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Research Article

Infinitely Many Solutions for a Superlinear Fractional p-Kirchhoff-Type Problem without the (AR) Condition

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In this paper, we investigate the existence of infinitely many solutions to a fractional p-Kirchhoff-type problem satisfying superlinearity with homogeneous Dirichlet boundary conditions as follows: $\{[a + b(\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^p K(x - y)dxdy)]\mathscr{L}_p^s u - \lambda|u|^{p-2}u = g(x,u)$, *in* Ω, $u = 0$, *in* $\mathbb{R}^N \setminus \Omega,\}$ where $\mathscr{L}_p^s$ is a nonlocal integrodifferential operator with a singular kernel K. We only consider the non-Ambrosetti-Rabinowitz condition to prove our results by using the symmetric mountain pass theorem.

1. Introduction

In recent years, the problems with fractional and nonlocal operator have attracted a lot of attention. These types of operators arise in many different contexts. We know that there are population dynamics, stratified materials, minimal surface, water waves, continuum mechanics, and so on. As far as we know, we are able to learn more about their association through referring to [1–6].

The problem we are going to deal with also involves fractional and nonlocal operator. Here, we will study the p-Kirchhoff-type problem as follows:

$$\begin{aligned} &\left[a + b\left(\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^p K(x-y)\,dxdy\right)\right]\mathscr{L}_p^s u \\ &\quad - \lambda |u|^{p-2} u = g(x,u) \quad \text{in } \Omega, \\ &u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{aligned} \tag{1}$$

Ω is an open bounded smooth domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. $a, b > 0$, $1 < p < +\infty$; $ps < N$ with $s \in (0,1)$. λ is a real parameter. g: $\Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function and $\mathscr{L}_p^s$ is usually called nonlocal operator. It is defined as follows:

$$\begin{aligned} \mathscr{L}_p^s u(x) &:= 2 \lim_{\varepsilon\longrightarrow 0^+} \int_{\mathbb{R}^N\setminus\mathscr{B}_\varepsilon(x)} |u(x) - u(y)|^{p-2} \\ &\quad \cdot (u(x) - u(y)) K(x-y)\,dy, \end{aligned} \tag{2}$$

for all $x \in \mathbb{R}^N$, where $\mathscr{B}_\varepsilon(x) = \{z \mid |x - z| < \varepsilon\}$. The function $K : \mathbb{R}^N \setminus \{0\} \longrightarrow (0,\infty)$ is measurable. It has the following properties:

$$\begin{aligned} &\ell K \in L^1(\mathbb{R}^N) \\ &\qquad \text{where } \ell(x) = \min\{|x|^p, 1\}, \\ &\text{there exists } \gamma > 0 \text{ such that } K(x) \ge \gamma |x|^{-(N+ps)} \\ &\qquad \text{for any } x \in \mathbb{R}^N \setminus \{0\}, \\ &K(x) = K(-x) \quad \text{for any } x \in \mathbb{R}^N \setminus \{0\}. \end{aligned} \tag{3}$$

As the singular kernel K satisfies $K(x) = |x|^{-(N+ps)}$, we call it a typical model. Hence, the fractional p-Laplace operator may be defined as follows:

$$(-\triangle)_p^s u(x) := 2\lim_{\varepsilon\longrightarrow 0^+}\int_{\mathbb{R}^N\setminus\mathscr{B}_\varepsilon(x)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}dy, \tag{4}$$

for $x\in\mathbb{R}^N$. Problem (1) also becomes

$$\begin{aligned}&\left[a+b\left(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x|^{N+ps}}dxdy\right)\right](-\triangle)_p^s u(x)\\&\quad-\lambda|u|^{p-2}u = g(x,u)\quad in\ \Omega,\\&u=0\quad in\ \mathbb{R}^N\setminus\Omega.\end{aligned} \tag{5}$$

Usually, we write the Kirchhoff function as M. Clearly, $M(t) = a+bt$ in problem (5). When $M = 1$, $p = 2$, $\lambda = 0$, problem (5) becomes the original problem with the following fractional Laplacian form:

$$\begin{aligned}(-\triangle)^s u &= g(x,u)\quad in\ \Omega,\\ u&=0\quad in\ \mathbb{R}^N\setminus\Omega.\end{aligned} \tag{6}$$

It is the nonlocality that is a typical characteristic of problem (6). In other words, the value of $(-\triangle)^s u(x)$ at any point $x\in\Omega$ relies not only on Ω but actually also on the whole space. We know that the Dirichlet boundary condition was applied to problem (6) in [5]. In [7], through the use of the mountain pass theorem, Servadei and Valdinoci obtained the existence of nontrivial weak solutions of problem (6). In [8], Pucci and Saldi studied the Kirchhoff-type eigenvalue problem in whole space. They proved the existence and multiplicity of nontrivial solutions. We also refer to [9] for related problems.

On the other hand, the Kirchhoff function M is transformable. So far, a variety of forms of function M are taken into account in many references on studying Kirchhoff-type problems; see [10–18]. In addition, we notice that more attention has been focused on p-Kirchhoff-type problems.

In [19], with the help of the Fountain Theorem, they studied the existence of infinitely many solutions for a fractional p-Kirchhoff equation. In [20], the authors showed the existence and multiplicity of solutions to a degenerate fractional p-Kirchhoff problem. However, we perceive that the *Ambrosetti − Rabinowitz* condition was used widely in these papers about p-Laplacian problems. We refer the interested readers to [12, 21–28]. The condition is usually called (AR) condition for short. It is described as follows:

$$\begin{aligned}&there\ exist\ three\ constants\ r>0,\mu>\eta>1,\\&\qquad such\ that\\&\qquad 0<\mu G(x,t)\le tg(x,t)\\&\qquad for\ any\ x\in\Omega, t\in\mathbb{R}\ and\ |t|\ge r.\end{aligned} \tag{7}$$

And $G(x,t) = \int_0^t g(x,s)ds$.

