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Research Article

Generalized Diffusion Equation Associated with a Power-Law Correlated Continuous Time Random Walk

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In this work, a generalization of continuous time random walk is considered, where the waiting times among the subsequent jumps are power-law correlated with kernel function $M(t) = t^{\rho}(\rho > -1)$. In a continuum limit, the correlated continuous time random walk converges in distribution a subordinated process. The mean square displacement of the proposed process is computed, which is of the form $\langle x^2(t)\rangle \propto t^H = t^{1/(1+\rho+1/\alpha)}$. The anomy exponent H varies from α to $\alpha/(1+\alpha)$ when $-1 < \rho < 0$ and from $\alpha/(1+\alpha)$ to 0 when $\rho > 0$. The generalized diffusion equation of the process is also derived, which has a unified form for the above two cases.

1. Introduction

Continuous time random walk (CTRW), which was originally introduced to physics by Montroll and Weiss [1], has been applied successfully in many fields (e.g., the reviews [2–4] and references therein).

In a continuum one-dimensional space, a CTRW is a process where the motion of a random walker is described by a sequence of independent identically distributed (IID) positive waiting times $T_1, T_2, T_3, \ldots$ and a sequence of IID random jumps $X_1, X_2, X_3, \ldots$. Set $t_0 = 0$ and $t_n = \sum_{i=1}^{n} T_i$ as the time of n–th jump. Then, the process $N(t) = \max\{n \geq 0 : t_n \leq t\}$ counts the number of jumps of the walker up to time t. Consequently, the CTRW process, defined as

$$x(t) = x(N(t)) = \sum_{i=1}^{N(t)} X_i, \tag{1}$$

describes the position of the walker at time t.

The independence among the waiting times given rise to a renewal process is not always justified. As soon as the random walk has some form of memory, the variables become nonindependent. Examples are found in financial market dynamics [5], human motion patterns [6], and so on. Recently, these facts impel one to introduce the correlated CTRWs [7–18].

There exists two simple approaches to CTRW with correlated waiting times. One was introduced by Chechkin et al. in Ref. [7]. Authors assumed the corresponding waiting times to be weighted sums of independent random variables in the following form:

$$T_i = \sum_{j=1}^{i} M(i-j+1)\,\xi_j, \tag{2}$$

where $M(t) = t^{-\mu}/\Gamma(1-\mu)$ with $0 < \mu < 1$ and $\{\xi_j\}$ is the sequence of IID α–stable random variables with the one-sided totally skewed probability density function (PDF). The characteristic function of the random variables $\{\xi_j\}$ is given by

$$\left\langle e^{ik\xi_j}\right\rangle = \exp\left\{-|k|^{\alpha}\exp\left(-\frac{i\pi\alpha}{2}\right)\operatorname{sgn}(k)\right\}, \quad 0 < \alpha < 1. \tag{3}$$

Another correlated temporal structure was introduced by Tejedor et al. in Ref. [9]. The authors assumed that the waiting times $\{T_i\}$ equaled

$$T_i = \left|\sum_{j=1}^{i} \xi_j\right|, \tag{4}$$

where the distribution of the IID random variables $\{\xi_j\}$ is symmetric and α−stable with Fourier transform:

$$\left\langle e^{ik\xi_j}\right\rangle = \exp\left\{-\frac{1}{2}|k|^{\alpha}\right\}, \quad 0 < \alpha \le 2. \tag{5}$$

Magdziarz et al. generalized these two models by combining the underlying correlation mechanisms for waiting times $\{T_i\}$ in the following manner [12]:

$$T_i = \left|\sum_{j=1}^{i} M(i-j+1)\,\xi_j\right|, \tag{6}$$

where $M(t) = t^{\rho}$ ($\rho \in R$) is the memory function and $\{\xi_j\}$ is the sequence of IID α−stable random variables with the Fourier transform:

$$\left\langle e^{ik\xi_j}\right\rangle = \exp\left\{-|k|^{\alpha}\left(1 - i\beta\tan\left(\frac{\pi\alpha}{2}\right)\operatorname{sgn}(k)\right)\right\}, \quad 0 < \alpha < 1,\ |\beta| \le 1. \tag{7}$$

Note that, for $\beta = 1$, random variables $\{\xi_j\}$ are positive. By choosing $\beta = 1$ and $\rho = -\mu$, one obtains the correlated CTRW introduced in the study by Chechkin et al. [7]. On the other hand, choosing $\beta = 0$ and $M(t) = 1$, one obtains the case $0 < \alpha < 1$ of the correlated CTRW derived in the study by Tejedor et al. [8].

