Research Article

Peakon Solutions of Alice-Bob b -Family Equation and Novikov Equation

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By requiring $B = \hat{P}_s \hat{T}_d A$ and substituting $u = A + B$ into the b -family equation and Novikov equation, we can obtain Alice-Bob peakon systems, where $\hat{P}_s$ and $\hat{T}_d$ are the arbitrary shifted parity transformation and delayed time reversal transformation, respectively. The nonlocal integrable Camassa-Holm equation and Degasperis-Procesi equation can be derived from the Alice-Bob b -family equations by choosing different parameters. Some new types of interesting solutions are solved including explicit one-peakons, two-peakons, and N -peakons solutions.

In shallow water theory, a lot of integrable equations are derived to model wave propagation in water of variable depth. We consider the b -family equations

$$m_t + m_x u + b m u_x = 0, \quad m = u - u_{xx}, \quad (1)$$

where b is a real constant.

When $b = 2$, (1) reduces to Camassa-Holm (CH) equation

$$m_t + m_x u + 2 m u_x = 0, \quad m = u - u_{xx}, \quad (2)$$

which was first reported by Fuchssteiner and Fokas in the context of hereditary symmetries [1]. It was rediscovered by Camassa and Holm as a model for shallow water waves. This equation shares most of the important properties of an integrable system of a Korteweg-de Vries type, for example, the existence of the bi-Hamiltonian structure [1, 2], Lax pair formalism [1], integrable by the inverse scattering transform [3], and other remarkable properties of integrable equations [4, 5]. Also, this equation admits peaked solitary wave solutions [3, 6], peaked soliton solutions [1, 7–9], and algebrogeometric solutions [10, 11]. The most interesting feature of the CH equation is that it admits peaked peakon solutions [1, 7]. A peakon is a weak solution in some Sobolev space with corner at its crest. The stability and analysis study of peakons were discussed in several references [7, 12–16].

When $b = 3$, (1) simplifies to Degasperis-Procesi (DP) equation

$$m_t + m_x u + 3 m u_x = 0, \quad m = u - u_{xx}, \quad (3)$$

which was proposed firstly by Degasperis and Procesi. This equation can also be considered as a model for shallow water wave and satisfied the asymptotic integrability to third order [17]; then in the complete integrability of the system it was proved because of the existence of a Lax pair and a bi-Hamiltonian structure based on a third order spectral problem [18]. The N -peakon solutions of the DP solution have been obtained [19]. In fact, every member of the b -family equation has peakon solutions for each b . However, it is not the case that every member is also integrable. In fact, the DP and CH equations are the only integrable members of the b -family equation. As we all know, the CH equation is widely used. Many new integrable equations can be regarded as a generalization of the well-known CH equation [6, 20] including FORQ (or MCH) system [21, 22] and Novikov equation [23]. In 2013, the integrable peakon systems with weak kink and kink-peakon interactional solutions of the FORQ system are first introduced by Qiao and Xia [24]. Next, we will study the Novikov equation as follows:

$$m_t + u^2 m_x + 3 u u_x m = 0, \quad m = u - u_{xx}. \quad (4)$$

FIGURE 1: (Colored online) One-peakon solution A of (5). (a) 3D-plot for $A(x, t)$ when $x_0 = 1, t_0 = 0, c = 1$. (b) The plot for $A(x, t)$ when $t = -1, t = 1, t = 3, t = 5$.

Like the CH equation, the Novikov equation has the Lax pair, bi-Hamiltonian structure, Lie symmetry, algebraic-geometric solutions [23, 25] and infinitely many conserved quantities. The Novikov equation also enjoys two other important properties of the CH equation; it admits peakon solutions and the Cauchy problem [26].

We all know that nonlocal integrable systems have attracted much attention in different nonlocal nonlinear equations, for example, the nonlocal nonlinear Schrödinger equation [27, 28], the nonlocal modified KdV systems [29], the (2+1)-dimensional KdV equation [30], KP equation [31], (2+1)-dimensional Sawada-Kotera equation [32, 33], nonlocal symmetry for the gKP equation [34], nonlocal symmetry of the (2+1)-dimensional breaking soliton equation [35], (2+1)-dimensional Gardner equation [36], and Drinfeld-Sokolov-Satsuma-Hirota system [37]. Recently, Lou introduced Alice-Bob (AB) models to study two-place physical problems [38]. In [38, 39], many possible models named AB systems are proposed; some special types of $\widehat{P}_s - \widehat{T}_d - C$ group invariant multisoliton solutions for the KdV-KP-Toda type, mKdV-sG type, discrete H_1 type, and NLS type AB systems are explicitly constructed. In [40], nonlocal integrable peakon equations are obtained and shown to have peakon solutions for some AB peakon systems. Now, we will investigate peakon solutions for nonlocal integrable b -family equation and Novikov equation.

