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Research Article

Helicity and Spin of Linearly Polarized Hermite-Gaussian Modes

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The angular momentum content and propagation of linearly polarized Hermite-Gaussian modes are analyzed. The helicity gauge invariant continuity equation reveals that the helicity and flow in the direction of propagation are zero. However, the helicity flow exhibits nonvanishing transverse components. These components have been recently described as photonic wheels. These intrinsic angular momentum terms, depending on the criterion, can be associated with spin or orbital momentum. The electric and magnetic contributions to the optical helicity will be shown to cancel out for Hermite-Gaussian modes. The helicity $\varrho_{\mathbf{AC}}$ here derived is consistent with the interpretation that it represents the projection of the angular momentum onto the direction of motion.

1. Introduction

Electromagnetic fields are capable of carrying angular momentum and transferring it to matter. The angular momentum (AM) content in the fields may be due to the polarization and/or to the helical wavefront of the beam. The former has been associated with the spin angular momentum (SAM) whereas the latter has been associated with the orbital angular momentum (OAM) of light. The field angular momentum has also been cataloged as intrinsic or extrinsic; polarization is an intrinsic property because it does not depend on the choice of origin. OAM was originally considered an extrinsic property because of its dependence on position with respect to the beam axis. However, it was later shown that OAM has an intrinsic as well as an extrinsic part [1]. It was recently shown that electromagnetic fields can also carry angular momentum orthogonal to the direction of propagation [2, 3]. A mechanical analogue is the angular momenta of the wheels in a vehicle that are perpendicular to the direction of propagation.

The angular momentum of the field is defined, in analogy to the mechanical angular momentum $\mathbf{J} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{r}$ is position and $\mathbf{p}$ is mechanical linear momentum. The electromagnetic field's linear momentum is given by Poynting's vector $\mathbf{p} \longrightarrow \mathbf{E} \times \mathbf{H}$, where $\mathbf{E}$ and $\mathbf{H}$ represent the electric and magnetic fields, respectively. This approach has the baffling prediction that the AM in the direction of propagation must be zero, since it must be perpendicular to $\mathbf{r}$ and $\mathbf{p}$ [4]. Furthermore, the AM, due to circular polarization in classical electromagnetism or spin in the quantized version, is not expected to be dependent on position. However, the total angular momentum $\mathbf{J} = \mathbf{r} \times (\mathbf{E} \times \mathbf{H})$ can be split into two parts $\mathbf{J} = \mathbf{S} + \mathbf{L}$, where $\mathbf{S} = (\mathbf{E} \times \mathbf{A})$ and $\mathbf{L} = \sum_i \mathrm{E}_i (\mathbf{r} \times \nabla) \mathrm{A}_i \widehat{\mathbf{e}}_i$ are identified with the spin and orbital angular momenta [5]. $\mathbf{A}$ represents the magnetic potential $\mathbf{B} = \nabla \times \mathbf{A}$, with A_i components and $\widehat{\mathbf{e}}_i$ represent unit Cartesian vectors throughout the text. This separation is not exempt from difficulties because it requires various assumptions to dismiss some of the extra terms involved [6]. The spin term has been objected on two grounds, its lack of gauge invariance and the fact that it does not satisfy the Heaviside-Larmor symmetry [7]. Gauge invariance has been remedied in two ways, (i) invoking the integral form $\mathbf{S}_{tot} = \int (\mathbf{E} \times \mathbf{A}) d^3 r$ [8, 9] and/or (ii) performing a Helmholtz decomposition and using only the transverse parts of the potentials $\mathbf{S} = (\mathbf{E} \times \mathbf{A}_\perp)$ [10, 11]. The Heaviside-Larmor symmetry, or more generally a duality transformation [12], is satisfied by invoking an ad hoc principle of electric-magnetic democracy [13]. The SAM is then written as $\mathbf{S} = (1/2)(\mathbf{E} \times \mathbf{A}_\perp + \mathbf{B} \times \mathbf{C}_\perp)$, where $\mathbf{C}$ is a second vector potential whose curl is proportional to the electric field.