It was introduced for the first time by Ambrosetti and Rabinowitz in [29]. Since then, the (AR) condition has been used far and wide in more and more works involving superlinear elliptic boundary. We know that the importance of (AR) condition is to guarantee the boundedness of familiar (PS) sequences for the energy functional associated with the problem. The nonlinearity function f satisfies superlinear growth under the (AR) condition.

Through (7), we can get

$$G(x,t)\ge a_1|t|^\mu - a_2\quad for\ any\ (x,t)\in\Omega\times\mathbb{R}, \tag{8}$$

where $\mu>\eta$, for two constants $a_1, a_2>0$.

However, there are still lots of functions that dissatisfy the (AR) condition, even though they are superlinear at infinity. We notice another form given by

$$\lim_{|t|\longrightarrow\infty}\frac{G(x,t)}{|t|^\eta} = +\infty\quad uniformly\ for\ x\in\Omega. \tag{9}$$

We find that the nonlinearity g is also superlinear at infinity under condition (9). Obviously, the functional

$$g(x,t) = t^{\eta-1}\ln(1+t) \tag{10}$$

satisfies condition (9) and dissatisfies condition (8). So it does not satisfy (7).

Motivated by the above works and [20, 24, 25, 30], we study the existence of infinitely many solutions of problem (1) without (AR) condition. Our results are extension of some problems studied by N. Van Thin in [30].

Now, we give some assumptions on the function g: $\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$.

($g1$) There exist $C>0$ and $q\in(p,p_s^*)$ such that $|g(x,t)|\le C(1+|t|^{q-1})$ for a.e. $x\in\Omega$ and all $t\in\mathbb{R}$, where $p_s^* = Np/(N-ps)$.

($g2$) $g(x,-t) = -g(x,t)$ for all $x\in\Omega$, $t\in\mathbb{R}$.

($g3$) $\lim_{|t|\longrightarrow\infty}(G(x,t)/|t|^{2p}) = +\infty$ uniformly for $x\in\Omega$.

($g4$) $\lim_{|t|\longrightarrow 0}(g(x,t)/|t|^{p-1}) = 0$ uniformly for $x\in\Omega$.

($g5$) There exists $\tilde{t}>0$ such that the function $t\longmapsto g(x,t)/t^{2p-1}$ is decreasing if $t\le-\tilde{t}<0$ and increasing if $t\ge\tilde{t}>0$ for all $x\in\Omega$.

($g6$) There exist $\sigma\ge 1$ and $T\in L^1(\Omega)$ satisfying $T(x)\ge 0$ such that $\mathscr{G}(x,s)\le\sigma\mathscr{G}(x,t)+T(x)$ for all $x\in\Omega$ and $0\le|s|\le|t|$, where $\mathscr{G}(x,t) = (1/2p)tg(x,t) - G(x,t)$.

Definition 1. We claim that a function $u\in X_0$ is a weak solution of problem (1), if

$$\begin{aligned}&\left(a+b\|u\|_{X_0}^p\right)\int_{\mathscr{Q}}|u(x)-u(y)|^{p-2}(u(x)-u(y))\\&\quad\cdot(\hbar(x)-\hbar(y))K(x-y)dxdy\\&\quad-\lambda\int_\Omega|u(x)|^{p-2}u(x)\hbar(x)dx - \int_\Omega g(x,u(x))\\&\quad\cdot\hbar(x)dx = 0,\quad for\ any\ \hbar\in X_0.\end{aligned} \tag{11}$$

See Section 2 for a detailed description for of X_0.

Theorem 2. *Let $K : \mathbb{R}^N \setminus \{0\} \longrightarrow (0,\infty)$ be a function satisfying (3). Let conditions $(g1) - (g5)$ hold. Then, for any $\lambda \in \mathbb{R}$, problem (1) has infinitely many nontrivial solutions $\{u_k\}_{k\in\mathbb{N}}$ in X_0 with unbounded energy.*

Corollary 3. *Let $K : \mathbb{R}^N \setminus \{0\} \longrightarrow (0,\infty)$ be a function satisfying (3). Let conditions $(g1)-(g4)$ hold. If condition $(g6)$ replaces $(g5)$, then the conclusion of Theorem 2 holds.*

Remark. Originally, Jeanjean put forward a condition that was similar to $(g6)$ in [31]. It is easy to see that condition $(g6)$ is equivalent to $(g5)$ when $\sigma = 1$. Actually, condition $(g6)$ is weaker than condition $(g5)$. We can find that there are some functions satisfying $(g1) - (g4), (g6)$ but dissatisfying $(g5)$. For example,

$$g(x,t) = 2p\,|t|^{2p-2}\,t\ln\left(1+t^{2p}\right) + p\sin t. \qquad (12)$$

This paper consists of the following parts. In Section 2, we give the definition and some properties for the space X_0 and some preliminary results. Section 3 verifies compactness conditions. In Section 4, we prove Theorem 2 and Corollary 3.

2. Preliminary Results

Firstly, we recall the functional space X and X_0 and some lemmas, which will be used in next section for problem (1). We appoint $\mathcal{Q} = \mathbb{R}^{2N} \setminus \Gamma$ where $\Gamma = \mathscr{C}(\Omega) \times \mathscr{C}(\Omega) \subset \mathbb{R}^{2N}$ and $\mathscr{C}(\Omega) = \mathbb{R}^N \setminus \Omega$. The space X is a linear of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in X belongs to $L^p(\Omega)$ and

$$\int_{\mathcal{Q}} |u(x)-u(y)|^p K(x-y)\,dxdy < \infty. \qquad (13)$$

X is endowed with the following norm:

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \left(\int_{\mathcal{Q}} |u(x)-u(y)|^p K(x-y)\,dxdy\right)^{1/p}. \qquad (14)$$

In addition, X_0 is endowed with the following norm:

$$\|u\|_{X_0} = \left(\int_{\mathbb{R}^{2N}} |u(x)-u(y)|^p K(x-y)\,dxdy\right)^{1/p}, \qquad (15)$$

and $(X_0, \|\cdot\|_{X_0})$ is known as the Hilbert space defined by the following scalar product (see [12], Lemma 7).

$$\langle u,\hbar\rangle_{X_0} := \int_{\mathbb{R}^{2N}} |u(x)-u(y)|^{p-2}(u(x)-u(y)) \cdot (\hbar(x)-\hbar(y))\,K(x-y)\,dxdy. \qquad (16)$$

We denote the usual fractional Sobolev space by $W^{s,p}(\Omega)$, which is endowed with norm (the so-called *Gagliardonorm*) as follows:

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_{\Omega\times\Omega} |u(x)-u(y)|^p |x-y|^{-(N+ps)}\,dxdy\right)^{1/p}. \qquad (17)$$

We observe that the norms (14) and (17) are not the same when $K(x) = 1/|x|^{N+ps}$, since $\Omega \times \Omega$ is contained strictly in $\mathcal{Q}$. It makes the space X_0 different from the usual classical fractional Sobolev space. Therefore, from the point of view of the variational method, the classical fractional Sobolev space is insufficient for dealing with our problem.