Based on the AB approach [40], take $u = A + B$ to (1) and (4); the following AB b -family equation and AB Novikov equation can be generated:

$$\begin{aligned} (1 - \partial_x^2) A_t + (A + B) ((b+1) A - A_{xx})_x \\ - b(A + B)_x A_{xx} + H = 0, \end{aligned} \quad (5)$$

$$B = A(-x + x_0, -t + t_0),$$

$$\begin{aligned} (1 - \partial_x^2) A_t + (A + B)^2 (4A - A_{xx})_x \\ - \frac{3}{2} \partial_x (A + B)^2 A_{xx} + H = 0, \end{aligned} \quad (6)$$

$$B = A(-x + x_0, -t + t_0),$$

where H is an arbitrary shifted parity ($\widehat{P}_s$) and delayed time reversal ($\widehat{T}_d$) invariant function

$$\widehat{P}_s \widehat{T}_d H = H, \quad (7)$$

and the definitions of $\widehat{P}_s$ and $\widehat{T}_d$ are $\widehat{P}_s x = -x + x_0$, $\widehat{T}_d t = -t + t_0$; x_0 and t_0 are constants.

The rest of the paper is organized as follows. Firstly, peakon solutions of Alice-Bob (AB) b -family equation are introduced; the dynamic properties of these peakon solutions are displayed vividly by some figures. The peakon solutions of the AB Camassa-Holm equation and the AB Degasperis-Procesi equation can be derived. Secondly, we are devoted to peakon solutions of AB Novikov equation. Finally, short conclusion remarks are made in the paper.

Next, let us solve the peakon solutions to (5) for some special functions H .

Example 1. Setting $H = 0$ in (5) yields the following system:

$$\begin{aligned} (1 - \partial_x^2) A_t + (A + B) ((b+1) A - A_{xx})_x \\ - b(A + B)_x A_{xx} = 0, \end{aligned} \quad (8)$$

$$B = A(-x + x_0, -t + t_0).$$

One-peakon solution of (8) is

$$A = c \exp \left[- \left| x - \frac{x_0}{2} - 2c \left(t - \frac{t_0}{2} \right) \right| \right], \quad (9)$$

where c is real parameters to be determined which are nontraveling solitary waves with a fast decayed standing peak, as shown in Figure 1.

Furthermore, we can obtain the N -peakon dynamical system of (5)

$$A = \sum_{j=1}^N p_j \exp\left(-\left|x - \frac{x_0}{2} - q_j\right|\right), \quad (10)$$

$$p_{it} = b \sum_{j=1}^N \left(p_i p_j \operatorname{sgn}(q_i - q_j) e^{-|q_i - q_j|} + p_i \overline{p_j} \operatorname{sgn}(q_i + \overline{q_j}) e^{-|q_i + \overline{q_j}|} \right), \quad (11)$$

$$q_{it} = \sum_{j=1}^N \left(p_j e^{-|q_i - q_j|} + \overline{p_j} e^{-|q_i + \overline{q_j}|} \right),$$

where $\overline{p_j} = \widehat{T_d} p_j$, $\overline{q_j} = \widehat{T_d} q_j$, requiring that $x - x_0/2 - q_j$, $x - x_0/2 + \overline{q_j}$ ($j = 1, \dots, N$) be positive or negative at the same time. We obtain the following explicit two-peakon solutions:

$$A = p_1 \exp\left(-\left|x - \frac{x_0}{2} - q_1\right|\right) + p_2 \exp\left(-\left|x - \frac{x_0}{2} - q_2\right|\right), \quad (12)$$

where p_1, p_2, q_1, q_2 are real function with respect to variable t , and $B = \widehat{P_s} \widehat{T_d} A$:

$$\begin{aligned} A_x &= -p_1 \operatorname{sgn}\left(x - \frac{x_0}{2} - q_1\right) e^{-|x - x_0/2 - q_1|} \\ &\quad - p_2 \operatorname{sgn}\left(x - \frac{x_0}{2} - q_2\right) e^{-|x - x_0/2 - q_2|}, \\ (b+1)A - A_{xx} &= bp_1 \exp\left(-\left|x - \frac{x_0}{2} - q_1\right|\right) \\ &\quad + 2p_1 \delta\left(x - \frac{x_0}{2} - q_1\right) \\ &\quad + bp_2 \exp\left(-\left|x - \frac{x_0}{2} - q_2\right|\right) \\ &\quad + 2p_2 \delta\left(x - \frac{x_0}{2} - q_2\right), \end{aligned} \quad (13)$$