Angular momentum should be a conserved vector quantity in a free field since there is neither absorption nor sources. In continuous media, a continuity equation is thus expected where a scalar quantity, the helicity in this case, is conserved whereas its flow is the SAM. The helicity conservation equation is the angular momentum analogue to Poynting's theorem, where energy is the scalar quantity and linear momentum its corresponding flow. The optical helicity density is defined as $\varrho_{\mathbf{AC}} = \mathbf{A} \cdot \mathbf{B} - \mathbf{C} \cdot \mathbf{E}$; it is a scalar quantity pertaining to angular momentum conservation. Orbital angular momentum of light beams was first described in terms of Laguerre-Gaussian (LG) modes [14]. LG modes are solutions to the paraxial scalar wave equation in cylindrical coordinates. The two-dimensional Hermite-Gaussian (HG) modes are the corresponding paraxial scalar wave equation solutions in Cartesian coordinates. The correct field polarization should be taken into account to insure at least lowest order consistency with the electromagnetic equations [15]. Nonetheless, the paraxial solutions are not entirely consistent with Maxwell equations [16]. This result is not unexpected, since the electromagnetic equations incorporate, in the coupled first order differential equations, the information of the second order vector (nonparaxial) wave equations.

The helicity continuity equation can be written in a gauge invariant way although the helicity and its flow are not gauge invariant [17]. This equation has been derived for arbitrary real electromagnetic free fields without restrictions, of neither monochromaticity nor transverse fields invoking the Helmholtz decomposition. The present paper is devoted to the conservation of the helicity and its corresponding flow in linearly polarized HG beams. Some results are of particular relevance: There is a nonzero transverse "spin flow" although the polarization is linear. The contributions of the magnetic helicity $\mathbf{A} \cdot \mathbf{B}$ and the electric helicity $\mathbf{C} \cdot \mathbf{E}$ to the optical helicity cancel out in linearly polarized HG modes. The scalar potential source terms are essential to produce the correct results. In Section 2, the field equations for HG modes are stated for various approximations. Section 3 is dedicated to the gauge invariant continuity equation while Section 4 is devoted to the analysis of the helicity and flow for linearly polarized HG modes. Conclusions are drawn in the last section.

2. Hermite-Gaussian Solutions

The HG solutions can be written with a magnetic vector potential $\mathbf{A}$ pointing in a single direction, say the $\widehat{\mathbf{e}}_x$ direction, as the starting point of the solution [18]. However, due to the finite transverse spatial extent of beams, Maxwell's equations couple the fields polarization with the spatial degrees of freedom [19]. In general, all three Cartesian components of the field associated with the beam need to be nonzero. Therefore, the electric field vector cannot lie in a single direction and thus, since $\mathbf{E} = -\nabla\phi_A - \partial_t \mathbf{A}$, the scalar potential must be nonzero. In the Coulomb gauge, the scalar potential is zero for sourceless fields; thus in this scheme, the Lorenz gauge has to be employed. Let a vector potential with wave vector k and frequency ω propagate in the z direction

$$\mathbf{A} = u_{mn} \cos(kz - \omega t)\, \widehat{\mathbf{e}}_x, \tag{1}$$

where u_{mn} are the Hermite-Gauss space dependent functions:

$$\begin{aligned} u_{mn} = {} & \sqrt{\frac{2}{\pi w^2(z)}} H_m\left(\frac{\sqrt{2}x}{w(z)}\right) H_n\left(\frac{\sqrt{2}y}{w(z)}\right) \\ & \cdot \operatorname{Re}\left\{\exp\left(-i(m+n+1)\arctan\left(\frac{z}{z_0}\right) + ik\frac{\sqrt{x^2+y^2}}{2q(z)}\right)\right\}, \end{aligned} \tag{2}$$

where H_m is the Hermite polynomial of degree m, $w(z) = w_0\sqrt{1 + z/z_0}$ is the beam width, w_0 the beam waist, $\operatorname{Re}\{\cdots\}$ is the real part of the expression, and the complex beam parameter is

