[\begin{array}{l}\frac{\alpha_{PH}(\alpha_{PH}-1)}{z}f_H(z)-\alpha_{PH}(\alpha_{PH}-1)f_H(z)+\frac{z\beta_{PH}(\beta_{PH}-1)}{(1-z)}f_H(z)\\ +z(1-z)f_H^{''}(z)-2\alpha_{PH}\beta_{PH}f_H(z)+2\alpha_{PH}(1-z)f_H^{'}(z)-2z\beta_{PH}f_H^{'}(z)\end{array}\right]\\ +\left[\frac{\alpha_{PH}}{z}f_H(z)-2\alpha_{PH}f_H(z)-(1-2z)\frac{\beta_{PH}}{(1-z)}f_H(z)+(1-2z)f_H^{'}(z)\right]\\ +\left[v_{PH}^{'}\left(v_{PH}^{'}-1\right)-\frac{4\alpha_{PH}^2}{4z}-\frac{4\beta_{PH}^2}{4(1-z)}\right]f_H(z)=0\end{array} \tag{53}$$

Solve a part of equation (53) is given

$$\begin{aligned}\frac{z\beta_{PH}(\beta_{PH}-1)}{(1-z)}&=\frac{(1-1+z)\beta_{PH}(\beta_{PH}-1)}{(1-z)}\\ &=\frac{\beta_{PH}(\beta_{PH}-1)}{(1-z)}-\frac{(1-z)\beta_{PH}(\beta_{PH}-1)}{(1-z)}\\ &=\frac{\beta_{PH}(\beta_{PH}-1)}{(1-z)}-\beta_{PH}(\beta_{PH}-1)\end{aligned} \tag{54}$$

$$(1-2z)\frac{\beta_{PH}}{(1-z)}=2\left(\frac{1}{2}-z\right)\frac{\beta_{PH}}{(1-z)}$$
$$=2\frac{\left(\frac{1}{2}+\frac{1}{2}-\frac{1}{2}-z\right)\beta_{PH}}{(1-z)}$$
$$=2\left(\frac{(1-z)\beta_{PH}}{(1-z)}-\frac{1}{2}\frac{\beta_{PH}}{(1-z)}\right)$$
$$=2\beta_{PH}-\frac{\beta_{PH}}{(1-z)} \tag{55}$$

Then , equation (54) and (55) is inserted in equation (53), we have

$$\left\{\begin{array}{l}\left[\begin{array}{l}\frac{\alpha_{PH}(\alpha_{PH}-1)}{z}+\frac{\alpha_{PH}}{z}-2\alpha_{PH}-\alpha_{PH}(\alpha_{PH}-1)-2\beta_{PH}-\beta_{PH}(\beta_{PH}-1)\\+\frac{\beta_{PH}(\beta_{PH}-1)}{(1-z)}+\frac{\beta_{PH}}{(1-z)}-\frac{4\beta_{PH}^2}{4(1-z)}-\frac{4\alpha_{PH}^2}{4z}-2\alpha_{PH}\beta_{PH}+v_{PH}^{'}(v_{PH}^{'}-1)\end{array}\right]f_H(z)\\+\left[2\alpha_{PH}(1-z)-2z\beta_{PH}+1-2z\right]f_H^{'}(z)+z(1-z)f_H^{'}(z)\end{array}\right\}=0 \tag{56}$$

where

$$\frac{\alpha_{PH}(\alpha_{PH}-1)}{z}+\frac{\alpha_{PH}}{z}-\frac{4\alpha_{PH}^2}{4z}=\frac{4\alpha_{PH}^2-4\alpha_{PH}+4\alpha_{PH}-4\alpha_{PH}^2}{4z}=0 \tag{57}$$

$$\frac{\beta_{PH}(\beta_{PH}-1)}{(1-z)}+\frac{\beta_{PH}}{(1-z)}-\frac{4\beta_{PH}^2}{4(1-z)}=\frac{4\beta_{PH}^2-4\beta_{PH}+4\beta_{PH}-4\beta_{PH}^2}{(1-z)}=0 \tag{58}$$

So equation (56) become,

$$\left\{\begin{array}{l}z(1-z)f_H^{"}(z)+\left[(2\alpha_{PH}+1)-(2\alpha_{PH}+2\beta_{PH}+2)z\right]f_H^{'}(z)\\+\left[v_{PH}^{'}(v_{PH}^{'}-1)-(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)\right]f_H(z)\end{array}\right\}=0 \tag{59}$$

Next, equation (59) to transform in AIM type is given

$$f_H^{"}(z)+\left[\frac{(2\alpha_{PH}+1)-(2\alpha_{PH}+2\beta_{PH}+2)y}{z(1-z)}\right]f_H^{'}(z)+\left[\frac{v_{PH}^{'}(v_{PH}^{'}-1)-(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)}{z(1-z)}\right]f_H(z)=0 \tag{60}$$