$$\begin{aligned} A + B &= p_1 \exp\left(-\left|x - \frac{x_0}{2} - q_1\right|\right) \\ &\quad + p_2 \exp\left(-\left|x - \frac{x_0}{2} - q_2\right|\right) \\ &\quad + \overline{p_1} \exp\left(-\left|x - \frac{x_0}{2} + \overline{q_1}\right|\right) \\ &\quad + \overline{p_2} \exp\left(-\left|x - \frac{x_0}{2} + \overline{q_2}\right|\right), \end{aligned}$$

where

$$\begin{aligned} p_{1t} &= bp_1 p_2 \operatorname{sgn}(q_1 - q_2) e^{-|q_1 - q_2|} \\ &\quad + bp_1 \overline{p_1} \operatorname{sgn}(q_1 + \overline{q_1}) e^{-|q_1 + \overline{q_1}|} \\ &\quad + bp_1 \overline{p_2} \operatorname{sgn}(q_1 + \overline{q_2}) e^{-|q_1 + \overline{q_2}|}, \\ p_{2t} &= bp_1 p_2 \operatorname{sgn}(q_2 - q_1) e^{-|q_2 - q_1|} \\ &\quad + bp_2 \overline{p_1} \operatorname{sgn}(q_2 + \overline{q_1}) e^{-|q_2 + \overline{q_1}|} \\ &\quad + bp_2 \overline{p_2} \operatorname{sgn}(q_2 + \overline{q_2}) e^{-|q_2 + \overline{q_2}|}, \\ p_1 &= -\frac{\overline{p_1} \operatorname{sgn}(q_2 + \overline{q_1})}{\operatorname{sgn}(q_2 - q_1)} e^{-|q_2 + \overline{q_1}| + |q_1 - q_2|} \\ &\quad - \frac{\overline{p_2} \operatorname{sgn}(q_2 + \overline{q_2})}{\operatorname{sgn}(q_2 - q_1)} e^{-|q_2 + \overline{q_2}| + |q_2 - q_1|}, \\ p_2 &= -\frac{\overline{p_1} \operatorname{sgn}(q_1 + \overline{q_1})}{\operatorname{sgn}(q_1 - q_2)} e^{-|q_1 + \overline{q_1}| + |q_1 - q_2|} \\ &\quad - \frac{\overline{p_2} \operatorname{sgn}(q_1 + \overline{q_2})}{\operatorname{sgn}(q_1 - q_2)} e^{-|q_1 + \overline{q_2}| + |q_1 - q_2|}, \end{aligned} \quad (14)$$

$$\begin{aligned} q_{1t} &= p_1 + p_2 e^{-|q_1 - q_2|} + \overline{p_1} e^{-|q_1 + \overline{q_1}|} + \overline{p_2} e^{-|q_1 + \overline{q_2}|}, \\ q_{2t} &= p_2 + p_1 e^{-|q_2 - q_1|} + \overline{p_1} e^{-|q_2 + \overline{q_1}|} + \overline{p_2} e^{-|q_2 + \overline{q_2}|}, \end{aligned}$$

which require that $x - x_0/2 - q_1$, $x - x_0/2 - q_2$, $x - x_0/2 + \overline{q_1}$, and $x - x_0/2 + \overline{q_2}$ be positive or negative at the same time which are nontraveling solitary waves with a fast decayed standing peak, as shown in Figure 2.

Example 2. Substituting $b = 2$, $H = 0$ into (5), we can get the following AB CH equation:

$$\begin{aligned} (1 - \partial_x^2) A_t + (A + B)(3A - A_{xx})_x - 2(A + B)_x A_{xx} \\ = 0, \quad B = A(-x + x_0, -t + t_0). \end{aligned} \quad (15)$$

We can derive the one-peakon solutions of the system

$$A = c \exp\left[-\left|x - \frac{x_0}{2} - 2c\left(t - \frac{t_0}{2}\right)\right|\right], \quad (16)$$

Here we discuss the two-peakon solutions to CH equation

$$\begin{aligned} A &= p_1 \exp\left(-\left|x - \frac{x_0}{2} - q_1\right|\right) \\ &\quad + p_2 \exp\left(-\left|x - \frac{x_0}{2} - q_2\right|\right), \end{aligned} \quad (17)$$