$$\frac{1}{q(z)} = \frac{1}{R(z)} + i\frac{2}{kw^2(z)}. \tag{3}$$

The radius of curvature is $R(z) = z + z_0^2/z$ and z_0 is the Rayleigh distance. Although a complex representation of the fields is possible, nonlinear terms are involved in the continuity equations. Thus care should be taken when generalizing the formalism to $\mathbb{C}$. Real fields will be employed throughout this work to stress the fact that no temporal averages are being evaluated unless explicitly stated. In the Lorenz gauge $\partial_t \phi_A = -c^2 \nabla \cdot \mathbf{A}$, the scalar potential from (1) is then $\phi_A = (c^2/\omega)\partial_x u_{mn} \sin(kz - \omega t)$. The gradient of the scalar potential involves all three directions, but if only first order derivatives in u_{mn} are retained

$$\nabla\phi_A \approx c\partial_x u_{mn} \cos(kz - \omega t)\, \widehat{\mathbf{e}}_z + \mathcal{O}^{2+}. \tag{4}$$

The electric field $\mathbf{E} = -\nabla\phi_A - \partial_t \mathbf{A}$, obtained from the potentials (1) and (4) is

$$\begin{aligned} \mathbf{E} = {} & -\omega u_{mn} \sin(kz - \omega t)\, \widehat{\mathbf{e}}_x - c\partial_x u_{mn} \cos(kz - \omega t)\, \widehat{\mathbf{e}}_z \\ & + \mathcal{O}^{2+}, \end{aligned} \tag{5}$$

whereas the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ is

$$\begin{aligned} \mathbf{B} = {} & \left(\partial_z u_{mn} \cos(kz - \omega t) - k u_{mn} \sin(kz - \omega t)\right) \widehat{\mathbf{e}}_y \\ & - \partial_y u_{mn} \cos(kz - \omega t)\, \widehat{\mathbf{e}}_z. \end{aligned} \tag{6}$$

It is possible to introduce a second vector potential $\mathbf{C}$ [20, 21], with its concomitant scalar potential ϕ_C, such that in SI units,

$$\mathbf{E} = -\nabla\phi_A - \partial_t \mathbf{A} = -\frac{1}{\mu\varepsilon}\nabla \times \mathbf{C}, \tag{7a}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\nabla\phi_C - \partial_t \mathbf{C}, \tag{7b}$$

where the inverse product of permeability times permittivity in vacuum is $1/\mu\varepsilon = c^2$. In a highly symmetrical fashion with its $\mathbf{A}$ counterpart, the time derivative of the vector potential $\mathbf{C}$ is now related to the magnetic field whereas its curl is proportional to the electric field. The two vector potentials are not independent; the above equations establish a relationship between them that is similar to the relations between the electric and magnetic fields in Maxwell's equations. Let the second vector potential be

$$\mathbf{C} = \frac{1}{c} u_{mn} \cos(kz - \omega t)\,\widehat{\mathbf{e}}_y. \tag{8}$$

It should be noted that (7a) is only approximately fulfilled if both potential fields have only one component. The scalar potential from the Lorenz condition $\partial_t \phi_C = -c^2 \nabla \cdot \mathbf{C}$ is $\phi_C = (c/\omega)\partial_y u_{mn} \sin(kz - \omega t)$. This set of equations are the real field solutions arising from the complex solutions used in the OAM seminal papers of Allen et al. (with some typos being corrected.) [14, 22].

The previous first order electromagnetic field solutions starting with the $\mathbf{A}$ potential are not symmetric in the electric and magnetic contributions. This asymmetry can be remedied by calculating the fields from the potentials with the same functional form; i.e., $\mathbf{E} = -\nabla\phi_A - \partial_t \mathbf{A}$ and $\mathbf{B} = -\nabla\phi_C - \partial_t \mathbf{C}$. The fields then have nonvanishing contributions in all the three spatial directions if second order u_{mn} derivatives are not dismissed,