The general AIM type as follow,

$$f''_{n_r}(z)=\lambda_0(z)f'_{n_r}(z)+s_0(z)f_{n_r}(z) \tag{61}$$

By compare equation (60) and (61) we have,

$$\lambda_{0_{PH}}=\frac{\left(2\alpha_{PH}+2\beta_{PH}+2\right)z-\left(2\alpha_{PH}+1\right)}{z\left(1-z\right)} \quad (62)$$

$$s_{0_{PH}}=\frac{\left(\alpha_{PH}+\beta_{PH}\right)\left(\alpha_{PH}+\beta_{PH}+1\right)-v'_{PH}\left(v'_{PH}-1\right)}{z\left(1-z\right)} \quad (63)$$

We can be written,

$$\lambda_{0_{PH}}=\frac{-\left(2\alpha_{PH}+1\right)}{z}+\frac{\left(2\beta_{PH}+1\right)}{1-z} \quad (64)$$

$$s_o=\left[\begin{array}{l}\frac{\left(\alpha_{PH}+\beta_{PH}\right)\left(\alpha_{PH}+\beta_{PH}+1\right)-v'_{PH}\left(v'_{PH}-1\right)}{z}\\+\frac{\left(\alpha_{PH}+\beta_{PH}\right)\left(\alpha_{PH}+\beta_{PH}+1\right)-v'_{PH}\left(v'_{PH}-1\right)}{\left(1-z\right)}\end{array}\right] \quad (65)$$

Next the general AIM type is given

$$f'''_{n_r}(z)=\lambda_1(z)f'_{n_r}(z)+s_1(z)f_{n_r}(z) \quad (66)$$

where

$$\lambda_1(z)=\lambda_0'(z)+s_0(z)+\lambda_0^{\,2}(z)$$
$$s_1(z)=s_0'(z)+s_0(z)\lambda_0(z) \quad (67)$$

We have differential of equation (64) and (65)

$$\lambda'_{0_{PH}}=\frac{\left(2\alpha_{PH}+1\right)}{z^2}+\frac{\left(2\beta_{PH}+1\right)}{\left(1-z\right)^2} \quad (68)$$

$$s'_{0_{PH}}=\left[\begin{array}{l}-\frac{\left(\alpha_{PH}+\beta_{PH}\right)\left(\alpha_{PH}+\beta_{PH}+1\right)-v'_{PH}\left(v'_{PH}-1\right)}{z^2}\\+\frac{\left(\alpha_{PH}+\beta_{PH}\right)\left(\alpha_{PH}+\beta_{PH}+1\right)-v'_{PH}\left(v'_{PH}-1\right)}{\left(1-z\right)^2}\end{array}\right] \quad (69)$$

So, equation (60) is substituted in equation (62)-(63) and (68)-(69), we get

$$\lambda_{1_{PH}}(z)=\left[\begin{array}{l}\left(\frac{(2\alpha_{PH}+1)}{z^2}+\frac{(2\beta_{PH}+1)}{(1-z)^2}\right)\\+\left(\begin{array}{l}\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)}{z}\\+\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)}{(1-z)}\end{array}\right)\\+\left(\frac{-(2\alpha_{PH}+1)}{z}+\frac{(2\beta_{PH}+1)}{(1-z)}\right)^2\end{array}\right] \tag{70}$$

$$s_{1_{PH}}(z)=\left[\begin{array}{l}\left(\begin{array}{l}-\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)}{z^2}\\+\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)}{(1-z)^2}\end{array}\right)\\+\left(\begin{array}{l}\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)}{z}\\+\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)}{(1-z)}\end{array}\right)\left(\begin{array}{l}\frac{-(2\alpha_{PH}+1)}{z}\\+\frac{(2\beta_{PH}+1)}{(1-z)}\end{array}\right)\end{array}\right] \tag{71}$$

To get eigen value, we use

$$\Delta_k(z)=\lambda_k(z)s_{k-1}(z)-\lambda_{k-1}(z)s_k(z)=0$$

$$\Delta_1(z)=\lambda_1(z)s_0(z)-\lambda_0(z)s_1(z)=0 \tag{72}$$

$$\Delta_1(z)=\left[\begin{array}{l}\left[\begin{array}{l}\left(\frac{(2\alpha_{PH}+1)}{z^2}+\frac{(2\beta_{PH}+1)}{(1-z)^2}\right)\\+\left(\begin{array}{l}\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)}{z}\\+\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)}{(1-z)}\end{array}\right)\\+\left(\frac{-(2\alpha_{PH}+1)}{z}+\frac{(2\beta_{PH}+1)}{(1-z)}\right)^2\end{array}\right]\\\left(\begin{array}{l}\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)}{z}\\+\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)}{(1-z)}\end{array}\right)\end{array}\right]-\left[\left(\begin{array}{l}\frac{-(2\alpha_{PH}+1)}{z}\\+\frac{(2\beta_{PH}+1)}{(1-z)}\end{array}\right)\left[\begin{array}{l}\left(\begin{array}{l}-\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)}{z^2}\\+\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)}{(1-z)^2}\end{array}\right)\\+\left(\begin{array}{l}\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)}{z}\\+\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)}{(1-z)}\end{array}\right)\left(\begin{array}{l}\frac{-(2\alpha_{PH}+1)}{z}\\+\frac{(2\beta_{PH}+1)}{1-z}\end{array}\right)\end{array}\right]\right]=0 \tag{73}$$