FIGURE 2: (Colored online) The two-peakon solution of (9). (a) 3D-plot for $A(x,t)$ when $p_1 = 1, p_2 = 1, x_0 = 0, q_1 = -t + 1, q_2 = t$. (b) 3D-plot for $A(x,t)$ when $p_1 = 1, p_2 = -1, x_0 = 0, q_1 = -t + 1, q_2 = t$. (c, d) Plot for $A(x,t)$ when $t = -1, t = 1, t = 3, t = 5$.

where p_1, p_2, q_1, q_2 are real function with respect to variable t

$$\begin{aligned}
 p_{1t} &= 2p_1 p_2 \operatorname{sgn}(q_1 - q_2) e^{-|q_1 - q_2|} \\
 &\quad + 2p_1 \bar{p}_1 \operatorname{sgn}(q_1 + \bar{q}_1) e^{-|q_1 + \bar{q}_1|} \\
 &\quad + 2p_1 \bar{p}_2 \operatorname{sgn}(q_1 + \bar{q}_2) e^{-|q_1 + \bar{q}_2|}, \\
 p_{2t} &= 2p_1 p_2 \operatorname{sgn}(q_2 - q_1) e^{-|q_2 - q_1|} \\
 &\quad + 2p_2 \bar{p}_1 \operatorname{sgn}(q_2 + \bar{q}_1) e^{-|q_2 + \bar{q}_1|} \\
 &\quad + 2p_2 \bar{p}_2 \operatorname{sgn}(q_2 + \bar{q}_2) e^{-|q_2 + \bar{q}_2|}, \\
 p_1 &= -\frac{\bar{p}_1 \operatorname{sgn}(q_2 + \bar{q}_1)}{\operatorname{sgn}(q_2 - q_1)} e^{-|q_2 + \bar{q}_1| + |q_1 - q_2|} \\
 &\quad - \frac{\bar{p}_2 \operatorname{sgn}(q_2 + \bar{q}_2)}{\operatorname{sgn}(q_2 - q_1)} e^{-|q_2 + \bar{q}_2| + |q_1 - q_2|},
 \end{aligned}$$

$$\begin{aligned}
 p_2 &= -\frac{\bar{p}_1 \operatorname{sgn}(q_1 + \bar{q}_1)}{\operatorname{sgn}(q_1 - q_2)} e^{-|q_1 + \bar{q}_1| + |q_1 - q_2|} \\
 &\quad - \frac{\bar{p}_2 \operatorname{sgn}(q_1 + \bar{q}_2)}{\operatorname{sgn}(q_1 - q_2)} e^{-|q_1 + \bar{q}_2| + |q_1 - q_2|}, \\
 q_{1t} &= p_1 + p_2 e^{-|q_1 - q_2|} + \bar{p}_1 e^{-|q_1 + \bar{q}_1|} + \bar{p}_2 e^{-|q_1 + \bar{q}_2|}, \\
 q_{2t} &= p_2 + p_1 e^{-|q_2 - q_1|} + \bar{p}_1 e^{-|q_2 + \bar{q}_1|} + \bar{p}_2 e^{-|q_2 + \bar{q}_2|},
 \end{aligned} \tag{18}$$

which needs to satisfy $x - x_0/2 - q_1, x - x_0/2 - q_2, x - x_0/2 + \bar{q}_1$, and $x - x_0/2 + \bar{q}_2$ being positive or negative at the same time. The N -peakon dynamical system of (15) is as follows:

$$A = \sum_{j=1}^N p_j \exp\left(-\left|x - \frac{x_0}{2} - q_j\right|\right), \tag{19}$$

where

$$\begin{aligned}
p_{it} &= 2 \sum_{j=1}^N \left(p_i p_j \operatorname{sgn}(q_i - q_j) e^{-|q_i - q_j|} \right. \\
&\quad \left. + p_i \overline{p_j} \operatorname{sgn}(q_i + \overline{q_j}) e^{-|q_i + \overline{q_j}|} \right), \\
q_{it} &= \sum_{j=1}^N \left(p_j e^{-|q_i - q_j|} + \overline{p_j} e^{-|q_i + \overline{q_j}|} \right),
\end{aligned} \quad (20)$$

where $x - x_0/2 - q_j$, $x - x_0/2 - \overline{q_j}$, $x - x_0/2 + \overline{q_j}$, and $x - x_0/2 + q_j$, $j = 1, \dots, N$, are positive or negative at the same time.