We recall that the space X_0 is nonempty due to $C_0^\infty \subseteq X_0$ (see [12], Lemma 2.1). The following conclusion is correct if a general kernel K satisfies (3):

$$X_0 \subset \left\{u \in W^{s,p}\left(\mathbb{R}^N\right) : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\}. \qquad (18)$$

Particularly, the following characterization holds when $K(x) = 1/|x|^{N+ps}$:

$$X_0 = \left\{u \in W^{s,p}\left(\mathbb{R}^N\right) : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\}. \qquad (19)$$

For more details about space X and X_0, we refer to [5, 32].

Considering future works, we recall the following eigenvalue problem:

$$\begin{aligned} \mathscr{L}_p^s u &= \lambda |u|^{p-2} u \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^{\mathbb{N}} \setminus \Omega. \end{aligned} \qquad (20)$$

It has a divergent sequence of positive eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \lambda_{j+1} \le \cdots \qquad (21)$$

whose homologous eigenfunctions are denoted by e_j. From Proposition 9 of [32], we know that $\{e_j\}_{j\in\mathbb{N}}$ can be chosen in such a way that this sequence provides an orthonormal basis in $L^p(\Omega)$ and an orthogonal basis in X_0.

Firstly, we define

$$\Psi(u) = J(u) - I(u) - H(u), \qquad (22)$$

where

$$\begin{aligned} J(u) &= a\|u\|_{X_0}^p + b\|u\|_{X_0}^{2p}, \\ I(u) &= \frac{\lambda}{p}\int_\Omega |u|^p\,dx, \\ H(u) &= \int_\Omega G(x,u)\,dx, \end{aligned} \qquad (23)$$

where $G(x,u) := \int_0^u g(x,s)ds$.

Clearly, the energy functional $\Psi : X_0 \longrightarrow \mathbb{R}$ associated with problem (1) is well defined.

For convenience, we write $\|u\|_{L^p(\Omega)}$ as $\|u\|_p$. From Lemma 3.1 of [25], clearly we know that functional $J \in C^1(X_0,\mathbb{R})$. And if $(g1)$ holds, $H \in C^1(X_0,\mathbb{R})$. So, we get that $\Psi \in C^1(X_0,\mathbb{R})$ and

$$\begin{aligned} \left\langle \Psi'(u),\hbar\right\rangle_{X_0} &= \left(a + b\|u\|_{X_0}^p\right)\int_{\mathcal{Q}} |u(x)-u(y)|^{p-2} \\ &\cdot (u(x)-u(y))(\hbar(x)-\hbar(y)) \\ &\cdot K(x-y)\,dxdy - \lambda\int_\Omega |u(x)|^{p-2}u(x) \\ &\cdot \hbar(x)\,dx - \int_\Omega g(x,u(x))\,\hbar(x)\,dx \end{aligned} \qquad (24)$$

for all $u, \hbar \in X_0$. In order to prove the conclusion of problem (1), we need some lemmas.

Lemma 4 (see [12]). *Assume that (3) holds. We have the following conclusions:*

(1) For any $\kappa \in [1, p_s^*)$, the embedding $X_0 \hookrightarrow L^\kappa(\Omega)$ is compact when Ω is a bounded domain with continuous boundary.

(2) For all $\kappa \in [1, p_s^*]$, the embedding $X_0 \hookrightarrow L^\kappa(\Omega)$ is continuous.

Definition 5. Let $\Psi \in C^1(X_0, \mathbb{R})$. The functional Ψ satisfies $(Ce)_c$ at the level $c \in \mathbb{R}$, if any sequence $\{u_k\} \subset X_0$, with

$$\begin{gathered} \Psi(u_k) \longrightarrow c \quad \text{in } X_0, \\ (1+\|u_k\|)\Psi'(u_k) \longrightarrow 0 \quad \text{in } X_0' \\ \text{as } k \longrightarrow \infty, \end{gathered} \tag{25}$$

has a strongly convergent subsequence in X_0. X_0' is the dual space of X_0.

Theorem 6 (symmetric mountain pass theorem [33]). *Assume that X is an infinite dimensional Banach space. $\widetilde{Y}$ is a finite dimensional Banach space and $X = \widetilde{Y} \oplus \widetilde{Z}$. For any $c > 0$, if $\Psi \in C^1(X, \mathbb{R})$, it satisfies $(Ce)_c$ condition, and*

(a) *Ψ is even and $\Psi(0) = 0$ for all $u \in X$.*

(b) *There exist constants $\delta, \rho > 0$ such that $\Psi|_{\partial B_\delta(X)} \geq \rho$, where*

$$B_\delta(X) = \{u \in X : \|u\| \leq \delta\}. \tag{26}$$

(c) *For any finite dimensional subspace $\overline{X} \subseteq X$, there exists $\mathcal{R} = \mathcal{R}(\overline{X}) > 0$ such that $\Psi(u) \leq 0$ on $\overline{X} \setminus B_{\mathcal{R}}(\overline{X})$.*

Then Ψ possesses an unbounded sequence of critical values characterized by a minimax argument.

3. Compactness Conditions

In this section, we are going to give some lemmas about the compactness of functional Ψ and prove them.