$$\begin{aligned}\mathbf{E} &= -\nabla\phi_A - \partial_t \mathbf{A} \\ &= \left(-\frac{c}{k}\partial_x^2 u_{mn} \sin\varphi - \omega u_{mn} \sin\varphi\right)\widehat{\mathbf{e}}_x \\ &\quad - \frac{c}{k}\partial_y \partial_x u_{mn} \sin\varphi\,\widehat{\mathbf{e}}_y \\ &\quad + \left(-\frac{c}{k}\partial_z \partial_x u_{mn} \sin\varphi - c\partial_x u_{mn} \cos\varphi\right)\widehat{\mathbf{e}}_z\end{aligned} \tag{9a}$$

and

$$\begin{aligned}\mathbf{B} &= -\nabla\phi_C - \partial_t \mathbf{C} \\ &= -\frac{1}{k}\partial_x \partial_y u_{mn} \sin\varphi\,\widehat{\mathbf{e}}_x \\ &\quad + \left(-\frac{1}{k}\partial_y^2 u_{mn} \sin\varphi - k u_{mn} \sin\varphi\right)\widehat{\mathbf{e}}_y \\ &\quad + \left(-\frac{1}{k}\partial_z \partial_y u_{mn} \sin\varphi - \partial_y u_{mn} \cos\varphi\right)\widehat{\mathbf{e}}_z,\end{aligned} \tag{9b}$$

where $\varphi = kz - \omega t$. The HG functions are solutions to the paraxial wave equation but not to the wave equation without approximations. It is therefore not surprising that strict results emanating from the wave equations are only approximately satisfied by the HG solutions. It is sometimes useful to retain higher order terms in the paraxial solution in order to improve the quality of the solution in a larger domain. These improved solutions reveal some new features regarding the relationship between helicity and flow as we shall see in Section 4.

The HG solutions obtained directly from the fields and Maxwell's equations without invoking the potentials are [15]

$$\begin{aligned}\mathbf{E} &= -\omega u_{mn} \sin\varphi\,\widehat{\mathbf{e}}_x - \frac{c}{k}\partial_y \partial_x u_{mn} \sin\varphi\,\widehat{\mathbf{e}}_y \\ &\quad - c\partial_x u_{mn} \cos\varphi\,\widehat{\mathbf{e}}_z\end{aligned} \tag{10}$$

and

$$\begin{aligned}\mathbf{B} &= -\frac{1}{k}\partial_x \partial_y u_{mn} \sin\varphi\,\widehat{\mathbf{e}}_x - k u_{mn} \sin\varphi\,\widehat{\mathbf{e}}_y \\ &\quad - \partial_y u_{mn} \cos\varphi\,\widehat{\mathbf{e}}_z,\end{aligned} \tag{11}$$

where the argument of the original solutions has been shifted by $\pi/2$ and an ω factor has been added for ease of comparison. These substitutions are most easily seen by looking at the $\widehat{\mathbf{e}}_x$ electric field direction; when the starting point is the field, $\mathbf{E}_x = u_{mn} \cos\varphi\,\widehat{\mathbf{e}}_x$, whereas when it is the magnetic vector potential, $\mathbf{E}_x = -\partial_t \mathbf{A}_x = -\omega u_{mn} \sin\varphi\,\widehat{\mathbf{e}}_x$. The solutions (10) and (11) are obtained from the previous symmetric case (9a) and (9b) by retaining only the lowest order term in each of the three directions.

3. Helicity Gauge Invariant Continuity Equation

If the helicity $\varrho_{\mathbf{AC}}$ and its flow $\boldsymbol{J}_{\mathbf{AC}}$ are defined without scalar potential terms,

$$\varrho_{\mathbf{AC}} \coloneqq \mu\varepsilon\,(\mathbf{A} \cdot \mathbf{B} - \mathbf{C} \cdot \mathbf{E}), \tag{12}$$

and

$$\boldsymbol{J}_{\mathbf{AC}} \coloneqq \mu\varepsilon \mathbf{E} \times \mathbf{A} + \mathbf{B} \times \mathbf{C}, \tag{13}$$

the gauge invariant continuity equation for free fields is [17],

$$\begin{aligned}&\nabla \cdot (\mu\varepsilon \mathbf{E} \times \mathbf{A} + \mathbf{B} \times \mathbf{C}) + \mu\varepsilon\partial_t (\mathbf{A} \cdot \mathbf{B} - \mathbf{C} \cdot \mathbf{E}) \\ &\quad = -\mu\varepsilon\,(\mathbf{E} \cdot \nabla\phi_C - \mathbf{B} \cdot \nabla\phi_A).\end{aligned} \tag{14}$$