Example 3. Setting $b = 3, H = 0$ to (5), we can get the AB system of the DP equation

$$\begin{aligned}
(1 - \partial_x^2) A_t + (A + B)(4A - A_{xx})_x - 3(A + B)_x A_{xx} \\
= 0, \quad B = A(-x + x_0, -t + t_0),
\end{aligned} \quad (21)$$

which admits the following one-peakon solution of (21):

$$A = c \exp \left(- \left| x - \frac{x_0}{2} - 2c \left(t - \frac{t_0}{2} \right) \right| \right), \quad (22)$$

The two-peakon solutions are given in the following form:

$$\begin{aligned}
A &= p_1 \exp \left(- \left| x - \frac{x_0}{2} - q_1 \right| \right) \\
&\quad + p_2 \exp \left(- \left| x - \frac{x_0}{2} - q_2 \right| \right),
\end{aligned} \quad (23)$$

where

$$\begin{aligned}
p_{1t} &= 3p_1 p_2 \operatorname{sgn}(q_1 - q_2) e^{-|q_1 - q_2|} \\
&\quad + 3p_1 \overline{p_1} \operatorname{sgn}(q_1 + \overline{q_1}) e^{-|q_1 + \overline{q_1}|} \\
&\quad + 3p_1 \overline{p_2} \operatorname{sgn}(q_1 + \overline{q_2}) e^{-|q_1 + \overline{q_2}|}, \\
p_{2t} &= 3p_1 p_2 \operatorname{sgn}(q_2 - q_1) e^{-|q_2 - q_1|} \\
&\quad + 3p_2 \overline{p_1} \operatorname{sgn}(q_2 + \overline{q_1}) e^{-|q_2 + \overline{q_1}|} \\
&\quad + 3p_2 \overline{p_2} \operatorname{sgn}(q_2 + \overline{q_2}) e^{-|q_2 + \overline{q_2}|},
\end{aligned}$$

$$\begin{aligned}
p_1 &= - \frac{\overline{p_1} \operatorname{sgn}(q_2 + \overline{q_1})}{\operatorname{sgn}(q_2 - q_1)} e^{-|q_2 + \overline{q_1}| + |q_2 - q_1|} \\
&\quad - \frac{\overline{p_2} \operatorname{sgn}(q_2 + \overline{q_2})}{\operatorname{sgn}(q_2 - q_1)} e^{-|q_2 + \overline{q_2}| + |q_2 - q_1|}, \\
p_2 &= - \frac{\overline{p_1} \operatorname{sgn}(q_1 + \overline{q_1})}{\operatorname{sgn}(q_1 - q_2)} e^{-|q_1 + \overline{q_1}| + |q_1 - q_2|} \\
&\quad - \frac{\overline{p_2} \operatorname{sgn}(q_1 + \overline{q_2})}{\operatorname{sgn}(q_1 - q_2)} e^{-|q_1 + \overline{q_2}| + |q_1 - q_2|}, \\
q_{1t} &= p_1 + p_2 e^{-|q_1 - q_2|} + \overline{p_1} e^{-|q_1 + \overline{q_1}|} + \overline{p_2} e^{-|q_1 + \overline{q_2}|}, \\
q_{2t} &= p_2 + p_1 e^{-|q_2 - q_1|} + \overline{p_1} e^{-|q_2 + \overline{q_1}|} + \overline{p_2} e^{-|q_2 + \overline{q_2}|},
\end{aligned} \quad (24)$$

which require that $x - x_0/2 - q_1$, $x - x_0/2 - q_2$, $x - x_0/2 + \overline{q_1}$, and $x - x_0/2 + \overline{q_2}$ be positive or negative at the same time. The N -peakon dynamical system of (21) can be obtained as follows:

$$A = \sum_{j=1}^N p_j \exp \left(- \left| x - \frac{x_0}{2} - q_j \right| \right), \quad (25)$$

$$\begin{aligned}
p_{it} &= 3 \sum_{j=1}^N \left(p_i p_j \operatorname{sgn}(q_i - q_j) e^{-|q_i - q_j|} \right. \\
&\quad \left. + p_i \overline{p_j} \operatorname{sgn}(q_i + \overline{q_j}) e^{-|q_i + \overline{q_j}|} \right),
\end{aligned} \quad (26)$$

$$q_{it} = \sum_{j=1}^N \left(p_j e^{-|q_i - q_j|} + \overline{p_j} e^{-|q_i + \overline{q_j}|} \right),$$

and requires that $x - x_0/2 - q_j$ and $x - x_0/2 + \overline{q_j}$ ($j = 1, \dots, N$) have the same positive or negative qualities.

Example 4. When $H = (1/2)(AB - A_x B_x)$ or $H = (1/2)(A - A_x)(B + B_x)$ of (5), we can obtain AB N -peakon solution as (8).