Lemma 7. *Let $(g1)$ hold. Any bounded sequence $\{u_k\}_{k\in\mathbb{N}} \subset X_0$, which satisfies $(1+\|u_k\|)\Psi'(u_k) \longrightarrow 0$ as $k \longrightarrow \infty$, possesses a strongly convergent subsequence in X_0*

Proof. Suppose that $\{u_k\}_k$ is bounded in X_0. From Lemma 2.4 of [12] and Theorem 1.21 of [34], we know that X_0 is reflexive. Combining with Lemma 4, we have

$$\begin{aligned} & u_k \rightharpoonup u \quad \text{in } X_0, \\ & u_k \longrightarrow u \quad \text{in } L^\kappa(\Omega), \ 1 \leq \kappa < p_s^*, \\ & u_k \longrightarrow u \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{27}$$

We just need to prove that $u_k \longrightarrow u$ strongly in X_0.

Through the Hölder inequality and $(g1)$, we obtain

$$\begin{aligned} \int_\Omega & |g(x,u_k)(u_k-u)|\,dx \\ & \leq \int_\Omega \left(C + C|u_k|^{q-1}\right)|u_k-u|\,dx \\ & \leq C\left(|\Omega|^{(q-1)/q} + \|u_k\|_q^{q-1}\right)\|u_k-u\|_q. \end{aligned} \tag{28}$$

By (27), we get

$$\lim_{k\longrightarrow\infty} \int_\Omega g(x,u_k)(u_k-u)\,dx = 0. \tag{29}$$

We consider the following formula with Hölder inequality:

$$\int_\Omega |u_k|^{p-2} u_k (u_k-u)\,dx \leq \|u_k\|_p^{p-1}\|u_k-u\|_p. \tag{30}$$

Hence, by (27), we get

$$\lim_{k\longrightarrow\infty} \int_\Omega |u_k|^{p-2} u_k(u_k-u)\,dx = 0. \tag{31}$$

Then, for convenience, we define a new linear functional on X_0 as follows:

$$\begin{aligned} B_w(v) := \int_{\mathbb{R}^{2N}} & |w(x)-w(y)|^{p-2}(w(x)-w(y)) \\ & \cdot (v(x)-v(y))K(x-y)\,dxdy, \end{aligned} \tag{32}$$

for all $w, v \in X_0$. By means of the Hölder inequality, we have that

$$|B_w(v)| \leq \|w\|_{X_0}^{p-1} \cdot \|v\|_{X_0} \quad \text{for all } v \in X_0. \tag{33}$$

Hence, $B_w(v)$ is a continuous functional on X_0. Hence, we obtain that

$$\lim_{k\longrightarrow\infty} B_u(u_k-u)\,dx = 0. \tag{34}$$

Clearly, $\langle \Psi'(u_k), u_k - u\rangle \longrightarrow 0$ as $k \longrightarrow \infty$, since $u_k \rightharpoonup u$ in X_0 and $(1+\|u_k\|_{X_0})\Psi'(u_k) \longrightarrow 0$ in X_0'. Hence, by (27), (29), and (31), we have

$$\begin{aligned} o(1) &= \left\langle \Psi'(u_k), u_k - u \right\rangle \\ &= \left(a + b\|u_k\|_{X_0}^p\right) B_{u_k}(u_k-u) \\ &\quad - \lambda \int_\Omega |u_k|^{p-2} u_k(u_k-u)\,dx \\ &\quad + \int_\Omega g(x,u_k)(u_k-u)\,dx \\ &= \left(a + b\|u_k\|_{X_0}^p\right) B_{u_k}(u_k-u) + o(1) \\ &\qquad \text{as } k \longrightarrow \infty. \end{aligned} \tag{35}$$

Hence, through the boundedness of $\|u_k\|$ in X_0 and (34), we have

$$\lim_{k\longrightarrow\infty} \left[B_{u_k}(u_k-u) - B_u(u_k-u)\right] = 0. \tag{36}$$

Now, we recall the Simon inequalities:

$$|\vartheta-\chi|^p \le Q_p\left(|\vartheta|^{p-2}\vartheta-|\chi|^{p-2}\chi\right)\cdot(\vartheta-\chi) \quad p\ge 2, \tag{37}$$

$$|\vartheta-\chi|^p \le \widetilde{Q_p}\left[\left(|\vartheta|^{p-2}\vartheta-|\chi|^{p-2}\chi\right)\right]^{p/2} \cdot\left(|\vartheta|^p+|\chi|^p\right)^{(2-p)/2} \quad 1<p<2, \tag{38}$$

for all $\vartheta,\chi\in\mathbb{R}^N$, where $Q_p,\widetilde{Q_p}>0$ relying on p. Then if $p\ge 2$, by (36) and (37), we have

$$\begin{aligned}\|u_k-u\|_{X_0}^p &\le Q_p\int_{\mathbb{R}^{2N}}\left[|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y)) - |u(x)-u(y)|^{p-2}(u(x)-u(y))\right] \\ &\times\left[(u_k(x)-u(x))-(u_k(y)-u(y))\right]K(x-y)\,dx\,dy = Q_p\left[B_{u_k}(u_k-u)-B_u(u_k-u)\right]\longrightarrow 0,\end{aligned} \tag{39}$$

as $k\longrightarrow\infty$. When $1<p<2$, by (36), (38), and the boundedness of $\|u_k\|$ in X_0, we have

$$\begin{aligned}\|u_k-u\|_{X_0}^p &\le \widetilde{Q_p}\left[B_{u_k}(u_k-u)-B_u(u_k-u)\right]^{p/2}\cdot\left(\|u_k\|_{X_0}^p+\|u\|_{X_0}^p\right)^{(2-p)/2} \\ &\le \widetilde{Q_p}\left[B_{u_k}(u_k-u)-B_u(u_k-u)\right]^{p/2}\cdot\left(\|u_k\|_{X_0}^{p(2-p)/2}+\|u\|_{X_0}^{p(2-p)/2}\right)\longrightarrow 0,\end{aligned} \tag{40}$$

as $k\longrightarrow\infty$. Hence, we get $u_k\longrightarrow u$ strongly in X_0. □