In the scaling limit, correlated CTRWs converge to the subordinated process $x(t) = x(s(t))$. Here, $s(t)$ is a continuum analog of the count process $N(t)$, defined by

$$s(t) = \inf\{s \ge 0 : t(s) > t\}. \tag{8}$$

The process $t(s)$ is a continuum analog of hitting time t_n.

One way to explore the statistical characteristics of the subordinated process $x(t) = x(s(t))$ is to consider its probability distribution. Here, we are interested in generalized diffusion equation associated with the subordinated process $x(t) = x(s(t))$. Note that reflecting boundary condition in waiting times is the disadvantage of deriving generalized diffusion equation. Therefore in this work, we limit $\beta = 1$ in Eq. (7) for our purpose. That is, we assume

$$T_i = \sum_{j=1}^{i}(i-j+1)^{\rho}\,\xi_j, \quad \rho > -1, \tag{9}$$

where $\{\xi_j\}$ is IID random variables with a one-sided Lévy α−stable PDF $L_{\alpha}(t)$, whose Laplace transform is

$$\widetilde{L}_{\alpha}(u) = \int_0^{\infty} L_{\alpha}(t)\,e^{-ut}dt = \exp\{-u^{\alpha}\}, \quad 0 < \alpha < 1. \tag{10}$$

The structure of this work is as follows. In Section 2, we introduce the Langevin description of power-law correlated CTRW model. In Section 3, we compute the MSD of the subordinated process $x(t)$ and derive its generalized diffusion equation. The conclusions are given in Section 4.

2. Model

The Langevin equations for the position x and the time t corresponding to the continuous-time limit process of the above-defined correlated CTRW have the form:

$$\begin{aligned} \frac{d}{ds}x(s) &= \frac{d}{ds}B(s), \\ \frac{d}{ds}t(s) &= \int_0^{s}(s-s')^{\rho}\,dl_{\alpha}(s'), \quad 0 < \alpha < 1,\ \rho > -1. \end{aligned} \tag{11}$$

Here, $B(s)$ is a Brownian motion with variance $2s$ and $l_{\alpha}(s)$ is an α−stable totally skewed Lévy motion with Laplace transform:

$$\left\langle e^{-ul_{\alpha}(s)}\right\rangle = \exp\{-su^{\alpha}\}. \tag{12}$$

Note that when $-1 < \rho < 0$ one obtains the correlated CTRW introduced in Ref. [7].

To solve Eqs. (11), one first solves the first equation to produce the driving process $B(s)$. Next, one solves the second equation to obtain the process $t(s)$, thus yielding the process $s(t)$, which is inverse to $t(s)$. Finally, one assembles both processes $B(s)$ and $s(t)$ to obtain the solution as the subordination

$$x(t) = B(s(t)). \tag{13}$$

Here the subordinator $s(t)$, defined by Eq. (8), is the inverse of

$$t(s) = \int_0^{s}\left[\int_0^{s'}(s'-s'')^{\rho}\,dl_{\alpha}(s'')\right]ds'. \tag{14}$$

By exchanging integral order for Eq. (14), the process $t(s)$ can be rewritten as

$$t(s) = \frac{1}{\rho+1}\int_0^{s}(s-s')^{\rho+1}\,dl_{\alpha}(s'). \tag{15}$$

3. Discussions

Proposition I. The second moment (MSD) of the process $x(t) = B(s(t))$ is finite and proportional to t^H with $H = 1/(1+\rho+1/\alpha)$.

Proof. Assume that $p(x,t)$, $f(x,s)$, and $g(s,t)$ are the PDFs of processes $x(t)$, $B(s)$, and $s(t)$, respectively. Using the total probability formula, we obtain

$$p(x,t) = \int_0^{\infty} f(x,s)\,g(s,t)\,ds. \tag{16}$$

Thus,

$$\begin{aligned} \left\langle x^2(t)\right\rangle &= \int_{-\infty}^{\infty} x^2 p(x,t)\,dx \\ &= \int_{-\infty}^{\infty} dx\left[\int_0^{\infty} x^2 f(x,s)\,g(s,t)\,ds\right] \\ &= 2\int_0^{\infty} sg(s,t)\,ds = 2\langle s(t)\rangle. \end{aligned} \tag{17}$$