To get simple equation (73), we use

$$A=(2\alpha_{PH}+1) \tag{74}$$

$$B=\left(2\beta_{PH}+1\right) \tag{75}$$

$$C=\left(\alpha_{PH}+\beta_{PH}\right)\left(\alpha_{PH}+\beta_{PH}+1\right)-v'_{PH}\left(v'_{PH}-1\right) \tag{76}$$

So equation (73) become,

$$\Delta_1(z)=\left[\left(\begin{array}{l}\left(\dfrac{A}{z^2}+\dfrac{B}{(1-z)^2}\right)\\+\left(\dfrac{C}{z}+\dfrac{C}{(1-z)}\right)\\+\left(\dfrac{-A}{z}+\dfrac{B}{1-z}\right)^2\end{array}\right)\left(\dfrac{C}{z}+\dfrac{C}{(1-z)}\right)-\left[\left(\dfrac{-A}{z}+\dfrac{B}{1-z}\right)\left(\begin{array}{l}\left(-\dfrac{C}{z^2}+\dfrac{C}{(1-z)^2}\right)\\+\left(\dfrac{C}{z}+\dfrac{C}{(1-z)}\right)\left(\dfrac{-A}{z}+\dfrac{B}{1-z}\right)\end{array}\right)\right]\right]=0 \tag{77}$$

$$\Delta_1(z)=\left(\dfrac{C}{z}+\dfrac{C}{(1-z)}\right)\left[\left(\begin{array}{l}\left(\dfrac{A}{z^2}+\dfrac{B}{(1-z)^2}\right)\\+\left(\dfrac{C}{z}+\dfrac{C}{(1-z)}\right)\\+\left(\dfrac{-A}{z}+\dfrac{B}{1-z}\right)^2\end{array}\right)-\left[\left(\dfrac{-A}{z}+\dfrac{B}{1-z}\right)\left(\begin{array}{l}\left(-\dfrac{1}{z}+\dfrac{1}{(1-z)}\right)\\+\left(\dfrac{-A}{z}+\dfrac{B}{1-z}\right)\end{array}\right)\right]\right]=0$$

$$\Delta_1(z)=\left(\dfrac{C}{z}+\dfrac{C}{(1-z)}\right)\left[\left(\begin{array}{l}\left(\dfrac{A}{z^2}+\dfrac{B}{(1-z)^2}\right)\\+\left(\dfrac{C}{z}+\dfrac{C}{(1-z)}\right)\\+\left(\dfrac{-A}{z}+\dfrac{B}{1-z}\right)^2\end{array}\right)-\left(\begin{array}{l}\left(\dfrac{-A}{z}+\dfrac{B}{1-z}\right)\left(-\dfrac{1}{z}+\dfrac{1}{(1-z)}\right)\\-\left(\dfrac{-A}{z}+\dfrac{B}{1-z}\right)^2\end{array}\right)\right]=0$$

$$\Delta_1(z)=\left(\dfrac{C}{z}+\dfrac{C}{(1-z)}\right)\left[\left(\left(\dfrac{A}{z^2}+\dfrac{B}{(1-z)^2}\right)+\left(\dfrac{C}{z}+\dfrac{C}{(1-z)}\right)\right)-\left(\dfrac{-A}{z}+\dfrac{B}{1-z}\right)\left(-\dfrac{1}{z}+\dfrac{1}{(1-z)}\right)\right]=0$$

$$\Delta_1(z)=\left(\frac{C}{z}+\frac{C}{(1-z)}\right)\left[\left(\left(\frac{A}{z^2}+\frac{B}{(1-z)^2}\right)+\left(\frac{C}{z}+\frac{C}{(1-z)}\right)\right)-\left(\frac{-A}{z}+\frac{B}{1-z}\right)\left(-\frac{(1-z)+z}{z(1-z)}\right)\right]=0$$

$$\Delta_1(z)=\left(\frac{C}{z}+\frac{C}{(1-z)}\right)\left[\left(\left(\frac{A}{z^2}+\frac{B}{(1-z)^2}\right)+\left(\frac{C}{z}+\frac{C}{(1-z)}\right)\right)-\left(\frac{-A}{z}+\frac{B}{1-z}\right)\left(\frac{2z-1}{z(1-z)}\right)\right]=0$$

$$\Delta_1(z)=\left(\frac{C}{z}+\frac{C}{(1-z)}\right)\left[\left(\left(\frac{A(1-z)^2+Bz^2}{z^2(1-z)^2}\right)+\left(\frac{Cz(1-z)^2+Cz^2(1-z)}{z^2(1-z)^2}\right)\right)-\left(\frac{-A(1-z)+B}{z^2(1-z)^2}\right)(2z-1)\right]=0$$

$$\Delta_1(z)=\left(\frac{C}{z}+\frac{C}{(1-z)}\right)\left[\left(\frac{A(1-z)^2+Bz^2}{z^2(1-z)^2}\right)+\left(\frac{Cz(1-z)^2+Cz^2(1-z)}{z^2(1-z)^2}\right)+\left(\frac{-A(1-z)+Bz}{z^2(1-z)^2}\right)-2z\left(\frac{-A(1-z)+Bz}{z^2(1-z)^2}\right)\right]=0$$