Lemma 8. *Let $(g1)$, $(g3)$, and $(g5)$ hold. Then, functional Ψ satisfies the $(Ce)_c$ condition.*

Proof. Let $(g5)$ hold. According to the monotonicity of $t\longmapsto g(x,t)/t^{2p-1}$, we find that there exists a positive constant L_1 such that

$$\mathcal{G}(x,s)\le\mathcal{G}(x,t)+L_1, \tag{41}$$

where $\mathcal{G}(x,t)=(1/2p)tg(x,t)-G(x,t)$, for all $x\in\Omega$ and $0\le|s|\le|t|$. Let $\{u_k\}_{k\in\mathbb{N}}$ be a Cecrami sequence in X_0. We know that it satisfies

$$\Psi(u_k)\longrightarrow c \quad in\ X_0, \tag{42}$$

$$\left(1+\|u_k\|_{X_0}\right)\Psi'(u_k)\longrightarrow 0 \quad in\ X_0' \tag{43}$$

as $k\longrightarrow\infty$. By means of Lemma 7, it suffices to prove the boundedness of $\{u_k\}$. Suppose that $\{u_k\}_{k\in\mathbb{N}}$ is unbounded in X_0. Then we have

$$\|u_k\|_{X_0}\longrightarrow+\infty. \tag{44}$$

By (43) and (44), we get

$$\Psi'(u_k)\longrightarrow 0. \tag{45}$$

Hence, we get

$$\|u_k\|_{X_0}\left\langle\Psi'(u_k),\frac{u_k}{\|u_k\|_{X_0}}\right\rangle\longrightarrow 0 \quad as\ k\longrightarrow\infty. \tag{46}$$

We define $\zeta_k=u_k/\|u_k\|_{X_0}$. Then $\|\zeta_k\|_{X_0}=1$. So $\{\zeta_k\}_{k\in\mathbb{N}}$ is a bounded sequence in X_0. Through Lemma 4, there exists ζ_∞ satisfying

$$\begin{aligned}&\zeta_k\longrightarrow\zeta_\infty \quad in\ L^p(\mathbb{R}^N),\\ &\zeta_k\longrightarrow\zeta_\infty \quad in\ L^q(\mathbb{R}^N),\\ &\zeta_k\longrightarrow\zeta_\infty \quad in\ \mathbb{R}^N,\end{aligned} \tag{47}$$

as $k\longrightarrow\infty$. What is more, through Lemma A.1 of [35], there exists a function $\varepsilon\in L^q(\mathbb{R}^N)$ satisfying

$$|\zeta_k(x)|\le\varepsilon(x) \quad in\ \mathbb{R}^N. \tag{48}$$

We only need to consider two cases: $\zeta_\infty=0$ and $\zeta_\infty\ne 0$. Firstly, we consider the case $\zeta_\infty=0$. Refer to [31]; for any $k\in\mathbb{N}$, we have $l_k\in[0,1]$ such that

$$\Psi(l_ku_k)=\max_{l\in[0,1]}\Psi(lu_k). \tag{49}$$

Because of the unboundedness of $\{u_k\}$, for any $\tau\in\mathbb{N}$, we select $h_\tau=((4p/b)\tau)^{1/2p}$ such that

$$\frac{h_\tau}{\|u_k\|_{X_0}}\in(0,1), \tag{50}$$

where k is large enough, say $k>\overline{k}$, with $\overline{k}=\overline{k}(\tau)$. By (47) and $\zeta_\infty=0$, we get

$$\int_\Omega|h(\tau)\zeta_k(x)|^p\,dx\longrightarrow 0. \tag{51}$$

Since the function G is continuous, we get that

$$G(x,h_\tau\zeta_k(x))\longrightarrow G(x,h_\tau\zeta_\infty(x)) \quad in\ \Omega, \tag{52}$$

as $k\longrightarrow\infty$, for any $\tau\in\mathbb{N}$. Through $(g1)$, (48), and Hölder inequality, we obtain that

$$\begin{aligned}|G(x,h_\tau\zeta_k(x))| &\le C|h_\tau\zeta_k(x)|+\frac{C}{q}|h_\tau\zeta_k(x)|^q\\ &\le C|h_\tau\varepsilon(x)|+\frac{C}{q}|h_\tau\varepsilon(x)|^q\\ &\in L^1(\Omega),\end{aligned} \tag{53}$$

for any $k,\tau\in\mathbb{N}$. Hence, we get that

$$G(\cdot,h_\tau\zeta_k(\cdot))\longrightarrow G(\cdot,h_\tau\zeta_\infty(\cdot)) \quad in\ L^1(\Omega), \tag{54}$$

as $k \longrightarrow \infty$, for any $\tau \in \mathbb{N}$, thanks to the Dominated Convergence Theorem and (51) and (52). As a consequence of $G(x,0) = 0$ for all $x \in \Omega$, by (54) and $\zeta_\infty = 0$, we have

$$\int_\Omega G(x, h_\tau \zeta_k(x))\, dx \longrightarrow 0, \tag{55}$$

as $k \longrightarrow \infty$, for any $\tau \in \mathbb{N}$. Therefore, by (50)–(52) and (55), we get

$$\begin{aligned} \Psi(l_k u_k) &\geq \Psi\left(\frac{h_\tau}{\|u_k\|_{X_0}} u_k\right) = \Psi(h_\tau \zeta_k) \\ &= \frac{a}{p}\|h_\tau \zeta_k(x)\|_{X_0}^p + \frac{b}{2p}\|h_\tau \zeta_k(x)\|_{X_0}^{2p} \\ &\quad - \frac{\lambda}{p}\int_\Omega |h_\tau \zeta_k(x)|^p\, dx \\ &\quad - \int_\Omega G(x, h_\tau \zeta_k(x))\, dx \\ &\geq \frac{b}{2p}\|h_\tau \zeta_k(x)\|_{X_0}^{2p} = 2\tau, \end{aligned} \tag{56}$$

as $k \longrightarrow \infty$, for any $\tau \in \mathbb{N}$. Hence, we infer that

$$\Psi(l_k u_k) \longrightarrow +\infty \quad as\ k \longrightarrow \infty. \tag{57}$$