Since

$$\begin{aligned} t(s) &= \frac{1}{\rho+1}\int_0^s \left(s-s'\right)^{\rho+1} dl_\alpha\left(s'\right) \\ &=^d \frac{l_\alpha(1)}{\alpha(\rho+1)}\int_0^s \left(s-s'\right)^{\rho+1} s'^{1/\alpha-1} ds' \\ &= \frac{l_\alpha(1)}{\alpha(\rho+1)} s^{1+\rho+1/\alpha}\int_0^1 (1-\tau)^{\rho+1}\tau^{1/\alpha-1} d\tau \\ &= As^{1+\rho+1/\alpha} l_\alpha(1) = As^{1/H} l_\alpha(1), \end{aligned} \tag{18}$$

where the sign "$=^d$" means the same finite-dimensional distributions, $A = B(1/\alpha, 2+\rho)/\alpha(\rho+1)$ and $B(\cdot,\cdot)$ is beta function. We obtain the PDF $h(t,s)$ of the process $t(s)$ in Laplace space

$$\begin{aligned} \tilde{h}(u,s) &= \left\langle e^{-ut(s)}\right\rangle = \left\langle e^{-uAs^{1/H} l_\alpha(1)}\right\rangle \\ &= \exp\left\{-s^{\alpha/H}A^\alpha u^\alpha\right\}. \end{aligned} \tag{19}$$

Considering the relation $g(s,t) = (d/ds)\int_t^\infty h(t',s)dt'$, we have

$$\begin{aligned} \tilde{g}(s,u) &= -\frac{d}{ds}\frac{1}{u}\tilde{h}(u,s) \\ &= \frac{\alpha}{H}A^\alpha s^{\alpha/H-1}u^{\alpha-1}\exp\left\{-s^{\alpha/H}A^\alpha u^\alpha\right\}. \end{aligned} \tag{20}$$

Thus,

$$\begin{aligned} \langle s\rangle(u) &= \int_0^\infty s\tilde{g}(s,u)\,ds \\ &= \frac{\alpha}{H}A^\alpha u^{\alpha-1}\int_0^\infty s^{\alpha/H}\exp\left\{-s^{\alpha/H}A^\alpha u^\alpha\right\} ds \\ &= A^\alpha u^{\alpha-1}\int_0^\infty \tau^{H/\alpha}\exp\left\{-A^\alpha u^\alpha \tau\right\} d\tau \\ &\qquad\qquad \left(\tau = s^{\alpha/H}\right) \\ &= \frac{\Gamma(H/\alpha+1)}{A^H}u^{-H-1}. \end{aligned} \tag{21}$$

Taking inverse Laplace transform on $u \longmapsto t$, we get

$$\langle s(t)\rangle = \frac{\Gamma(H/\alpha)}{\alpha A^H\Gamma(H)}t^H. \tag{22}$$

Substituting Eq. (22) into Eq. (17), we obtain

$$\left\langle x^2(t)\right\rangle \propto t^H. \tag{23}$$

Note that anomy exponent $H = 1/(1+\rho+1/\alpha)$ decreases from α to $\alpha/(1+\alpha)$ with the parameter ρ varying from -1 to 0 and decreases from $\alpha/(1+\alpha)$ to 0 with the parameter ρ varying from 0 to ∞. □

Proposition II. The PDF $p(x,t)$ of $x(t)$ satisfies the generalized diffusion equation:

$$\frac{\partial}{\partial t}p(x,t) = \frac{\partial^2}{\partial x^2}Gp(x,t), \quad t>0, \tag{24}$$

where the operator G acts on variable t and is defined by

$$\begin{aligned} Gp(x,t) &= L^{-1}_{u\longrightarrow t}\left[\frac{1}{2\pi i}\right. \\ &\quad \left.\cdot\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(H/\alpha+1)}{(\Psi(u)-v)^{H/\alpha}}\frac{\Psi^{-1}(v)}{v}\tilde{p}\left(x,\Psi^{-1}(v)\right)dv\right]. \end{aligned} \tag{25}$$

Here $\Psi^{-1}(u)$ denotes the inverse in point u of $\Psi(u) = (Au)^\alpha$ and $L^{-1}_{u\longrightarrow t}$ is the inverse Laplace transform $u \longrightarrow t$.