$$\Delta_1(z)=\left(\frac{C}{z}+\frac{C}{(1-z)}\right)\left[\begin{array}{l}\left(\frac{A(1-2z+z^2)+Bz^2}{z^2(1-z)^2}\right)+\left(\frac{Cz(1-2z+z^2)+Cz^2(1-z)}{z^2(1-z)^2}\right)\\+\left(\frac{-A(1-z)+Bz}{z^2(1-z)^2}\right)-2z\left(\frac{-A(1-z)+Bz}{z^2(1-z)^2}\right)\end{array}\right]=0$$

$$\Delta_1(z)=\left(\frac{C}{z}+\frac{C}{(1-z)}\right)\left[\begin{array}{l}\left(\frac{A(1-2z+z^2)+Bz^2}{z^2(1-z)^2}\right)+\left(\frac{Cz-2Cz^2+Cz^3+Cz^2-Cz^3}{z^2(1-z)^2}\right)\\+\left(\frac{-A(1-z)+Bz}{z^2(1-z)^2}\right)-2z\left(\frac{-A(1-z)+Bz}{z^2(1-z)^2}\right)\end{array}\right]=0$$

$$\Delta_1(z)=\left(\frac{C}{z}+\frac{C}{(1-z)}\right)\left[\begin{array}{l}\left(\frac{A(1-2z+z^2)+Bz^2}{z^2(1-z)^2}\right)+\left(\frac{Cz-Cz^2}{z^2(1-z)^2}\right)\\+\left(\frac{-A(1-z)+Bz}{z^2(1-z)^2}\right)-2z\left(\frac{-A(1-z)+Bz}{z^2(1-z)^2}\right)\end{array}\right]=0$$

$$\Delta_1(z)=\left(\frac{C}{z}+\frac{C}{(1-z)}\right)\left[\left(\frac{A-2Az+Az^2+Bz^2+Cz-Cz^2-A+Az+Bz+2Az-2Az^2-2Bz^2}{z^2(1-z)^2}\right)\right]=0$$

$$\Delta_1(z)=\left(\frac{C}{z}+\frac{C}{(1-z)}\right)\left[\left(\frac{-Az^2-Bz^2-Cz^2+Cz+Az+Bz}{z^2(1-z)^2}\right)\right]=0$$

$$\Delta_1(z)=\left(\frac{C}{z}+\frac{C}{(1-z)}\right)\left[\left(\frac{(A+B+C)z-(A+B+C)z^2}{z^2(1-z)^2}\right)\right]=0$$

$$\Delta_1(z)=\left(\frac{C}{z}+\frac{C}{(1-z)}\right)\left[\left(\frac{(A+B+C)z(1-z)}{z^2(1-z)^2}\right)\right]=0$$

$$\Delta_1(z)=\left(\frac{C}{z}+\frac{C}{(1-z)}\right)\left[\left(\frac{(A+B+C)}{z(1-z)}\right)\right]=0$$

$$\Delta_1(z)=\left(\frac{C(1-z)+Cz}{z(1-z)}\right)\left[\left(\frac{(A+B+C)}{z(1-z)}\right)\right]=0$$

$$\Delta_1(z)=\left(\frac{C}{z(1-z)}\right)\left[\left(\frac{(A+B+C)}{z(1-z)}\right)\right]=0 \tag{78}$$

Then inserting equation (74)-(76) in equation (78), we obtain

$$\Delta_1(z)=\left\{\begin{array}{l}\left(\dfrac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)}{z(1-z)}\right)\\ \left[\left(\dfrac{((2\alpha_{PH}+1)+(2\beta_{PH}+1)+(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1))}{z(1-z)}\right)\right]\end{array}\right\}=0 \tag{79}$$

So, we obtain eigen value is given

$$\begin{aligned}&(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1)=0\\&(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)=v'_{PH}(v'_{PH}-1)\end{aligned} \tag{80}$$

and

$$\begin{aligned}&((2\alpha_{PH}+1)+(2\beta_{PH}+1)+(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v'_{PH}(v'_{PH}-1))=0\\&v'_{PH}(v'_{PH}-1)=(2\alpha_{PH}+2\beta_{PH}+2+\alpha_{PH}{}^2+\alpha_{PH}\beta_{PH}+\alpha_{PH}+\beta_{PH}{}^2+\alpha_{PH}\beta_{PH}+\beta_{PH})\\&v'_{PH}(v'_{PH}-1)=(\alpha_{PH}{}^2+\beta_{PH}{}^2+2\alpha_{PH}\beta_{PH}+3\alpha_{PH}+3\beta_{PH}+2)\\&v'_{PH}(v'_{PH}-1)=(\alpha_{PH}+\beta_{PH}+1)(\alpha_{PH}+\beta_{PH}+2)\end{aligned} \tag{81}$$

From equation (80) and (81), we have

$$v'_{PH}\left(v'_{PH}-1\right)=\left(\alpha_{PH}+\beta_{PH}+n\right)\left(\alpha_{PH}+\beta_{PH}+\left(n+1\right)\right) \tag{82}$$