Example 5. When $H = (1/2)(AA_x - BB_x)$ of (5), this system has only one-peakon solutions

$$A = c \exp \left[- \left| x - \frac{x_0}{2} - 2c \left(t - \frac{t_0}{2} \right) \right| \right], \quad (27)$$

No multipeakon solution is found for this special example.

Just like b -family equation, taking $H = 0$ to AB Novikov (6),

$$\begin{aligned}
(1 - \partial_x^2) A_t + (A + B)^2 (4A - A_{xx})_x \\
- \frac{3}{2} \partial_x (A + B)^2 A_{xx} = 0,
\end{aligned} \quad (28)$$

$$B = A(-x + x_0, -t + t_0).$$

We can obtain the following one-peakon solution:

$$A = c \exp \left[- \left| x - \frac{x_0}{2} - 4c^2 \left(t - \frac{t_0}{2} \right) \right| \right], \quad (29)$$

where c is arbitrary constant. The two-peakon solutions can be derived

where

$$A = h_1 \exp \left(- \left| x - \frac{x_0}{2} - s_1 \right| \right) + h_2 \exp \left(- \left| x - \frac{x_0}{2} - s_2 \right| \right), \quad (30)$$

$$\begin{aligned} h_{1t} &= 3h_1 \left(h_1 h_2 \operatorname{sgn}(s_1 - s_2) e^{-|s_1 - s_2|} + h_1 \overline{h_1} \operatorname{sgn}(s_1 + \overline{s_1}) e^{-|s_1 + \overline{s_1}|} + h_1 \overline{h_2} \operatorname{sgn}(s_1 + \overline{s_2}) e^{-|s_1 + \overline{s_2}|} \right) \\ &\quad + 3h_1 \left(h_2^2 \operatorname{sgn}(s_1 - s_2) e^{-|s_1 - s_2|} + h_2 \overline{h_1} [\operatorname{sgn}(s_1 - s_2) + \operatorname{sgn}(s_1 + \overline{s_1})] e^{-|s_1 - s_2| - |s_1 + \overline{s_1}|} \right) \\ &\quad + 3h_1 \left(h_2 \overline{h_2} [\operatorname{sgn}(s_1 - s_2) + \operatorname{sgn}(s_1 + \overline{s_2})] e^{-|s_1 - s_2| - |s_1 + \overline{s_2}|} \right) \\ &\quad + 3h_1 \left(\overline{h_1} \overline{h_2} [\operatorname{sgn}(s_1 + \overline{s_1}) + \operatorname{sgn}(s_1 + \overline{s_2})] e^{-|s_1 + \overline{s_1}| - |s_1 + \overline{s_2}|} \right) \\ &\quad + 3h_1 \left[\overline{h_1}^2 \operatorname{sgn}(s_1 + \overline{s_1}) e^{-2|s_1 + \overline{s_1}|} + \overline{h_2}^2 \operatorname{sgn}(s_1 + \overline{s_2}) e^{-2|s_1 + \overline{s_2}|} \right], \\ h_{2t} &= 3h_2 \left(h_1^2 \operatorname{sgn}(s_2 - s_1) e^{-|s_2 - s_1|} + h_1 \overline{h_1} [\operatorname{sgn}(s_2 - s_1) + \operatorname{sgn}(s_2 + \overline{s_1})] e^{-|s_2 - s_1|} \right) \\ &\quad + 3h_2 \left(h_1 h_2 \operatorname{sgn}(s_2 - s_1) e^{-|s_2 - s_1|} + h_2 \overline{h_1} \operatorname{sgn}(s_2 + \overline{s_1}) e^{-|s_2 + \overline{s_1}|} + h_2 \overline{h_2} \operatorname{sgn}(s_2 + \overline{s_2}) e^{-|s_2 + \overline{s_2}|} \right) \\ &\quad + 3h_2 \left(h_1 \overline{h_2} [\operatorname{sgn}(s_2 - s_1) + \operatorname{sgn}(s_2 + \overline{s_2})] e^{-|s_2 - s_1| - |s_2 + \overline{s_2}|} \right) \\ &\quad + 3h_2 \left(\overline{h_1} \overline{h_2} [\operatorname{sgn}(s_2 + \overline{s_1}) + \operatorname{sgn}(s_2 + \overline{s_2})] e^{-|s_2 + \overline{s_1}| - |s_2 + \overline{s_2}|} \right) \\ &\quad + 3h_2 \left(\overline{h_1}^2 \operatorname{sgn}(s_2 + \overline{s_1}) e^{-2|s_2 + \overline{s_1}|} + \overline{h_2}^2 \operatorname{sgn}(s_2 + \overline{s_2}) e^{-2|s_2 + \overline{s_2}|} \right), \\ h_1 &= - \frac{h_2^2 \operatorname{sgn}(s_1 - s_2) e^{-2|s_1 - s_2|} + h_2 \overline{h_1} (\operatorname{sgn}(s_1 - s_2) + \operatorname{sgn}(s_1 + \overline{s_1})) e^{-|s_1 - s_2| - |s_1 + \overline{s_1}|}}{h_2 \operatorname{sgn}(s_1 - s_2) e^{-|s_1 - s_2|} + \overline{h_1} \operatorname{sgn}(s_1 + \overline{s_1}) e^{-|s_1 + \overline{s_1}|} + \overline{h_2} \operatorname{sgn}(s_1 + \overline{s_2}) e^{-|s_1 + \overline{s_2}|}} \\ &\quad - \frac{h_2 \overline{h_2} (\operatorname{sgn}(s_1 - s_2) + \operatorname{sgn}(s_1 + \overline{s_2})) e^{-|s_1 - s_2| - |s_1 + \overline{s_2}|} + \overline{h_1} \overline{h_2} (\operatorname{sgn}(s_1 + \overline{s_1}) + \operatorname{sgn}(s_1 + \overline{s_2})) e^{-|s_1 + \overline{s_1}| - |s_1 + \overline{s_2}|}}{h_2 \operatorname{sgn}(s_1 - s_2) e^{-|s_1 - s_2|} + \overline{h_1} \operatorname{sgn}(s_1 + \overline{s_1}) e^{-|s_1 + \overline{s_1}|} + \overline{h_2} \operatorname{sgn}(s_1 + \overline{s_2}) e^{-|s_1 + \overline{s_2}|}} \\ &\quad - \frac{\overline{h_1}^2 \operatorname{sgn}(s_1 + \overline{s_1}) e^{-2|s_1 + \overline{s_1}|} + \overline{h_2}^2 \operatorname{sgn}(s_1 + \overline{s_2}) e^{-2|s_1 + \overline{s_2}|}}{h_2 \operatorname{sgn}(s_1 - s_2) e^{-|s_1 - s_2|} + \overline{h_1} \operatorname{sgn}(s_1 + \overline{s_1}) e^{-|s_1 + \overline{s_1}|} + \overline{h_2} \operatorname{sgn}(s_1 + \overline{s_2}) e^{-|s_1 + \overline{s_2}|}}, \\ h_2 &= - \frac{h_1^2 \operatorname{sgn}(s_2 - s_1) e^{-2|s_2 - s_1|} + h_1 \overline{h_1} (\operatorname{sgn}(s_2 - s_1) + \operatorname{sgn}(s_2 + \overline{s_1})) e^{-|s_2 - s_1| - |s_2 + \overline{s_1}|}}{h_1 \operatorname{sgn}(s_2 - s_1) e^{-|s_2 - s_1|} + \overline{h_1} \operatorname{sgn}(s_2 + \overline{s_1}) e^{-|s_2 + \overline{s_1}|} + \overline{h_2} \operatorname{sgn}(s_2 + \overline{s_2}) e^{-|s_2 + \overline{s_2}|}} \\ &\quad - \frac{h_1 \overline{h_2} (\operatorname{sgn}(s_2 - s_1) + \operatorname{sgn}(s_2 + \overline{s_2})) e^{-|s_2 - s_1| - |s_2 + \overline{s_2}|} + \overline{h_1} \overline{h_2} (\operatorname{sgn}(s_2 + \overline{s_1}) + \operatorname{sgn}(s_2 + \overline{s_2})) e^{-|s_2 + \overline{s_1}| - |s_2 + \overline{s_2}|}}{h_1 \operatorname{sgn}(s_2 - s_1) e^{-|s_2 - s_1|} + \overline{h_1} \operatorname{sgn}(s_2 + \overline{s_1}) e^{-|s_2 + \overline{s_1}|} + \overline{h_2} \operatorname{sgn}(s_2 + \overline{s_2}) e^{-|s_2 + \overline{s_2}|}} \\ &\quad - \frac{\overline{h_1}^2 \operatorname{sgn}(s_2 + \overline{s_1}) e^{-2|s_2 + \overline{s_1}|} + \overline{h_2}^2 \operatorname{sgn}(s_2 + \overline{s_2}) e^{-2|s_2 + \overline{s_2}|}}{h_1 \operatorname{sgn}(s_2 - s_1) e^{-|s_2 - s_1|} + \overline{h_1} \operatorname{sgn}(s_2 + \overline{s_1}) e^{-|s_2 + \overline{s_1}|} + \overline{h_2} \operatorname{sgn}(s_2 + \overline{s_2}) e^{-|s_2 + \overline{s_2}|}}, \\ s_{1t} &= h_1^2 + 2h_1 h_2 e^{-|s_1 - s_2|} + 2h_1 \overline{h_1} e^{-|s_1 + \overline{s_1}|} + 2h_1 \overline{h_2} e^{-|s_1 + \overline{s_2}|} + h_2^2 e^{-2|s_1 - s_2|} + 2h_2 \overline{h_1} e^{-|s_1 - s_2| - |s_1 + \overline{s_1}|} + 2h_2 \overline{h_2} e^{-|s_1 - s_2| - |s_1 + \overline{s_2}|} \\ &\quad + \overline{h_1}^2 e^{-2|s_1 + \overline{s_1}|} + 2\overline{h_1} \overline{h_2} e^{-|s_1 + \overline{s_1}| - |s_1 + \overline{s_2}|} + \overline{h_2}^2 e^{-2|s_1 + \overline{s_2}|}, \\ s_{2t} &= h_1^2 e^{-2|s_2 - s_1|} + 2h_1 h_2 e^{-|s_2 - s_1|} + 2h_1 \overline{h_1} e^{-|s_2 - s_1| - |s_2 + \overline{s_1}|} + 2h_1 \overline{h_2} e^{-|s_2 - s_1| - |s_2 + \overline{s_2}|} + h_2^2 + 2h_2 \overline{h_1} e^{-|s_2 + \overline{s_1}|} + 2h_2 \overline{h_2} e^{-|s_2 + \overline{s_1}|} \\ &\quad + \overline{h_1}^2 e^{-2|s_2 + \overline{s_1}|} + 2\overline{h_1} \overline{h_2} e^{-|s_2 + \overline{s_1}| - |s_2 + \overline{s_2}|} + \overline{h_2}^2 e^{-2|s_2 + \overline{s_2}|}, \end{aligned} \quad (31)$$