This equation is gauge invariant, although the helicity $\varrho_{\mathbf{AC}}$ and its flow $\boldsymbol{J}_{\mathbf{AC}}$ are not. This state of affairs is similar to the well known $\mathbf{A}, \phi_A$ potentials wave equations, where the wave equations are gauge invariant although the potentials themselves are gauge dependent. The helicity is in general not conserved due to the $\mu\varepsilon(\mathbf{E}\cdot\nabla\phi_C - \mathbf{B}\cdot\nabla\phi_A)$ source terms on the right hand side of the equation. For arbitrary fields, it is only in the Coulomb gauge that these terms become zero (since $\partial_t \nabla \cdot \mathbf{A} = 0 = -\nabla^2\phi_A$, then $\phi_A = 0$ if the potential falls to zero at infinity, and a similar argument holds for ϕ_C), then, the helicity for free fields is conserved. However, as we shall presently see, these terms will be zero in the Lorenz gauge for HG functions.

The gauge invariant continuity equation (14) has been derived with the complementary fields formalism. The potential fields $\mathbf{A}$ and $\mathbf{C}$ are considered complementary in the sense that energy is being exchanged between these two fields and their time derivatives. There is a mechanical analogue to the complementary fields approach in terms of kinetic and potential energy [23]. In the mechanics of a one-dimensional harmonic oscillator, say a simple pendulum in the linear regime, the complementary fields invariant is the sum of kinetic and potential energies $\mathscr{E} = m\omega^2\rho^2\cos^2\omega t + g\rho^2\sin^2\omega t$. This result evinces the fact that invariants may be obtained in the complementary fields approach without performing any time averaging.

4. Helicity and Flow of HG Modes

The helicity terms $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{C}\cdot\mathbf{E}$ evaluated from the Hermite-Gauss solutions (5) and (6) in Section 2 are zero up to first order derivatives in u_{mn}. The helicity is then equal to zero $\varrho_{\mathbf{AC}}^{(\text{first order})} = 0$. However, the terms involved in the flow are

$$\mathbf{E}\times\mathbf{A} = -cu_{mn}\partial_x u_{mn}\cos^2(kz-\omega t)\,\hat{\mathbf{e}}_y + \mathcal{O}^{2+},$$
$$\mathbf{B}\times\mathbf{C} = \frac{u_{mn}}{c}\partial_y u_{mn}\cos^2(kz-\omega t)\,\hat{\mathbf{e}}_x, \tag{15}$$

so that the flow is

$$\boldsymbol{J}_{\mathbf{AC}}^{(\text{first order})} = \frac{u_{mn}}{c}\partial_y u_{mn}\cos^2(kz-\omega t)\,\hat{\mathbf{e}}_x - \frac{u_{mn}}{c}\partial_x u_{mn}\cos^2(kz-\omega t)\,\hat{\mathbf{e}}_y. \tag{16}$$