The eigen value is used to obtain energy of Klein Gordon equation in minimal length effect. To get energy follow as

$$v'_{PH}\left(v'_{PH}-1\right)=\left(\alpha_{PH}+\beta_{PH}+1\right)\left(\alpha_{PH}+\beta_{PH}+\left(n+1\right)\right)$$
$$v'_{PH}\left(v'_{PH}-1\right)=\left(\alpha_{PH}+\beta_{PH}+\left(n+1\right)\right)^2-\left(\alpha_{PH}+\beta_{PH}+\left(n+1\right)\right) \tag{83}$$

where

$$v'_{PH}\left(v'_{PH}-1\right)=\xi_H$$

$$\mathrm{X}=\left(\alpha+\beta+\left(n+1\right)\right)$$

So we have

$$\xi_H=\mathrm{X}^2-\mathrm{X}$$
$$\xi_H=\mathrm{X}\left(\mathrm{X}-1\right)$$
$$\xi_H=\left(\mathrm{X}-\frac{1}{2}\right)^2-\frac{1}{4}$$
$$\xi_H+\frac{1}{4}=\left(\mathrm{X}-\frac{1}{2}\right)^2$$
$$\sqrt{\xi_H+\frac{1}{4}}=\left(\mathrm{X}-\frac{1}{2}\right)$$

$$\sqrt{\xi_H+\frac{1}{4}}=\alpha_{PH}+\beta_{PH}+\left(n+1\right)-\frac{1}{2}$$
$$\sqrt{\xi_H+\frac{1}{4}}=\alpha_{PH}+\beta_{PH}+n+\frac{1}{2}$$
$$\alpha_{PH}+\beta_{PH}=\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)$$
$$\left(\alpha_{PH}+\beta_{PH}\right)^2=\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2$$
$$\alpha_{PH}^2+2\alpha_{PH}\beta_{PH}+\beta_{PH}^2=\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2 \tag{84}$$

Then, inserting equation $-2q'_{PH}+\kappa'^2_{PH}=4\alpha^2_{PH}$ and $2q'_{PH}+\kappa'^2_{PH}=4\beta^2_{PH}$ in equation (86), we obtain

$$\alpha^2{}_{PH}+\beta^2{}_{PH}+2\alpha_{PH}\beta_{PH}=\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2$$

$$\frac{-2q'_{PH}+\kappa'^2_{PH}}{4}+\frac{2q'_{PH}+\kappa'^2_{PH}}{4}+2\left(\sqrt{\frac{-2q'_{PH}+\kappa'^2_{PH}}{4}}\right)\left(\sqrt{\frac{2q'_{PH}+\kappa'^2_{PH}}{4}}\right)=\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2$$

$$\frac{\kappa'^2_{PH}}{2}+2\left(\sqrt{\frac{-2q'_{PH}+\kappa'^2_{PH}}{4}}\right)\left(\sqrt{\frac{2q'_{PH}+\kappa'^2_{PH}}{4}}\right)=\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2$$

$$\frac{\kappa'^2_{PH}}{2}+2\left(\sqrt{\frac{\kappa'^4_{PH}-4q'^2_{PH}}{16}}\right)=\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2$$

$$\frac{\kappa'^2_{PH}}{2}+\frac{1}{2}\sqrt{\kappa'^4_{PH}-4q'^2_{PH}}=\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2$$

$$\frac{1}{2}\sqrt{\kappa'^4_{PH}-4q'^2_{PH}}=\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2-\frac{\kappa'^2_{PH}}{2}$$

$$\left(\frac{1}{2}\sqrt{\kappa'^4_{PH}-4q'^2_{PH}}\right)^2=\left(\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2-\frac{\kappa'^2_{PH}}{2}\right)^2$$

$$\frac{\kappa'^4_{PH}}{4}-\frac{4q'^2_{PH}}{4}=\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^4-\kappa'^2_{PH}\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2+\frac{\kappa'^4_{PH}}{4}$$

$$-\frac{4q'^2_{PH}}{4}=\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^4-\kappa'^2_{PH}\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2$$

$$\kappa'^2_{PH}\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2=\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^4+\frac{4q'^2_{PH}}{4}$$

$$\kappa'^2_{PH}=\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2+\frac{4q'^2_{PH}}{4\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2}$$

$$\kappa'^2_{PH}=\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2+\frac{q'^2_{PH}}{\left(\sqrt{\xi_H+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2}$$

So we have energy equation of Klein Gordon equation in minimal length effect is given

$$\kappa'^2_{PH}=\left(\sqrt{v'_{PH}\left(v'_{PH}-1\right)+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2+\frac{q'^2_{PH}}{\left(\sqrt{v'_{PH}\left(v'_{PH}-1\right)+\frac{1}{4}}-\left(n+\frac{1}{2}\right)\right)^2} \qquad (85)$$