Now, we will show that

$$\limsup_{k \longrightarrow \infty} \Psi(l_k u_k) \leq \varpi, \tag{58}$$

where $\varpi > 0$. Because $\Omega \subset \mathbb{R}^N$ is an open bounded smooth domain with Lipschitz boundary, we deduce that there exists a positive constant $C_{p,\lambda}$, which depends on λ and p such that

$$\left|\frac{\lambda}{2p}\int_\Omega |u_k(x)|^p\, dx\right| < C_{p,\lambda} < +\infty. \tag{59}$$

We notice that $\Psi(0) = 0$. Through (42), (49), and (57), we get that there exists $l_k \in (0,1)$ such that

$$\left.\frac{d}{dl}\right|_{l=l_k} \Psi(l u_k) = 0, \tag{60}$$

for any $k \in \mathbb{N}$. Then we have

$$\left\langle \Psi'(l_k u_k), l_k u_k \right\rangle = l_k \left.\frac{d}{dl}\right|_{l=l_k} \Psi(l u_k) = 0. \tag{61}$$

Through (41), (59), and $l_k \in [0,1]$, we have

$$\begin{aligned} \Psi(l_k u_k) &= \Psi(l_k u_k) - \frac{1}{2p}\left\langle \Psi'(l_k u_k), l_k u_k \right\rangle \\ &= \frac{a}{2p}\|t_k u_k(x)\|_{X_0}^p - \frac{\lambda}{2p}\int_\Omega |l_k u_k(x)|^p\, dx \\ &\quad - \int_\Omega G(x, l_k u_k(x))\, dx \\ &\quad + \int_\Omega \frac{1}{2p} g(x, l_k u_k(x))\, l_k u_k(x)\, dx \\ &\leq \frac{a}{2p}\|t_k u_k(x)\|_{X_0}^p + \int_\Omega \mathcal{G}(x, l_k u_k)\, dx \\ &\quad + C_{p,\lambda} \\ &\leq \frac{a}{2p}\|u_k(x)\|_{X_0}^p + \int_\Omega \mathcal{G}(x, u_k)\, dx + L_1|\Omega| \\ &\quad + C_{p,\lambda} \\ &= \frac{a}{2p}\|u_k(x)\|_{X_0}^p - \frac{\lambda}{2p}\int_\Omega |u_k(x)|^p\, dx \\ &\quad - \int_\Omega G(x, u_k(x))\, dx \\ &\quad + \int_\Omega \frac{1}{2p} g(x, u_k(x))\, u_k(x)\, dx \\ &\quad + \frac{\lambda}{2p}\int_\Omega |u_k(x)|^p\, dx + L_1|\Omega| + C_{p,\lambda} \\ &\leq \Psi(u_k) - \frac{1}{2p}\left\langle \Psi'(u_k), u_k \right\rangle + 2C_{p,\lambda} \\ &\quad + L_1|\Omega| = c + 2C_{p,\lambda} + L_1|\Omega| < +\infty. \end{aligned} \tag{62}$$

And $k \longrightarrow \infty$. It contradicts with (57). Hence, the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in X_0.

Now we consider the case $\zeta_\infty \neq 0$. We define $\overline{\Omega} = \{x \in \Omega : \zeta_\infty(x) \neq 0\}$. Clearly, $\overline{\Omega}$ is Lebesgue measureable. Through (44), (47), and $\zeta_\infty \neq 0$, we have

$$|u_k(x)| = |\zeta_k(x)| \cdot \|u_k\|_{X_0} \longrightarrow +\infty \quad in\ \overline{\Omega}, \tag{63}$$

as $k \longrightarrow \infty$. Through (47), (63), and $(g3)$, it holds true that

$$\begin{aligned} \frac{G(x, u_k(x))}{\|u_k\|_{X_0}^{2p}} &= \frac{G(x, u_k(x))}{|u_k|^{2p}} \cdot \frac{|u_k|^{2p}}{\|u_k\|_{X_0}^{2p}} \\ &= \frac{G(x, u_k(x))}{|u_k(x)|^{2p}}|\zeta_k(x)|^{2p} \longrightarrow +\infty, \end{aligned} \tag{64}$$

in $\overline{\Omega}$, as $k \longrightarrow \infty$. Through the Fatou Lemma and (64), we have

$$\int_{\overline{\Omega}} \frac{G(x, u_k(x))}{\|u_k\|_{X_0}^{2p}} dx \longrightarrow +\infty. \tag{65}$$

Next, we discusses the case in $\Omega \setminus \overline{\Omega}$. By $(g3)$, we get

$$\lim_{|t|\longrightarrow\infty} G(x,t) = +\infty, \quad x \in \Omega. \tag{66}$$

Therefore, there exist constants $t_1, A > 0$ such that

$$G(x,t) \geq A, \tag{67}$$

for all $x \in \Omega$ and $|t| > t_1$. Moreover, by means of the continuity of G in $\Omega \times \mathbb{R}$, we get

$$G(x,t) \geq \min_{(x,t)\in\Omega\times[-t_1,t_1]} G(x,t), \tag{68}$$

for all $|t| \leq t_1$. Hence, we obtain that

$$G(x,t) \geq \min\left\{A, \min_{(x,t)\in\Omega\times[-t_1,t_1]} G(x,t)\right\} := \widetilde{A}, \tag{69}$$

for any $(x,t) \in \Omega \times \mathbb{R}$, thanks to (67) and (68). So we get

$$\lim_{|t|\longrightarrow\infty} \int_{\Omega\backslash\overline{\Omega}} \frac{G(x, u_k(x))}{\|u_k\|_{X_0}^{2p}} dx \geq 0. \tag{70}$$