Therefore, although the mode is linearly polarized, there is a contribution to the helicity flow in a direction transverse to propagation. This transverse spin angular momentum density has been suggestively dubbed *photonic wheel* [3]. The $\cos^2(kz-\omega t)$ reveals that this term oscillates in spacetime. Nonetheless, its average has a nonzero contribution $\langle \boldsymbol{J}_{\mathbf{AC}}^{(\text{first order})}\rangle = (u_{mn}/2c)\partial_y u_{mn}\hat{\mathbf{e}}_x - (u_{mn}/2c)\partial_x u_{mn}\hat{\mathbf{e}}_y$. However, it is somewhat surprising that the field helicity vanishes because then the flow cannot be written in the form $\boldsymbol{J}_{\mathbf{AC}} = \varrho_{\mathbf{AC}}\mathbf{v}$, where $\mathbf{v}$ is the velocity of propagation of $\varrho_{\mathbf{AC}}$ since $\varrho_{\mathbf{AC}}^{(\text{first order})} = 0$. Straightforward computation shows that $\nabla\cdot\boldsymbol{J}_{\mathbf{AC}}^{(\text{first order})} = 0$, whereas the source terms are $\mathbf{E}\cdot\nabla\phi_C = (c^2/\omega)(-k\partial_x u_{mn}\partial_y u_{mn}\cos^2\varphi)$ and $\mathbf{B}\cdot\nabla\phi_A = (c^2/\omega)(-k\partial_y u_{mn}\partial_x u_{mn}\cos^2\varphi)$, so that $\Upsilon_{\mathbf{AC}}^{(\text{first order})} = \mu\varepsilon(\mathbf{E}\cdot\nabla\phi_C - \mathbf{B}\cdot\nabla\phi_A) = 0$; thus the conservation equation is fulfilled.

The helicity in elementary particle physics is conceived as the projection of the spin onto the direction of motion [24]. It has been conjectured by Afanasiev and Stepanovsky that the helicity defined in this way and the helicity $\varrho_{\mathbf{AC}}$ are not equivalent. This statement is at the core of their explanation to avoid conflict with the Weinberg-Witten theorem [25]. However, if the helicity $\varrho_{\mathbf{AC}}$ is considered to be proportional to the projection of the spin in the direction of propagation, then $\varrho_{\mathbf{AC}} \propto \boldsymbol{J}_{\mathbf{AC}}^{(\text{first order})}\cdot\hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ is the unit wavevector. The argument $kz-\omega t$ in the trigonometric functions evinces that $\hat{\mathbf{k}} = \hat{\mathbf{e}}_z$ is the direction of propagation in the HG paraxial solutions. Therefore, a reasonable explanation for zero helicity is that since there is no flow in the direction of propagation, i.e., $\boldsymbol{J}_{\mathbf{AC}}^{(\text{first order})}$ has null $\hat{\mathbf{e}}_z$ component, the projection of the spin in this direction is zero.

Let us appraise the rotational content when three component electromagnetic fields are considered. In contrast with the first order solution, if second order derivatives are retained, the helicity terms are nonzero,

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{C}\cdot\mathbf{E} = -\frac{1}{2k}u_{mn}\partial_x\partial_y u_{mn}\sin(2(kz-\omega t)). \tag{17}$$

Nonetheless, since both terms are equal, the helicity is zero. The two flow terms now have a nonvanishing $\hat{\mathbf{e}}_z$ component equal to $\mp(u_{mn}/2\omega)\partial_y\partial_x u_{mn}\sin(2(kz-\omega t))\hat{\mathbf{e}}_z$. However, they are equal but with opposite signs so that there is still no net contribution in the $\hat{\mathbf{e}}_z$ direction,

$$\boldsymbol{J}_{\mathbf{AC}}^{(2s)} = \left(\frac{u_{mn}}{2\omega}\partial_z\partial_y u_{mn}\sin(2\varphi) + \frac{u_{mn}}{c}\partial_y u_{mn}\cos^2\varphi\right)\hat{\mathbf{e}}_x - \left(\frac{u_{mn}}{2\omega}\partial_z\partial_x u_{mn}\sin(2\varphi) + \frac{u_{mn}}{c}\partial_x u_{mn}\cos^2\varphi\right)\hat{\mathbf{e}}_y, \tag{18}$$

where $\varphi = kz-\omega t$. The superscript label $(2s)$ is placed to distinguish between the second order derivative terms in u_{mn} which are retained and the electromagnetic fields which are evaluated in a symmetric way so that they both have three components (9a)-(9b). The divergence of the flow is zero $\nabla\cdot\boldsymbol{J}_{\mathbf{AC}}^{(2s)} = 0$, and so are the source terms up to second order $\Upsilon_{\mathbf{AC}}^{(2s)} = \mu\varepsilon(\mathbf{E}\cdot\nabla\phi_C - \mathbf{B}\cdot\nabla\phi_A) = 0$, so it is reassuring that the continuity equation is fulfilled. The above results suggest that the magnetic helicity,