To get approximate follow as,

$$v'_{PH}\left(v'_{PH}-1\right)=\frac{v_{PH}\left(v_{PH}-1\right)}{\omega_H^2}=\frac{1}{\omega_H^2}\left(\omega_H^2 L\left(L+1\right)+\alpha_{ML}\left(E+M_o\right)^2\frac{\left(V_o\omega_H\right)^2}{2}\right)$$

$$v'_{PH}\left(v'_{PH}-1\right)+\frac{1}{4}=\frac{\left(\omega_H^2 L\left(L+1\right)+\alpha_{ML}\left(E+M_o\right)^2\frac{\left(V_o\omega_H\right)^2}{2}\right)}{\omega_H^2}+\frac{1}{4}$$

$$\left(v'_{PH}-\frac{1}{2}\right)^2=\frac{\left(\omega_H^2 L\left(L+1\right)+\alpha_{ML}\left(E+M_o\right)^2\frac{\left(V_o\omega_H\right)^2}{2}\right)}{\omega_H^2}+\frac{1}{4}$$

$$\left(v'_{PH}-\frac{1}{2}\right)=\sqrt{\frac{\left(\omega_H^2 L\left(L+1\right)+\alpha_{ML}\left(E+M_o\right)^2\frac{\left(V_o\omega_H\right)^2}{2}\right)}{\omega_H^2}+\frac{1}{4}}$$

$$v'_{PH}=\frac{1}{2}+\sqrt{\frac{\left(\omega_H^2 L\left(L+1\right)+\alpha_{ML}\left(E+M_o\right)^2\frac{\left(V_o\omega_H\right)^2}{2}\right)}{\omega_H^2}+\frac{1}{4}} \tag{86}$$

Next equation,

$$2q'=\frac{2q_{PH}}{\omega_H^2}=\frac{1}{\omega_H^2}\left(\left(E+M_o\right)\frac{\left(V_o\omega_H\right)}{2}+\alpha_{ML}\left(E+M_o\right)^2\left(V_o\omega_H\right)^2-2\alpha_{ML}\left(E^2-M_o{}^2\right)\left(E+M_o\right)\left(V_o\omega_H\right)\right)$$

$$q'=\frac{2q}{2\omega_H^2}=\frac{1}{2\omega_H^2}\left(\left(E+M_o\right)\frac{\left(V_o\omega_H\right)}{2}+\alpha_{ML}\left(E+M_o\right)^2\left(V_o\omega_H\right)^2-2\alpha_{ML}\left(E^2-M_o{}^2\right)\left(E+M_o\right)\left(V_o\omega_H\right)\right) \tag{87}$$

and equation,

$$\kappa'^2_{PH}=\frac{\kappa_{PH}^2}{\omega_H^2}=-\frac{1}{\omega_H^2}\left(\begin{array}{l}\left(E^2-M_o{}^2\right)-\left(E+M_o\right)\frac{\left(V_o\omega_H\right)}{2}+2\alpha_{ML}\left(E^2-M_o{}^2\right)\left(E+M_o\right)\left(V_o\omega_H\right)\\-\alpha_{ML}\left(E+M_o\right)^2\left(V_o\omega_H\right)^2-2\alpha_{ML}\left(E^2-M_o{}^2\right)^2\end{array}\right) \tag{88}$$

So equation (86)-(88) is substituted in equation (85), so we have

$$\left(E^2-M_o{}^2\right)=\left\{\begin{array}{l}-\omega_H^2\left[\left(\sqrt{v'_{PH}\left(v'_{PH}-1\right)+\frac{1}{4}}-n-\frac{1}{2}\right)^2+\frac{\left(\frac{1}{2\omega_H^2}\left(\begin{array}{l}\left(E+M_o\right)\frac{\left(V_H\omega_H\right)}{2}+\alpha_{ML}\left(E+M_o\right)^2\left(V_H\omega_H\right)^2\\-2\alpha_{ML}\left(E^2-M_o{}^2\right)\left(E+M_o\right)\left(V_H\omega_H\right)\end{array}\right)\right)^2}{\left(\sqrt{v'_{PH}\left(v'_{PH}-1\right)+\frac{1}{4}}-n-\frac{1}{2}\right)^2}\right]\\+\left(E+M_o\right)\frac{\left(V_H\omega_H\right)}{2}+\alpha_{ML}\left(E+M_o\right)^2\left(V_H\omega_H\right)^2+2\alpha_{ML}\left(E^2-M_o{}^2\right)^2-2\alpha_{ML}\left(E^2-M_o{}^2\right)\left(E+M_o\right)\left(V_H\omega_H\right)\end{array}\right\} \tag{89}$$

Where,

$$v'_{PH} = \frac{1}{2} + \sqrt{\frac{\left(\omega_H^2 L(L+1) + \alpha_{ML}(E+M_o)^2 \frac{(V_o\omega_H)^2}{2}\right)}{\omega_H^2} + \frac{1}{4}} \tag{90}$$

$$\zeta_{PH} = \begin{pmatrix} (E+M_o)\frac{(V_H\omega_H)}{2} + \alpha_{ML}(E+M_o)^2 (V_H\omega_H)^2 \\ -2\alpha_{ML}(E^2 - M_o{}^2)(E+M_o)(V_H\omega_H) \end{pmatrix} \tag{91}$$

$$\tau_{PH} = \alpha_{ML}(E+M_o)^2 (V_H\omega_H)^2 + 2\alpha_{ML}(E^2 - M_o{}^2)^2 - 2\alpha_{ML}(E^2 - M_o{}^2)(E+M_o)(V_H\omega_H) \tag{92}$$