By (42) and (44), we get

$$\begin{aligned} o(1) &= \frac{\Psi(u_k)}{\|u_k\|_{X_0}^{2p}} \\ &= \frac{a}{p\|u_k\|_{X_0}^{2p}} + \frac{b}{2p} - \frac{\lambda}{p}\int_\Omega \frac{|u_k(x)|^p}{\|u_k\|_{X_0}^{2p}} dx \\ &\quad - \int_{\overline{\Omega}} \frac{G(x,u_k(x))}{\|u_k\|_{X_0}^{2p}} dx \\ &\quad - \int_{\Omega\backslash\overline{\Omega}} \frac{G(x,u_k(x))}{\|u_k\|_{X_0}^{2p}} dx, \end{aligned} \tag{71}$$

as $k \longrightarrow \infty$. Consider (63), (65), (70), and the variational characterization of λ_j defined as follows:

$$\lambda_j = \min_{u\in X_0\backslash\{0\}} \frac{\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^p K(x-y)\, dx dy}{\int_\Omega |u(x)|^p dx}. \tag{72}$$

We have

$$\begin{aligned} o(1) &= \frac{a}{p\|u_k\|_{X_0}^{2p}} + \frac{b}{2p} - \frac{\lambda}{p}\int_\Omega \frac{|u_k(x)|^p}{\|u_k\|_{X_0}^{2p}} dx \\ &\quad - \int_{\overline{\Omega}} \frac{G(x,u_k)}{\|u_k\|_{X_0}^{2p}} dx - \int_{\Omega\backslash\overline{\Omega}} \frac{G(x,u_k)}{\|u_k\|_{X_0}^{2p}} dx \\ &\leq \frac{b}{2p} + \max\left\{0, -\frac{\lambda}{p\lambda_1}\int_\Omega \frac{\|u_k\|_{X_0}^p}{\|u_k\|_{X_0}^{2p}} dx\right\} \\ &\quad - \int_{\overline{\Omega}} \frac{G(x,u_k)}{\|u_k\|_{X_0}^{2p}} dx - \int_{\Omega\backslash\overline{\Omega}} \frac{G(x,u_k)}{\|u_k\|_{X_0}^{2p}} dx \\ &\leq -\infty. \end{aligned} \tag{73}$$

We find that (73) is a contradictory result. Thus, the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in X_0. This ends the proof of Lemma 8. □

4. Proof of Theorem 2 and Corollary 3

We know that X_0 is a Hilbert space. Hence, we divide X_0 into two parts. Let $X_0 = \bigoplus_{j\geq1} X_j$, where $X_j = \text{span}\{e_j\}_{j\in\mathbb{N}}$. Now we define

$$\begin{aligned} Y_m &= \bigoplus_{j=1}^{m} X_j, \\ Z_m &= \overline{\bigoplus_{j=m}^{\infty} X_j}. \end{aligned} \tag{74}$$

Obviously, Y_m is a finite-dimensional space.

Lemma 9. *Let $\nu \in [1, p_s^*)$. We have*

$$\xi_m(\nu) := \sup\left\{\|u\|_\nu : u \in Z_m, \|u\|_{X_0} = 1\right\} \longrightarrow 0, \quad m \longrightarrow \infty. \tag{75}$$

Proof. Clearly $\xi_{m+1} \geq \xi_m \geq 0$, so that $\xi_m \longrightarrow \xi \geq 0$, as $m \longrightarrow \infty$. For every $m \in \mathbb{N}$, there exists $u_m \in Z_m$ such that $\|u_m\|_\nu > \xi_m/2$ and $\|u_m\|_{X_0} = 1$. Through the definition of Z_m, we can get $u_m \rightharpoonup 0$ in X_0. According to Lemma 4, the embedding $X_0 \hookrightarrow\hookrightarrow L^\kappa(\Omega)$ $(1 \leq \kappa < p_s^*)$ implies that $u_m \longrightarrow 0$ in $L^\kappa(\Omega)$. Therefore, we get $\xi = 0$, as $m \longrightarrow \infty$. This implies that the proof is complete. □

Proof of Theorem 2. Obviously, all norms are equivalent in Y_m. Hence, there exist two constants $C_1, C_2 > 0$ such that

$$C_1\|u\|_{X_0} \leq \|u\|_\nu \leq C_2\|u\|_{X_0} \quad \text{for any } \nu \in [1, p_s^*). \tag{76}$$

According to Theorem 6 and Lemma 8, we just need to prove (a), (b), and (c) of Theorem 6. Let $(g2)$ hold. Clearly, $\Psi(0) = 0$. Hence, condition (a) of Theorem 6 is true. Then, we take into account the range of values of λ. According to (72), if $\lambda/a > \lambda_1$, we can find $j \in \mathbb{N}$ and $j > 2$ such that $\lambda/a \in [\lambda_{j-1}, \lambda_j)$. So, for all j, there exists $j \in \mathbb{N}$ and $j \geq 1$ such that $\lambda/a < \lambda_j$ for any $\lambda \in \mathbb{R}$.

By $(g1)$ and $(g4)$, for any $\varrho > 0$, there exists $C_\varrho > 0$ such that

$$G(x,t) \leq \frac{\varrho}{p}|t|^p + \frac{C_\varrho}{q}|t|^q, \tag{77}$$

for any $(x,t) \in \Omega\times\mathbb{R}$. By Lemma 9, for any fixed $\varrho > 0$, choose an integer $\widetilde{m} \geq 1$ such that

$$\begin{aligned} \|u\|_p^p &\leq \frac{\min\{a, a - \lambda/\lambda_j\}}{2\varrho}\|u\|_{X_0}^p, \\ \|u\|_q^q &\leq \frac{q\min\{a, a-\lambda/\lambda_j\}}{2pC_\varrho}\|u\|_{X_0}^q, \end{aligned} \tag{78}$$

for any $u \in Z_{\widetilde{m}}$. Choose $\delta := \|u\|_{X_0} = 1/2$, for all $u \in Z_{\widetilde{m}}$. Note that $q > p$; from (77) and (78), we have