$$\varrho_{\mathbf{A}} \equiv \mathbf{A}\cdot\mathbf{B} = -\frac{1}{2k}u_{mn}\partial_x\partial_y u_{mn}\sin(2\varphi), \tag{19}$$

has a corresponding flow in the z direction given by

$$\boldsymbol{J}_{\mathbf{A}z}^{(2s)} = \left(c^{-2}\mathbf{E}\times\mathbf{A}\right)_z^{(2s)} = -\frac{u_{mn}}{2\omega}\partial_y\partial_x u_{mn}\sin(2\varphi)\,\hat{\mathbf{e}}_z. \tag{20}$$

The electric helicity $\varrho_{\mathbf{C}} = -\mathbf{C}\cdot\mathbf{E}$ fulfills analogous expressions but with opposite signs. The ratio of the flow over the helicity for each pair is

$$\frac{\left(c^{-2}\mathbf{E}\times\mathbf{A}\right)_z}{c^{-2}\mathbf{A}\cdot\mathbf{B}} = \frac{(\mathbf{B}\times\mathbf{C})_z}{-c^{-2}\mathbf{C}\cdot\mathbf{E}} = c^2\frac{k}{\omega}\hat{\mathbf{e}}_z = c\hat{\mathbf{e}}_z, \tag{21}$$

so that, for each term, the flow is equal to the helicity times the velocity. These results reinforce the interpretation that $\varrho_{\mathbf{AC}}$ is indeed the projection of the spin onto the direction of motion. It is possible to establish continuity type equations for the magnetic helicity and the electric helicity separately [26]. However, an extra $\pm 2\mathbf{E}\cdot\mathbf{B}$ term crops up in the differential equation. The magnetic helicity differential equation is $\nabla\cdot(\mathbf{E}\times\mathbf{A}) + \partial_t(\mathbf{A}\cdot\mathbf{B}) = -2\mathbf{E}\cdot\mathbf{B} - \mathbf{B}\cdot\nabla\phi_A$, whereas the electric helicity differential equation is $\nabla\cdot(\mathbf{B}\times\mathbf{C}) - \mu\varepsilon\partial_t(\mathbf{C}\cdot\mathbf{E}) = 2\mu\varepsilon\mathbf{E}\cdot\mathbf{B} + \mu\varepsilon\mathbf{E}\cdot\nabla\phi_C$. Their sum with the appropriate factors reproduces (14).

Since the early days of AM conservation, it was recognized that the flow could be negative for a positive z propagating wave. This result was interpreted as a ‘retrograde’ flow; that is, it was conjectured that the assessed density could be transported opposite to the direction of propagation [27]. Recall AM results for a plane wave propagating in the z direction with amplitude E_0 and polarization in terms of the eccentricity e [28],

$$\varrho_{\mathbf{AC}} \propto \mp E_0^2\sqrt{1-e^2},$$
$$\boldsymbol{J}_{\mathbf{AC}} \propto \mp cE_0^2\sqrt{1-e^2}\hat{\mathbf{e}}_z. \tag{22}$$

Linear polarization corresponds to $e = 1$; in this case the magnetic and electric helicities are both zero; $\mathbf{A}\cdot\mathbf{B} = -\mathbf{C}\cdot\mathbf{E} = 0$. Circular polarization is obtained for $e = 0$; right (left) hand elliptically polarized light has negative (positive) helicity in accordance with the terminology of particle physics. For plane wave solutions, the electric and magnetic helicities are always equal (sign included), and thus each of them contributes equally to the total helicity. The assessed quantity $\varrho_{\mathbf{AC}}$, unlike energy in Poynting's theorem, is not positive definite. Thus the helicity conservation equation is akin to an electric charge conservation equation rather than a mass conservation equation. There is no retrograde flow, but flow of the negative or positive helicity always in the direction of propagation as seen from the ratio of flow over helicity (21), either for HG modes or plane waves. A different although akin problem is the energy backflow in the focal region of a beam with a polarization singularity [29]. While the helicity density is not positive definite, the energy density in Poynting's theorem is positive definite. Thus, a detailed analysis of this problem within the present framework is not straightforward. However, it seems plausible that, with the appropriate modifications, this novel prediction could be undertaken with the complementary fields approach.