The energy equation of klein Gordon equation in minimal length effect is given,

$$(E^2 - M_o{}^2) = -\omega_H^2 \left[\left(\sqrt{v'_{PH}(v'_{PH}-1) + \frac{1}{4}} - n - \frac{1}{2} \right)^2 + \frac{\zeta_{PH}}{4\left(\omega_H^2 \sqrt{v'_{PH}(v'_{PH}-1) + \frac{1}{4}} - n - \frac{1}{2} \right)^2} \right] + (E+M_o)\frac{V_H}{2} + \tau_{PH} \tag{93}$$

Wave Function of Klein-Gordon equation

Equation (60) is reduce to,

$$f_H''(z) = -\left[\frac{(2\alpha_{PH}+1)-(2\alpha_{PH}+2\beta_{PH}+2)z}{z(1-z)}\right]f_H'(z) - \left[\frac{v_{PH}'(v_{PH}'-1)-(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)}{z(1-z)}\right]f_H(z) \tag{94}$$

$$f''(z) = \left[\frac{(2\alpha_{PH}+2\beta_{PH}+2)z-(2\alpha_{PH}+1)}{z(1-z)}\right]f'(z) + \left[\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v_{PH}'(v_{PH}'-1)}{z(1-z)}\right]f(z)$$

$$f''(z) = \left[\frac{(2\alpha_{PH}+2\beta_{PH}+2)z}{z(1-z)} - \frac{(2\alpha_{PH}+1)}{z(1-z)}\right]f'(z) + \left[\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)}{z(1-z)} - \frac{v_{PH}'(v_{PH}'-1)}{z(1-z)}\right]f(z)$$

$$f''(z) = \left[\frac{(2\alpha_{PH}+2\beta_{PH}+2)z}{z(1-z)} - \frac{(2\alpha_{PH}+1)}{z} - \frac{(2\alpha_{PH}+1)}{(1-z)}\right]f'(z) + \left[\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)}{z(1-z)} - \frac{v_{PH}'(v_{PH}'-1)}{z(1-z)}\right]f(z)$$

$$f''(z) = \left[\frac{(2\alpha_{PH}+2\beta_{PH}+2)}{(1-z)} - \frac{(2\alpha_{PH}+1)}{z} - \frac{(2\alpha_{PH}+1)}{(1-z)}\right]f'(z) + \left[\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)}{z(1-z)} - \frac{v_{PH}'(v_{PH}'-1)}{z(1-z)}\right]f(z)$$

$$f''(z) = \left[\frac{(2\alpha_{PH}+2\beta_{PH}+2-2\alpha_{PH}-1)}{(1-z)} - \frac{(2\alpha_{PH}+1)}{z}\right]f'(z) + \left[\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)}{z(1-z)} - \frac{v_{PH}'(v_{PH}'-1)}{z(1-z)}\right]f(z)$$

$$f''(z) = \left[\frac{(2\beta_{PH}+1)}{(1-z)} - \frac{(2\alpha_{PH}+1)}{z}\right]f'(z) + \left[\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)}{z(1-z)} - \frac{v_{PH}'(v_{PH}'-1)}{z(1-z)}\right]f(z)$$

$$f_H''(z) = 2\left[\frac{\left(\beta_{PH}+\frac{1}{2}\right)}{(1-z)} - \frac{\left(\alpha_{PH}+\frac{1}{2}\right)}{z}\right]f_H'(z) + \left[\frac{(\alpha_{PH}+\beta_{PH})(\alpha_{PH}+\beta_{PH}+1)-v_{PH}'(v_{PH}'-1)}{z(1-z)}\right]f_H(z) \tag{95}$$

Equation (95) is eigen function,

$$g_{n_r}''(z_p) = 2\left\{\frac{ax^{N+1}}{1-bz^{N+1}} - \frac{t-1}{z}\right\}g_{n_r}'(z_p) + \frac{Wz^N}{(1-bz)}g_{n_r}(z_p) = 0 \tag{96}$$

By comparing equation (96) and (95), we have

$$a_{PH} = \left(\beta_{PH}+\frac{1}{2}\right) \tag{97}$$

$$N_{PH} = -1 \tag{98}$$

$$b_{PH} = 1 \tag{99}$$

$$t_{PH} = \left(\alpha_{PH}-\frac{1}{2}\right) \tag{100}$$

$$\left(\sigma_{A_{PH}}\right)_n = \frac{\Gamma\left(\sigma_A + n\right)}{\Gamma\left(\sigma_A\right)}$$

with,

$$\sigma_{A_{PH}} = \frac{2t_{PH} + N_{PH} + 3}{N_{PH} + 2}$$
$$= \frac{2\left(\alpha_{PH} - \frac{1}{2}\right) + (-1) + 3}{-1+2}$$
$$\sigma_{A_{PH}} = 2\alpha_{PH} + 1 \qquad (101)$$

and

$$\rho_{PH} = \frac{(2t_{PH}+1)b_{PH} + 2a_{PH}}{(N_{PH}+2)b_{PH}}$$
$$= \frac{\left(2\left(\alpha_{PH} - \frac{1}{2}\right)+1\right)1 + 2\left(\beta_{PH} + \frac{1}{2}\right)}{(-1+2)1}$$
$$\rho_{PH} = 2\alpha_{PH} + 2\beta_{PH} + 1 \qquad (102)$$