$$\begin{aligned}
\Psi(u) &= \frac{a}{p}\|u\|_{X_0}^p + \frac{b}{2p}\|u\|_{X_0}^{2p} - \frac{\lambda}{p}\int_\Omega |u(x)|^p dx \\
&\quad - \int_\Omega G(x,u(x))\,dx \\
&\geq \frac{a}{p}\|u\|_{X_0}^p + \min\left\{0, -\frac{\lambda}{p\lambda_j}\right\}\|u\|_{X_0}^p - \frac{\varrho}{p}\|u\|_p^p \\
&\quad - \frac{C_\varrho}{q}\|u\|_q^q \\
&\geq \frac{1}{p}\min\left\{a, a-\frac{\lambda}{\lambda_j}\right\}\|u\|_{X_0}^p \\
&\quad - \frac{1}{2p}\min\left\{a, a-\frac{\lambda}{\lambda_j}\right\}\|u\|_{X_0}^p \\
&\quad - \frac{1}{2p}\min\left\{a, a-\frac{\lambda}{\lambda_j}\right\}\|u\|_{X_0}^q \\
&\geq \frac{1}{2p}\min\left\{a, a-\frac{\lambda}{\lambda_j}\right\}\left(\|u\|_{X_0}^p - \|u\|_{X_0}^q\right) \\
&= \frac{1}{2p}\min\left\{a, a-\frac{\lambda}{\lambda_j}\right\}\left(\frac{1}{2^p} - \frac{1}{2^q}\right) := \rho > 0.
\end{aligned} \tag{79}$$

Then, condition (b) of Theorem 6 is true.

In the end, we demonstrate condition (c) of Theorem 6. In view of $(g3)$, there exist constants $\alpha > b/2pC_1^{2p}$, $\beta > 0$ such that

$$G(x,t) \geq \alpha |t|^{2p}, \tag{80}$$

for any $x \in \Omega$ and $|t| > \beta$. Considering condition $(g1)$, we have

$$|G(x,t)| \leq C\left(1+\beta^{q-1}\right)|t|, \tag{81}$$

for any $x \in \Omega$ and $|t| \leq \beta$. Take $C_* = C(1+\beta^{q-1})$, where $C_* > 0$. We obtain

$$G(x,t) \geq \alpha |t|^{2p} - C_* |t|, \tag{82}$$

for any $(x,t) \in \Omega \times \mathbb{R}$. Then, by (76) and (82), we have

$$\begin{aligned}
\Psi(u) &= \frac{a}{p}\|u\|_{X_0}^p + \frac{b}{2p}\|u\|_{X_0}^{2p} - \frac{\lambda}{p}\int_\Omega |u(x)|^p dx \\
&\quad - \int_\Omega G(x,u(x))\,dx \\
&\leq \frac{a}{p}\|u\|_{X_0}^p + \frac{b}{2p}\|u\|_{X_0}^{2p} \\
&\quad + \max\left\{0, -\frac{\lambda}{p\lambda_j}\right\}\|u\|_{X_0}^p - \alpha\|u\|_{2p}^{2p} \\
&\quad - C_*\|u\|_1 \\
&\leq \frac{1}{p}\max\left\{a, a-\frac{\lambda}{\lambda_j}\right\}\|u\|_{X_0}^p \\
&\quad + \left(\frac{b}{2p} - C_1^{2p}\alpha\right)\|u\|_{X_0}^{2p} - C_1 C^*\|u\|_{X_0}.
\end{aligned} \tag{83}$$

Let $\mathscr{R} = \mathscr{R}(\overline{X}) > 0$ be large enough. When $\|u\|_{X_0} > \mathscr{R}$, $\Psi(u) \leq 0$. Hence, condition (c) of Theorem 6 is true. In conclusion, Theorem 2 is proven. □

Proof of Corollary 3. Let $(g6)$ hold. Similar to Theorem 2, we only need to prove inequality (62).

$$\begin{aligned}
\frac{1}{\sigma}\Psi(l_k u_k) &= \frac{1}{\sigma}\left(\Psi(l_k u_k) - \frac{1}{2p}\left\langle \Psi'(l_k u_k), l_k u_k\right\rangle\right) \\
&= \frac{1}{\sigma}\left(\frac{a}{2p}\|l_k u_k(x)\|_{X_0}^p - \frac{\lambda}{2p}\int_\Omega |l_k u_k(x)|^p dx\right. \\
&\quad - \int_\Omega G(x, l_k u_k(x))\,dx \\
&\quad \left. + \int_\Omega \frac{1}{2p} g(x, l_k u_k(x))\, l_k u_k(x)\,dx\right) \\
&\leq \frac{1}{\sigma}\left(\frac{a}{2p}\|l_k u_k(x)\|_{X_0}^p + \int_\Omega \mathscr{G}(x, l_k u_k)\,dx\right) \\
&\quad + C_{p,\lambda} \leq \frac{a}{2p}\|u_k(x)\|_{X_0}^p + \int_\Omega \mathscr{G}(x,u_k)\,dx + \frac{1}{\sigma} \\
&\quad \cdot \int_\Omega T(x)\,dx + C_{p,\lambda} = \frac{a}{2p}\|u_k(x)\|_{X_0}^p - \frac{\lambda}{2p} \\
&\quad \cdot \int_\Omega |u_k(x)|^p dx - \int_\Omega G(x,u_k(x))\,dx \\
&\quad + \int_\Omega \frac{1}{2p} g(x,u_k(x))\, u_k(x)\,dx + \frac{\lambda}{2p} \\
&\quad \cdot \int_\Omega |u_k(x)|^p dx + \frac{1}{\sigma}\int_\Omega T(x)\,dx + C_{p,\lambda} \\
&\quad \leq \Psi(u_k) - \frac{1}{2p}\left\langle \Psi'(u_k), u_k\right\rangle + 2C_{p,\lambda} + \frac{1}{\sigma} \\
&\quad \cdot \int_\Omega T(x)\,dx \leq c + 2C_{p,\lambda} + \frac{1}{\sigma}\int_\Omega T(x)\,dx \\
&\quad < +\infty.
\end{aligned} \tag{84}$$

The proof is completed. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The contribution of Jiabin Zuo to this paper is the same as that of the first author. All authors read and approved the final manuscript.

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