Proof. From Eqs. (16) and (20), we obtain in Laplace space

$$\begin{aligned} \tilde{p}(x,u) &= \int_0^\infty p(x,t)e^{-ut}dt \\ &= \int_0^\infty f(x,s)\tilde{g}(s,u)\,ds \\ &= \frac{\alpha}{H}A^\alpha u^{\alpha-1}\int_0^\infty f(x,s)s^{\alpha/H-1}\exp\left\{-s^{\alpha/H}A^\alpha u^\alpha\right\} ds \\ &= A^\alpha u^{\alpha-1}\int_0^\infty f\left(x,\tau^{H/\alpha}\right)\exp\left\{-(Au)^\alpha\tau\right\} d\tau \\ &\qquad\qquad \left(\tau = s^{\alpha/H}\right) \\ &= \frac{\Psi(u)}{u}\tilde{f}_1(x,\Psi(u)), \end{aligned} \tag{26}$$

where $f_1(x,\tau) = f(x,\tau^{H/\alpha})$ and $\Psi(u) = (Au)^\alpha$.

Since $\Psi(u)$ is monotone, we notice that after transformation $u \longrightarrow \Psi^{-1}(u)$ one obtains

$$\tilde{f}_1(x,u) = \frac{\Psi^{-1}(u)}{u}\tilde{p}\left(x,\Psi^{-1}(u)\right). \tag{27}$$

Since

$$\begin{aligned} \frac{\partial}{\partial\tau}f_1(x,\tau) &= \frac{\partial}{\partial\tau}f\left(x,\tau^{H/\alpha}\right) \\ &= \frac{\partial}{\partial s}f(x,s)\cdot\frac{ds}{d\tau} \quad \left(s=\tau^{H/\alpha}\right) \\ &= \frac{\partial^2}{\partial x^2}f(x,s)\cdot\frac{H}{\alpha}\tau^{H/\alpha-1} \\ &= \frac{\partial^2}{\partial x^2}f_1(x,\tau)\cdot\frac{H}{\alpha}\tau^{H/\alpha-1}, \end{aligned} \tag{28}$$

Eq. (28) in Laplace space takes the form

$$\begin{aligned} &u\tilde{f}_1(x,u) - f_1(x,0) \\ &\quad = \frac{\partial^2}{\partial x^2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{f}_1(x,v)\frac{\Gamma(H/\alpha+1)}{(u-v)^{H/\alpha}}dv. \end{aligned} \tag{29}$$

After changing the variable $u \longrightarrow \Psi(u)$ in Eq. (29) and using the initial condition $p(x,0) = f(x,0) = f_1(x,0) = \delta(x)$, we obtain that in the Laplace domain the PDF $p(x,t)$ of $x(t)$ obeys

$$\begin{aligned} u\widetilde{p}(x,u) - p(x,0) &= \Psi(u)\,\widetilde{f}_1(x,\Psi(u)) - f_1(x,0) \\ &= \frac{\partial^2}{\partial x^2}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\widetilde{f}_1(x,v)\frac{\Gamma(H/\alpha+1)}{(\Psi(u)-v)^{H/\alpha}}dv \\ &= \frac{\partial^2}{\partial x^2}\frac{1}{2\pi i} \\ &\cdot\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(H/\alpha+1)}{(\Psi(u)-v)^{H/\alpha}}\frac{\Psi^{-1}(v)}{v}\widetilde{p}\left(x,\Psi^{-1}(v)\right)dv. \end{aligned} \tag{30}$$

Thus, after taking inverse Laplace transform on $u \longmapsto t$ for Eq. (30), we obtain the generalized diffusion equation:

$$\begin{aligned} \frac{\partial}{\partial t}p(x,t) &= \frac{\partial^2}{\partial x^2}L^{-1}_{u\longrightarrow t}\left[\frac{1}{2\pi i}\right. \\ &\left.\cdot\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(H/\alpha+1)}{(\Psi(u)-v)^{H/\alpha}}\frac{\Psi^{-1}(v)}{v}\widetilde{p}\left(x,\Psi^{-1}(v)\right)dv\right] \\ &= \frac{\partial^2}{\partial x^2}Gp(x,t). \end{aligned} \tag{31}$$

□

4. Conclusions

In this work, a CTRW model with power-law correlated waiting times is considered. The kernel function $M(t) = t^{\rho}$ contains two cases: $-1 < \rho < 0$ and $\rho > 0$. The MSD of the subordinated process $x(t) = x(s(t))$ based on the proposed correlated CTRW is computed, which is proportional to $t^H = t^{1/(1+\rho+1/\alpha)}$. One can observe that in the case $-1 < \rho < 0$ the exponent H decreases from α to $\alpha/(1+\alpha)$ and in the case $\rho \geq 0$ the exponent H decreases from $\alpha/(1+\alpha)$ to 0. The proposed model supplements the discussions in Ref. [7] on CTRW with correlated waiting times. A generalized diffusion equation associated with the subordinated process $x(t) = x(s(t))$ is also derived by using subordination and Laplace transform technique.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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