By using equation (97)-(102), we get,

$$f_H(z) = (-1)^n C'(1)^n \left(2\alpha_{PH}+1\right)_n {}_2F_1\left(-n, 2\alpha_{PH} + 2\beta_{PH} + 1 + n, 2\alpha_{PH} + 1, z\right) \qquad (103)$$

$$U_H(r) = z^{\alpha_{PH}}(1-z)^{\beta_{PH}} (-1)^n C'(1)^n \left(2\alpha_{PH}+1\right)_n {}_2F_1\left(-n, 2\alpha_{PH} + 2\beta_{PH} + 1 + n, 2\alpha_{PH} + 1, z\right) \qquad (104)$$

The variable is change become

$\coth \rho r = (1-2y)$ $2y = 1 - \coth \rho r$ $y = \frac{1-\coth \rho r}{2}$ (105)	$1 - y = 1 - \frac{1-\coth \rho r}{2}$ $= \frac{2-1+\coth \rho r}{2}$ (106) $= \frac{1+\coth \rho r}{2}$

By applying equation (104)-(106), Wave function for $n = 0$

$${}_2F_1\left(-n, 2\alpha_{PH} + 2\beta_{PH} + 1 + n, 2\alpha_{PH} + 1, z\right) = 1 \qquad (107)$$

$$U_H(r) = z^{\alpha_{PH}}(1-z)^{\beta_{PH}} f_0(z) \qquad (108)$$

$$U_{H_0}(z) = C'\left(\frac{1-\coth \omega_H r}{2}\right)^{\alpha_{PH}} \left(\frac{1+\coth \omega_H r}{2}\right)^{\beta_{PH}} \qquad (109)$$

Wave function $n=1$,

$$f_1(z)=(-1)^1 C'(1)^1 (2\alpha_{PH}+1)_1\, {}_2F_1(-1,2\alpha_{PH}+2\beta_{PH}+2,2\alpha_{PH}+1,z) \tag{110}$$

$${}_2F_1(-1,2\alpha_{PH}+2\beta_{PH}+2,2\alpha_{PH}+1,z)=1+\frac{(-1)(2\alpha_{PH}+2\beta_{PH}+2)\left(\frac{1-\coth\omega_H r}{2}\right)}{(2\alpha_{PH}+1)} \tag{111}$$

$$U_{H_1}(z)=\left\{\begin{array}{l} -C'\left(\frac{1-\coth\omega_H r}{2}\right)^{\alpha_{PH}}\left(\frac{1+\coth\omega_H r}{2}\right)^{\beta_{PH}}(2\alpha_{PH}+1) \\ \left[1+\frac{(-1)(2\alpha_{PH}+2\beta_{PH}+2)\left(\frac{1-\coth\omega_H r}{2}\right)}{(2\alpha_{PH}+1)1!}\right] \end{array}\right\} \tag{112}$$

Wave function $n=2$,

$$f_2(z)=(-1)^2 C'(1)^2 (2\alpha_{PH}+1)(2\alpha_{PH}+2)\, {}_2F_1(-2,2\alpha_{PH}+2\beta_{PH}+3,2\alpha_{PH}+1,z) \tag{113}$$

$${}_2F_1(-2,2\alpha_{PH}+2\beta_{PH}+3,2\alpha_{PH}+1,z)=\left[\begin{array}{l} 1+\frac{(-1)(2\alpha_{PH}+2\beta_{PH}+2)\left(\frac{1-\coth\omega_H r}{2}\right)}{(2\alpha_{PH}+1)1!} \\ +\frac{(-2)(-1)(2\alpha_{PH}+2\beta_{PH}+)(2\alpha_{PH}+2\beta_{PH}+4)\left(\frac{1-\coth\omega_H r}{2}\right)^2}{(2\alpha_{PH}+1)(2\alpha_{PH}+2)2!} \end{array}\right] \tag{114}$$

$$U_{H_2}(r)=\left\{\begin{array}{l} -C'\left(\frac{1-\coth\omega_H r}{2}\right)^{\alpha_{PH}}\left(\frac{1+\coth\omega_H r}{2}\right)^{\beta_{PH}}(2\alpha_{PH}+1)(2\alpha_{PH}+2) \\ \left[\begin{array}{l} 1+\frac{(-1)(2\alpha_{PH}+2\beta_{PH}+2)\left(\frac{1-\coth\omega_H r}{2}\right)}{(2\alpha_{PH}+1)1!} \\ +\frac{(-2)(-1)(2\alpha_{PH}+2\beta_{PH}+)(2\alpha_{PH}+2\beta_{PH}+4)\left(\frac{1-\coth\omega_H r}{2}\right)^2}{(2\alpha_{PH}+1)(2\alpha_{PH}+2)2!} \end{array}\right] \end{array}\right\} \tag{115}$$