which require that $x - x_0/2 - s_1$, $x - x_0/2 - s_2$, $x - x_0/2 + \bar{s}_1$, and $x - x_0/2 + \bar{s}_2$ be positive or negative at the same time. Furthermore, we can obtain the N -peakon dynamical system of (28):

$$A = \sum_{j=1}^N h_j \exp \left(- \left| x - \frac{x_0}{2} - s_j \right| \right), \quad (32)$$

where

$$\begin{aligned} h_{it} = & 3h_i \sum_{j=1}^N \left((h_i h_j + h_j^2) \operatorname{sgn}(s_i - s_j) e^{-|s_i - s_j|} + h_i \bar{h}_j \right. \\ & \cdot \operatorname{sgn}(s_i + \bar{s}_j) e^{-|s_i + \bar{s}_j|} + \bar{h}_j^2 \operatorname{sgn}(s_i + \bar{s}_j) e^{-2|s_i + \bar{s}_j|} \\ & + 3h_i \sum_{j \neq i}^N ((\bar{h}_i h_j + h_j \bar{h}_j) \\ & \cdot [\operatorname{sgn}(s_i - s_j) + \operatorname{sgn}(s_i + \bar{s}_j)] e^{-|s_i - s_j| - |s_i + \bar{s}_j|} \\ & + \bar{h}_i \bar{h}_j [\operatorname{sgn}(s_i + \bar{s}_i) + \operatorname{sgn}(s_i + \bar{s}_j)] e^{-|s_i + \bar{s}_i| - |s_i + \bar{s}_j|} \Big), \\ s_{it} = & \left(\sum_{j=1}^N (h_j e^{-|s_i - s_j|} + \bar{h}_j e^{-|s_i + \bar{s}_j|}) \right)^2, \end{aligned} \quad (33)$$

where $x - x_0/2 - s_j$ and $x - x_0/2 + \bar{s}_j$ ($j = 1, \dots, N$) are positive or negative at the same time. Similar to the b -family equation when the H takes the different functions of A and B , we can get the same single peakon and two-peakon forms.

In this study, we have obtained the Alice-Bob (AB) peakon system for b -family equation and Novikov equation, from which the nonlocal integrable Camassa-Holm equation and Degasperis-Procesi equation can be derived by choosing different parameters. Then the peakon solutions of these AB peakon systems are concretely established, and the dynamic properties of these peakon solutions are displayed vividly by some figures. For every peakon systems, there may exist different versions of integrable AB peakon systems, such as Novikov's cubic nonlinear equation, generalized peakon system, and Li-Liu-Popowicz's system. We believe that these systems deserve further investigation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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