For linearly polarized HG modes, the magnetic helicity is given by (19) and an electric helicity with equal magnitude but opposite sign. This possibility is not present in plane waves where both contributions are always the same. The electric and magnetic helicities have their flow counterparts in the propagation direction in HG modes. In addition, there are spin terms orthogonal to the propagation direction (18). The average flow is the same to first or second order in the u_{mn} derivatives,

$$\begin{aligned}\left\langle \boldsymbol{J}_{\mathbf{AC}}^{(2s)} \right\rangle &= \left\langle \boldsymbol{J}_{\mathbf{AC}}^{(\text{first order})} \right\rangle \\ &= \frac{u_{mn}}{2c}\partial_y u_{mn}\widehat{\mathbf{e}}_x - \frac{u_{mn}}{2c}\partial_x u_{mn}\widehat{\mathbf{e}}_y.\end{aligned} \tag{23}$$

These photonic wheel type terms carry angular momentum perpendicular to the propagation direction. Strictly speaking, there should be transverse helicity terms corresponding to the transverse spin components. However, the paraxial approximation ignores this contribution and, thus, it will not be present unless exact solutions are implemented. It is not a matter of retaining terms to higher order in u_{mn} derivatives; the problem resides in the fact that u_{mn} is time independent.

5. Conclusions

Linearly polarized Hermite-Gauss modes do not carry angular momentum in the direction of propagation. However, they carry a transverse angular momentum that arises from terms like $\mathbf{E}\times\mathbf{A}$, usually associated with SAM. However, this transverse flow is dependent on the amplitude structure of the mode, usually associated with OAM. Therefore, identifying these nonvanishing transverse components as spin or orbital is not unequivocal. HG modes have been used in this manuscript since their rotational content has been extensively studied. The present analysis can thus be readily compared with previous results. However, other modes will carry a transverse angular momentum provided that the amplitude modulation is spatially dependent on the transverse coordinates, although not necessarily with a modulation given by the Hermite polynomials. Barnett and Allen have already mentioned, in the context of nonparaxial beams, that the identification of terms as spin or orbital is not unique [30]. The AM decomposition is equivalent in quantum electrodynamics and chromodynamics (QCD), with the latter only requiring an additional sum over colors [6]. In QCD, there is a quark spin transverse to the target momentum and the impact parameter. This transverse flow is due to the spin-orbit coupling [31]. The similitude with the present case deserves further elucidation. The spin and orbit AM concepts are again at the forefront of the discussion. The helicity and flow $\varrho_{\mathbf{AC}}$, $\boldsymbol{J}_{\mathbf{AC}}$ are intrinsic, in the sense that they do not depend on properties other than those coming from the fields, such as the choice of origin. The separation of AM in intrinsic and extrinsic is perhaps less unambiguous than the spin/orbital splitting.

There is consistency when the helicity and its flow are evaluated to various degrees of approximation. However, as it has been shown, the less crude approximation gives rise to nonvanishing terms in the magnetic and electric helicities. In linearly polarized HG modes, these contributions cancel out to yield zero helicity but this is never the case for plane waves. Therefore, the contributions of the magnetic helicity $\mathbf{A}\cdot\mathbf{B}$ and the electric helicity $\mathbf{C}\cdot\mathbf{E}$ to the optical helicity may add up or cancel out depending on the wave functions involved.

The present results reinforce the interpretation that $\varrho_{\mathbf{AC}}$ is indeed the projection of the angular momentum onto the direction of motion. It is thus in this sense equal to the helicity concept used in particle physics, and its flow corresponds to the spin concept in quantized theories.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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